MATH1013 Calculus I

Functions I [1]

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[1] Steward, James, "Single Variable Calculus, Early Transcendentals", 7th edition, Brooks/Coles, 2012
Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

1994 Northridge, LA earthquake (a)

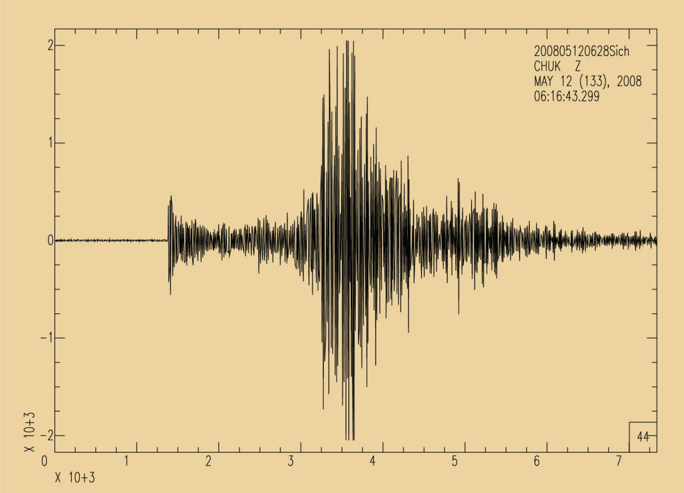


Figure: Stewart: Chap. 1, Figure 0

Definition of functions

- **Definition** A function is a rule f that assigns to each x (independent valuable) in a set D a unique value denoted by $f(x)$ (dependent valuable).
- **Definitions** The set D is called the domain of the function f, and the set of values of $f(x)$ assumes, as x varies over the domain, is called the range of the function $f(x)$.
- $$x \longmapsto f(x), \qquad \text{or} \qquad y = f(x),$$
- One can think of this as a model of

  one input $\rightarrow$ **one output**

- Important point: for **each** x in D, one can find (there exists) **one** value $f(x)$ (or y) that corresponds to it.
- However, depending on the f under consideration, one could have **two or more** x that correspond to the **same** $f(x)$.

Steward: Chapter 1, Slide 4

The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

Examples

- The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r.
- The cost C of mailing a large envelope depends on the weight w of the envelope. Although there is no simple explicit formula that connects w and C, the post office has a rule for determining C when w is known.
- The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t.

Figure of a function

FIGURE 1
Vertical ground acceleration during the Northridge earthquake

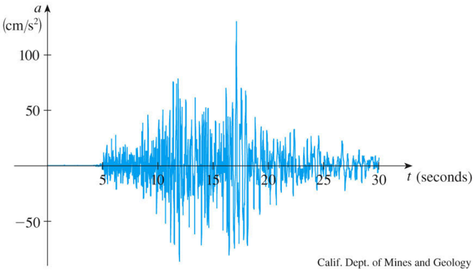


Figure: Stewart: Chap. 1, Figure 1

Represent a function

Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

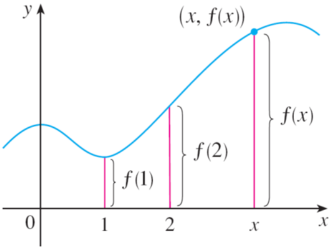


Figure 4

Figure: Stewart: Chap. 1, Figure 4

Domain and range

The graph of *f* also allows us to picture the domain of *f* on the *x*-axis and its range on the *y*-axis as in Figure 5.

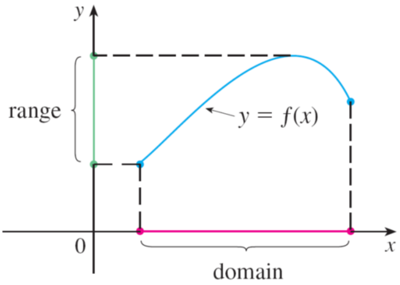


Figure 5

Figure: Stewart: Chap. 1, Figure 5

Example of domain and range

The graph of a function *f* is shown in Figure 6.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f ?

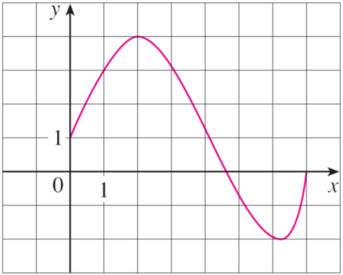


Figure 6

The notation for intervals is given in Appendix A.

