MATH150 Introduction to Ordinary Differential Equations, Fall 2010-11
Solutions to Week 03 worksheet: First Order Differential Equations

1. **(Demonstration)** (Ex. 2.1, Q. 29 (B & D)) Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

   (a) Find the solution of this IVP and describe its behaviour for large t.

   (b) Determine the value of t for which the solution first intersects the line $y = 12$.

   Sol:

   (a)

$$y(t) = 12 + \frac{64}{65}(\sin 2t + \frac{1}{8}\cos 2t) - \frac{8}{65}e^{-\frac{1}{4}t}.$$

   As t grows larger and tends to ∞, the solution oscillates about the line $y = 12$. (b) $t = 10.066$

2. **(Demonstration)** (Ex. 1.2, Q. 7 (B & D)) The field mouse population in Example 1 of Chapter 1.1 (B & D) satisfies the equation

$$dp/dt = 0.5p - 450.$$

   (a) Find the time at which the population becomes extinct if $p(0) = 850$.

   (b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

   (c) Find the initial population p_0 if the population is to become extinct in one year.

   Sol:

   (a)

$$p(t) = 900 - 50e^{\frac{1}{2}t}.$$

   The population becomes extinct at $t = 2\ln 18 = 5.78$ months.

   (b)

$$p(t) = 900 - (900 - p_0)e^{\frac{1}{2}t}.$$

   The population becomes extinct at $t = 2\ln\frac{900}{900-p_0}$ months.

   (c) $p_0 \leqslant 900(1 - e^{-6})$

3. **(Demonstration)** (Ex. 2.3, Q. 7 (B & D)) Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

   (a) Find the T required for the original sum to double in value as a function of r.

   (b) Determine T if $r = 7\%$.

   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

   Sol:

   (a) The model equation is given by

$$S'(t) = rS(t), \qquad S(0) = S_0.$$

   The solution is given by

$$S(t) = S_0e^{rt}.$$

   At time $T = \frac{1}{r}\ln 2$, the sum doubles.

   (b) $T = 9.9$ years.

   (c) $r \geqslant \frac{1}{8}\ln 2 = 8.66\%$.

4. **(Demonstration)** (Ex. 2.3, Q. 7 (B & D)) A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/mins, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amounting of salt in the tank as $t \to \infty$.

   Sol: Let $y(t)$ g be the amount of salt in the tank at time t. Note that the volume of the mixture in the tank does not change, so the concentration of salt at time t is given by $\frac{y(t)}{120}$ g/liter. Two parts could contribute to the change of the amount of salt: The concentration in the mixture flow entering the tank and the concentration in mixture flow going out. They are $2\times\gamma$ and $2\times\frac{y(t)}{120} = \frac{y(t)}{60}$, respectively. Therefore, the model equation can be

   $$y'(t) = 2\gamma - \frac{y(t)}{60}$$

   This is a first order ODE which can be solved by introducing integrating factor $\mu(t) = e^{\frac{1}{60}t}$. Combining the initial condition $y(0) = 0$, the solution can be obtained:

   $$y(t) = 120\gamma(1 - e^{-\frac{1}{60}t})$$

   This solution tends to 120γ as $t \to \infty$.

5. **(Class work)** (Ex. 2.1, Q. 31 (B & D)) Consider the IVP:

   $$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

   Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$.

   Sol: First note that this is a first order linear ODE and that direct separation of variable method does not apply. We may try to use the method of integrating factor to solve it. Recall that the integrating factor can be found as follows:

   $$p(t) = -\frac{3}{2}, \qquad \mu(t) = e^{\int p(t)\mathrm{d}t} = e^{-\frac{3}{2}t}.$$

   Multipling $\mu(t)$ on the both sides of the DE yields

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{3}{2}t}y(t)\right) = 3te^{-\frac{3}{2}t} + 2e^{-\frac{1}{2}t}.$$

   Integrating on both sides and using integration by parts for the first term, we have

   $$\begin{aligned} e^{-\frac{3}{2}t}y(t) - y_0 &= \int_0^t (3se^{-\frac{3}{2}s} + 2e^{-\frac{1}{2}s})\mathrm{d}s \\ &= \int_0^t -2s\mathrm{d}(e^{-\frac{3}{2}s}) + \int -4\mathrm{d}(e^{-\frac{1}{2}s}) \\ &= -2se^{-\frac{3}{2}s}\Big|_0^t - \int_0^t (-2)e^{-\frac{3}{2}s}\mathrm{d}s - 4e^{-\frac{1}{2}s}\Big|_0^t \\ &= -(2t + \frac{4}{3})e^{-\frac{3}{2}t} - 4e^{-\frac{1}{2}t} + \frac{16}{3}. \end{aligned}$$

