MATH150 Introduction to Ordinary Differential Equations, 2010-11 Fall
Midterm Exam. Solution

1. [10 pts] Find the solution $y = y(x)$ of the following initial value problems:

   (a)
$$\sqrt{1+x^2}y' = xy^3, \quad y(0) = -1.$$

   `Separating variables and integrating, we find`
$$\int_{-1}^{y} \frac{dy}{y^3} = \int_0^x \frac{x}{\sqrt{1+x^2}} dx.$$

   `Integrating,`
$$-\frac{1}{2}y^{-2}|_{-1}^{y} = \sqrt{1+x^2}|_0^x.$$

   `or`
$$-\frac{1}{2}(y^{-2}-1) = \sqrt{1+x^2}-1.$$

   `Solving for` y^2`:`
$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}.$$

   `Since only the negative root satisfies the initial conditions, we have`
$$y = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}. \quad [5\text{pts}]$$

   (b)
$$x^4y' + 4x^3y = e^{-x}, \quad y(1) = -1/e, \quad x \geq 1.$$

   `Rewriting the equation in standard form:`
$$y' + \frac{4}{x}y = \frac{e^{-x}}{x^4}.$$

   `The integrating factor is`
$$\mu = \exp\,(\int_1^x \frac{4}{x}dx = \exp\,(4\ln x) = x^4.$$

   `The solution is therefore`
$$y = \frac{1}{x^4}\left(-\frac{1}{e} + \int_1^x x^4\frac{e^{-x}}{x^4}dx\right),$$

   `or`
$$y = \frac{1}{x^4}\left(-\frac{1}{e} - (e^{-x} - \frac{1}{e})\right)$$

   `or`
$$y = -\frac{e^{-x}}{x^4}. \quad [5\text{pts}]$$

2. [10 pts]

(a) Given α, find the solution of the following initial value problem:

$$\ddot{x} - 4\dot{x} + 5x = 0, \quad x(0) = 1/2, \quad \dot{x}(0) = \alpha.$$

Since the characteristic equation is $r^2 - 4r + 5 = 0$. So the complex conjugate roots are $2 \pm i$. Thus the general solution is

$$x(t) = e^{2t}\big(A\cos t + B\sin t\big).$$

The initial condition $x(0) = 1/2$ gives $A = 1/2$, and $\dot{x}(0) = \alpha$ gives $B = \alpha - 1$.
The solution is ____. [6 pts]

(b) Determine α such that $x(\pi/2) = 0$. We deduce

$$0 = x(\pi/2) = e^{\pi}\big(\frac{1}{2}\cos\frac{\pi}{2} + (\alpha - 1)\sin\frac{\pi}{2}\big).$$

Thus $\alpha = 1$.
The solution is ____. [4 pts]

3. [10 pts] Assume that a spherical raindrop of volume $V = \frac{4}{3}\pi r^3$ evaporates at a rate proportional to its surface area $S = 4\pi r^2$, where r is the radius of the raindrop. That is, $\frac{dV}{dt} = -kS$.

(a) Determine a differential equation for $r = r(t)$ that depends on k.

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k\left(4\pi r^2\right),$$

or using the chain rule with $r = r(t)$

$$4\pi r^2\frac{dr}{dt} = -k\left(4\pi r^2\right).$$

The differential equation for r is therefore

$$\frac{dr}{dt} = -k. \quad \text{[5pts]}$$

(b) If the initial radius of the raindrop is r_0, solve for r to show that the radius decreases linearly with time.
Integrating the differential equation

$$\int_{r_0}^{r} dr = -k\int_0^t,$$

or

$$r(t) = r_0 - kt. \quad \text{[3pts]}$$

(c) At what time does the drop evaporate?
The drop evaporates when $r(T) = 0$. Therefore

$$T = \frac{r_0}{k}. \quad \text{[2pts]}$$

4. [10 pts] Find the solution $x = x(t)$ of the following initial value problem:

$$\ddot{x} + 3\dot{x} - 10x = 7e^{-5t} + 30e^{5t}, \qquad x(0) = 1, \ \dot{x}(0) = 4.$$

We immediately observe that the characteristic equation of the homogeneous equation

is $0 = r^2 + 3r - 10 = (r+5)(r-2)$ implying the characteristic roots to be $r = 2, -5$ and $\{e^{2t}, \ e^{-5t}\}$. Hence one of the two linearly independent solutions is identical to a non-homogeneous term e^{-5t}. So

$$y_p = Ae^{5t} + Bte^{-5t}.$$

Substitute this into the equation yields $A = 1$ and $B = -1$. So the general solution is

$$y = y_h + y_p = c_1e^{2t} + c_2e^{-5t} + e^{5t} - te^{-5t}.$$

Applying the initial condition that $y(0) = 1$ and $y'(0) = 4$ yields $c_1 = 0 = c_2$.
The solution is ____. [10 pts]

5. [10 pts] Match each of the four equations below to only one of the four graphs (I) to (IV) below. Please write down your choice in each of the boxes provided below.

- $25y'' - 10y' + y = 0$;

  The choice is **(II)** since the char. double roots are $\frac{1}{5}$.

- $4y'' + 4y' + 17y = 0$;

  The choice is **(IV)** since the char. roots are $-\frac{1}{2} \pm i2$.

- $25y'' - 10y' + 101y = 0$;

  The choice is **(I)** since the char. roots are $\frac{1}{5} \pm i2$.

- $y'' + 4y' + 3y = 0$

  The choice is **(III)** since the char. roots are $\{-1, -3\}$.