Solution of last example

(a) We see from Figure 6 that the point (1, 3) lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval [0, 7]. Notice that f takes on all values from –2 to 4, so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

Figure: Stewart: Chap. 1, Figure 6 (soln)

Domain and Range figure I

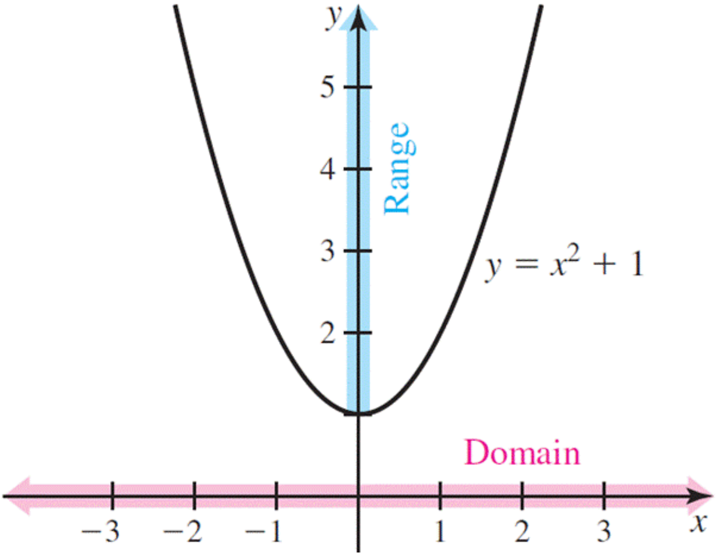


Figure: 1.4 (source: Briggs, et al)

Domain and Range figure II

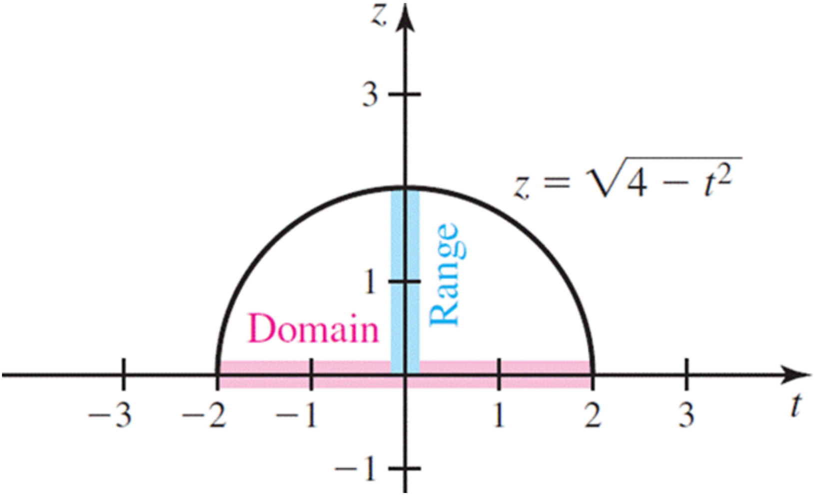


Figure: 1.5 (source: Briggs, et al)

Domain and Range figure III

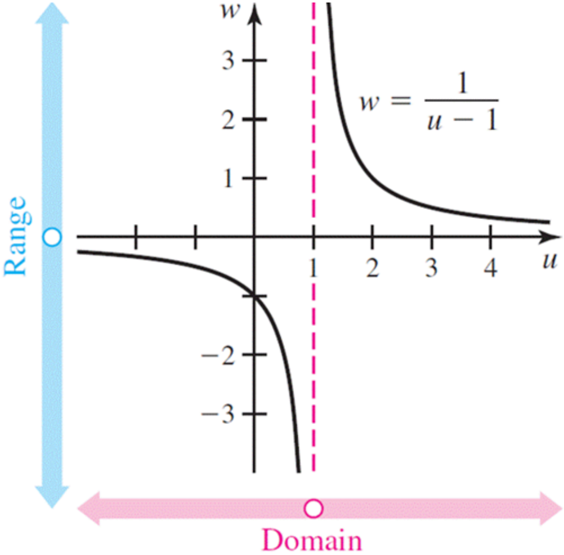


Figure: 1.6 (source: Briggs, et al)