   Hence,

   $$y(t) = -(2t + \frac{4}{3}) - 4e^t + (y_0 + \frac{16}{3})e^{\frac{3}{2}t}.$$

   Since as $t \to \infty$, the first two terms go to $-\infty$, and $e^{\frac{3}{2}t}$ goes to $+\infty$ faster than the first two terms, then: When $y_0 + \frac{16}{3} > 0$ or $y_0 > -\frac{16}{3}$, $y(t) \to +\infty$; when $y_0 + \frac{16}{3} < 0$ or $y_0 < -\frac{16}{3}$, $y(t) \to -\infty$; when $y_0 + \frac{16}{3} = 0$ or $y_0 = -\frac{16}{3}$ which is a critical point, we drop the third term and obtain that $y(t) \to -\infty$.

6. **(Class work)** (Ex. 1.2, Q. 8 (B & D)) Consider a population p of a field mice that grows at a rate proportional to the current population, so that $\frac{\mathrm{d}p}{\mathrm{d}t} = rp$.

   (a) Find the rate constant r if the population doubles in 30 days.

   (b) Find r if the population doubles in N days.

Sol: This is separable ODE. We separate the variable t and the unknown function p,

$$\frac{\mathrm{d}p}{p} = r\mathrm{d}t$$

Take integration on both sides,

$$\ln p = rt + C_1, \text{ or } p = Ce^{rt},$$

where $C = p(0)$ is the initial population at $t = 0$. When $t = 30$, according to the given requirement, $p(30) = Ce^{30r}$ is double as $p(0) = C$, i.e., $Ce^{30r} = 2C$. Then we can find that $r = \frac{1}{30}\ln 2$.

In general, if the population doubles in N days, we have $p(N) = Ce^{Nr} = 2p(0) = 2C$, and therefore, $r = \frac{1}{N}\ln 2$.

7. **(Class work)** (Ex. 2.3, Q. 8 (B & D)) A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

   (a) Determine the sum $S(t)$ accumulated at any time t.

   (b) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

   (c) If $k = \$2000$ per year, determine the return rate r that must be obtained to have \$1 million available in 40 years.

   Sol: The investment increment consists of two parts: One is the return of the previous year; the other is the investment of the current year. Thus,

   $$S'(t) = rS(t) + k.$$

   This is a first order linear ODE that the separation of variable method does not work. Find the integrating factor to be $\mu(t) = e^{\int -r\mathrm{d}t} = e^{-rt}$. Multiplying both sides of the DE by $\mu(t)$, and integrating both sides of the resulting equation yield, given that $S(0) = 0$,

   $$\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(e^{-rs}S(s))\mathrm{d}s = \int_0^t ke^{-rs}\mathrm{d}s$$

   Then the sum $S(t) = \frac{k}{r}(e^{rt} - 1)$.

   Suppose we are given $r = 7.5\%$, $k = \frac{rS(40)}{e^{40r}-1} = 0.075 \times 10^6/(\exp(40 \times 0.075) - 1) = 3929.68$ dollars.

   If $k = \$2000$, $S(40)/k = 500 = (e^{40r} - 1)/r$. Solve this algebraic equation by taking $x = 40r$ and using Newton's iterative method $x_{k+1} = x_k - f(x_k)/f'(x_k)$ to find the solution of $f(x) = 0$. Here $f(x) = e^x - 1 - 12.5x$ and $f'(x) = e^x - 12.5$. Take $x_0 = 5$, repeat the above iterations: $x_1 = 4.38$, $x_2 = 4.02$, $x_3 = 3.92$,$x_4 = 3.91$,$x_5 = 3.91$,$\cdots$. So the solution can be $r = x/40 = 3.91/40 = 0.0977 = 9.77\%$. In fact we can choose the initial guess x_0 to be any value and two solution can be found: $x = 0$ or $x = 3.91$. Of course the second one is what we need.

   Finally, we find that $r = 9.77\%$.