Real numbers $\mathbb{R}$

- Let us recall about real numbers, and the following notation about intervals:

-

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

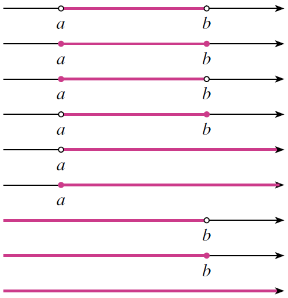


Figure: Stewart: Appendix A, figure A4

Inequalities

- We say that two numbers $a < b$ if $b - a > 0$. We take this as the definition, that whenever we see $a < b$, we interpret it this way.
  So we have
- Let m be positive. Then $ma < mb$.
  - Since $b - a > 0$, so $0 < m(b - a) = mb - ma$ so that $ma < mb$ by definition.
- Let m be negative. Then $ma > mb$.
  - Since $b - a > 0$, so $0 > m(b - a) = mb - ma$ so that $ma > mb$ by definition.
- Let $0 < c < d$. Then $1/d < 1/c$. The converse is also true, that is, if $1/d < 1/c$, then $c < d$.
  - Since c, d are positive, so $\frac{1}{c} - \frac{1}{d} = \frac{d-c}{cd} >$, so $1/d < 1/c$ holds. Conversely, if $1/d < 1/c$ holds, then $c = cd \cdot \frac{1}{d} < cd \cdot \frac{1}{c} = d$, as required.

Solving inequalities (p. A5)

- Solve $4 \leq 3x - 2 < 13$.

SOLUTION Here the solution set consists of all values of x that satisfy both inequalities. Using the rules given in [2], we see that the following inequalities are equivalent:

$$4 \leqslant 3x - 2 < 13$$

$$6 \leqslant 3x < 15 \qquad \text{(add 2)}$$

$$2 \leqslant x < 5 \qquad \text{(divide by 3)}$$

Therefore the solution set is $[2, 5)$.

Figure: Stewart: Appendix A, page A5

Solving inequalities (p. A5)

- Solve $x^2 - 5x + 6 \leq 0$.
  We consider for which x, we have

$$(x-2)(x-3) \leq 0.$$

Interval	$x-2$	$x-3$	$(x-2)(x-3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

Figure: Stewart: Appendix A, page A5

Solving inequalities (p. A6)

- Solve $x^3 + 3x^2 > 4x$.
  We consider for which x, we have

$$x^3 + 3x^2 - 4x = x(x-1)(x+4) > 0.$$

Interval	x	$x-1$	$x+4$	$x(x-1)(x+4)$
$x < -4$	$-$	$-$	$-$	$-$
$-4 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

Figure: Stewart: Appendix A, page A6

Absolute value (I)

- Let a be any real number. We define the absolute value of a to be

$$|a| = \begin{cases} a, & \text{if} \quad a \geq 0; \\ -a & \text{if} \quad a < 0. \end{cases}$$

- Solve $|3x - 2| = 1$

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

Figure: Stewart: Appendix A, page A7

- In that first case, we solve $3x - 2 = 1$ to obtain $x = 1$. In the second case, we need to solve $2 - 3x = 1$. That is, $x = 1/3$.

Properties of absolute value

$s^2 = r$ and $s \geqslant 0$. Therefore the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of $\boxed{3}$, we then have the equation

$$\sqrt{a^2} = |a|$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

Figure: Stewart: Appendix A, page A7

Absolute value inequalities

Suppose $a > 0$. Then

$$|x| = a \quad \text{if and only if} \quad x = \pm a$$

$$|x| < a \quad \text{if and only if} \quad -a < x < a$$

$$|x| > a \quad \text{if and only if} \quad x > a \text{ or } x < -a$$

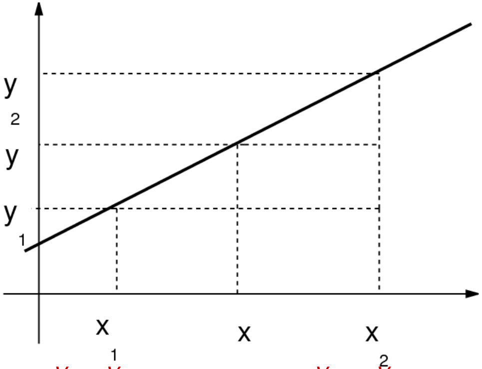
Figure: Stewart: Appendix A, page A7

- Solve $|x - 5| < 2$ and $|3x + 2| \geq 4$.
- Triangle inequality $|a + b| \leq |a| + |b|$.

Straight line equations (a)

- Some mathematics textbooks give many different forms of straight line equations, such as two-point form, point-slop form, slope-intercept form, etc. This could be confusing and does not necessary help us to gain better understanding about the equations.
- In plane geometry, we known that two points determine a straight line. When considering straight line equation in the $xy-$coordinate plane, the crucial quantity is the gradient of the line. For example, if we are given two points, then we shall use similar triangles to form the equation.
- Let (x_1, y_1) and (x_2, y_2) be two points on a line and (x, y) an arbitrary point lying on the straight line between the points (x_1, y_1) and (x_2, y_2). Then

Straight line equations (b)



$$\frac{y - y_1}{x - x_1} = \text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

That is,

$$(x_2 - x_1)y + (y_1 - y_2)x = x_2 y_1 - x_1 y_2.$$

Straight line equations (c)

- Show how we could have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y}{x_2 - x},$$

  and verify this relation also gives raise to same equation on last page.
- Do the same for the following expression as in the last equation:

$$\frac{y_2 - y}{x_2 - x} = \frac{y_2 - y_1}{x_2 - x_1},$$

  and verify this equation is again the same as above.

Two straight lines (a)

- Consider

$$a_1 x + a_2 y + a_3 = 0 \tag{1}$$

$$b_1 x + b_2 y + b_3 = 0. \tag{2}$$

Multiply the equation (1) by $-a_1/b_1$ gives

$$-a_1 x - a_1(b_2/b_1)y - b_3(a_1/b_1) = 0. \tag{3}$$

Adding the equation (1) and equation (3) gives

$$(a_2 - a_1(b_2/b_1))y + a_3 - b_3(a_1/b_1) = 0.$$

That is

$$y = \frac{a_1 b_3 - b_1 a_3}{a_2 b_1 - a_1 b_2}, \qquad x = \frac{a_3 b_2 - a_2 b_3}{a_2 b_1 - a_1 b_2}.$$

Two straight lines (b)

- Geometrically, the two equations represent two straight lines in the $xy-$coordinate plane. The quantity $a_2 b_1 - a_1 b_2$ is called the discriminant of the system. It is a quantity that measures the difference of gradients between the two lines.
- The gradient of first equation is $-\frac{a_1}{a_2}$ and the gradient of the second equation is $-\frac{b_1}{b_2}$.
- If the two lines are not parallel, then they are not equal. That is equivalent to $a_2 b_1 - a_1 b_2 \neq 0$. That is the two lines must intersect at a point whose coordinate is given on last page.
- If the two lines are parallel, then $-\frac{a_1}{a_2} = -\frac{b_1}{b_2}$. That is, the discriminant $a_2 b_1 - a_1 b_2 = 0$. But then there are two possibilities.
- Perpendicular lines: $m_1 m_2 = -1$.

Application: Stewart, 1, E.g. 5, p. 14 (Ex. 61, p. 21)

- A rectangular storage container with an open top has a volume of $10\ m^3$. The length of its base is twice its width. Material for the base costs $10/m^2$; material for the sides costs $6/m^2$. Express the cost of materials as a function of the width of the base.
- Let w and $2w$ be the width and length of its base respectively, and h its height. The volume $V = 10 = 2w^2h$. Let C stands for the total costs function, which makes up of the base cost and side costs. That is,

$$\begin{aligned} C &= 10 \times 2w^2 + 6 \times [2(2wh) + 2wh] = 20w^2 + 36wh \\ &= 20w^2 + 36w(5/w^2) = 20w^2 + 180/w. \end{aligned}$$

- It is clear that C is a function of w and that the domain of C is $w > 0$.
- We note that the range of C is more difficult to determine which is left to curve sketching later.

Application: Briggs, et al (p. 3)

- At time $t = 0$ a stone is thrown vertically upward from the ground at a speed of $30m/s$. Its height above the ground in meters is approximated by the function $h = f(t) = 30t - 5t^2$, where t is in seconds. Find the domain and range of this function as they apply to this particular problem.

Application: Stewart, 1, E.g. 6, p. 14

- Find the domains of the functions:
- E.g. 6 (a)

$$f(x) = \sqrt{x+2},$$

- E .g. 6 (b)

$$g(x) = \frac{1}{x^2 - x}.$$

- (Ex. 35)

$$h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}.$$

- (Ex. 37)

$$F(p) = \sqrt{2 - \sqrt{p}}.$$

Other functions (domains)

- $y = f(x) = x^2 + 1$,
- $y = g(t) = \sqrt{4 - t^2}$,
- $y = h(u) = \dfrac{1}{u - 1}$,
- $F(w) = \sqrt[4]{2 - w}$,
- $g(y) = \dfrac{y + 1}{(y + 2)(y - 3)}$

Piecewise defined functions, E.g. 7, p. 16 (Ex. ?? , p. 21)

- Sketch

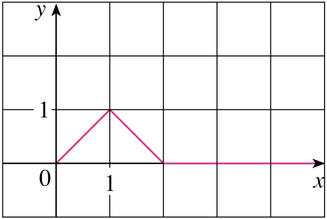
$$f(x) = \begin{cases} 1 - x, & \text{if } x \leq -1, \\ x^2, & \text{if } x > -1. \end{cases}$$

- Compute $f(-2)$, $f(-1)$, $f(0)$.
- (Ex. 49, p. 21) Sketch

$$f(x) = \begin{cases} x + 2, & \text{if } x \leq -1, \\ x^2, & \text{if } x > -1. \end{cases}$$

Absolute valued functions

- (Eg 9, p. 17) Find a formula that represents the following f:

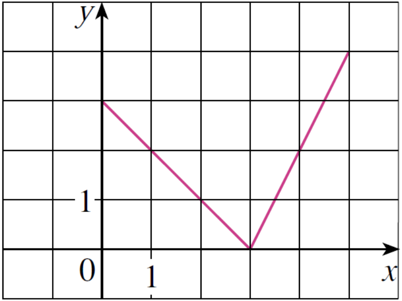


- We can represent the above function by x for $0 \leq x \leq 1$.
- It's straight line for $1 \leq x \leq 2$ with negative gradient -1 passing through the point $(1, 1)$.
- It equals $y = 0$ for $x \geq 2$.
- Putting all these cases together yields

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1; \\ 2 - x, & \text{if } 1 \leq x \leq 2; \\ 0, & \text{if } x \geq 2. \end{cases}$$

Absolute valued functions (Chap 1.1, Ex. 55)

- Find a formula that represents the following f:



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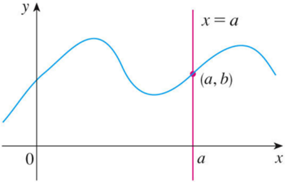
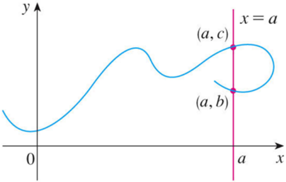
$$f(x) = \begin{cases} & \text{if } \quad \leq x \leq \\ & \text{if } x \geq \end{cases}$$

What is NOT a function

- **Definition** A function is a rule f that assigns to each x (independent valuable) in a set D a unique value denoted by $f(x)$ (dependent valuable).





- $y^2 = 1 - x^2$. Since for each x input, there always correspond to two outputs of $f(x) = \pm\sqrt{1 - x^2}$ within the domain of f.
- (Eg revisited) $f(x) = x^2 - 2x$ is not injective, as two different x can correspond to the same $f(x_1) = y = f(x_2)$

A quick test

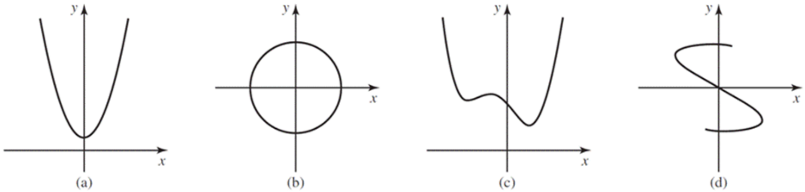


Figure: 1.3 (source Briggs, et al)

Even and odd functions

- **Definition** A function f is (i) even if $f(-x) = f(x)$ for all x lying within the domain of f, and (i) odd if $f(-x) = -f(x)$ for all x lying within the domain of f
- x^2, $x^2 + x^4$, $x^2 + 1/x^2$, $|x|$ are all even functions.
- x^3, x^{-1}, x^{-3} are all odd functions
- (Ex. 73, 74, 75, 76)

$$\frac{x}{x^2+1}, \qquad \frac{x^2}{x^4+1}, \qquad x|x|, \qquad \frac{x}{x+1}$$

- (Ex. 79) If f, g are both even (resp. odd) functions, is $f + g$ even (resp. odd)?

Figures of even and odd functions

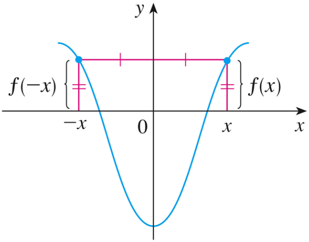
- 

Figure: Stewart: Figure 19

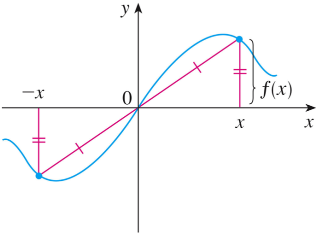
- 

Figure: Stewart: Figure 20

Different types of functions

- Polynomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the $a_{n-1}, \ldots a_1, a_0$ are some constants.

- Rational functions

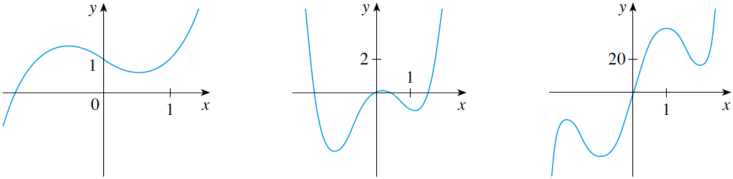
$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where the $a_n, \ldots a_1, a_0, b_m, \ldots b_1, b_0$ are constants

- Piecewise function

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ -\frac{1}{2}x + 5 & \text{if } x > 2. \end{cases}$$

Figures of polynomials

- 

(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Figure: Stewart: Chap 1.2, Figure 8

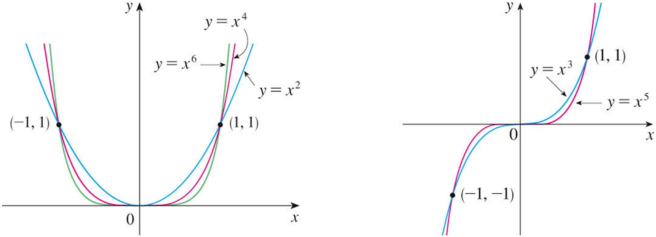
- 

Figure: Stewart: Chap 1.2, Figure 12

Figures of algebraic functions

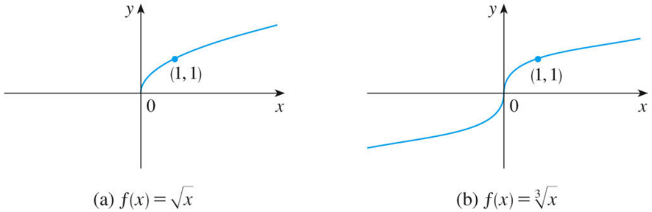
- Algebraic functions are those functions that carry *n*th-root
- 

Figure: Stewart: Chap 1.2, Figure 13

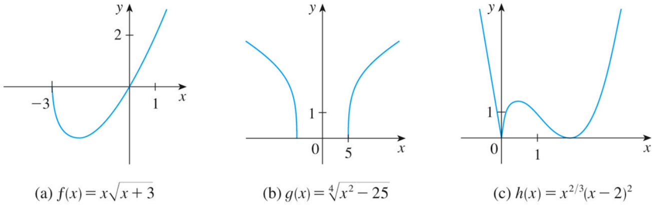
- 

Figure: Stewart: Chap 1.2, Figure 17