MATH150 Introduction to Ordinary Differential Equations, Fall 2010
Hints to Week 04 Worksheet: Applications

1. A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

   (a) Determine the maximum amount that this buyer can afford to borrow.

   (b) Determine the total interest paid during the term of the mortgage.

   Hints: (a) First assume that the buyer spend $\$k$/month on mortgage payments where k is a constant. Then we can construct the differential equation:

   $$\frac{dS}{dt} = rS - k$$

   with the initial condition $S(0) = S_0$, for which the DE is a separable equation, and hence

   $$\int_{S_0}^{S} \frac{dS}{rS - k} = \int_0^t dt$$

   This gives the sum to be $S = S_0 e^{rt} + \frac{k}{r} - \frac{k}{r}e^{rt}$. When $S = 0$, we can deduce the relation between original mortgage payment and the total mortgage term: $S_0 = \frac{k}{r}(1 - e^{-rT})$, where T is the total mortgage term. This function is a monotone increasing function with respect to each of k and T, so the maximum amount for S_0 will be reached when $k = \$800/\text{month}$ and $T = 20\ years$.

   So we can get $S_{0\max} = \frac{800 \times 12}{0.09}(1 - e^{-0.09\times 20}) \approx 8.90 \times 10^4$.

   (b) From the previous equations, we know $S_0 = \frac{k}{r}(1 - e^{-rT})$, so $T = \frac{1}{r}\ln\frac{k}{k - rS_0}$. So the total interest paid is $\int_0^T k dt - S_0 = kT - S_0 = \frac{k}{r}\ln\frac{k}{k - rS_0} - S_0$.

2. A sky diver weighing 82 kg (including equipment) falls vertically downward from an altitude of 1500 meters, and opens the parachute after 10 sec of free fall. Assume that the force of air resistance is $k_1|v|$, with $k_1 = 1.65$ kg/s, when the parachute is closed and $k_2|v|$, with $k_2 = 26.4$ kg/s, when the parachute is open.

   (a) Find the speed of the sky diver when the parachute opens.

   (b) Find the distance fallen before the parachute opens.

   (c) What is the limiting velocity v_L after the parachute opens?

   (d) Determine how long the sky diver is in the air after the parachute opens.

   (e) Sketch the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

   Hints:

   (a) Let us assume that the direction of velocity is the same as the direction of gravitational acceleration. Thus we can write down the following equations

   $$m\frac{dv_b}{dt} = mg - k_1 v_b, \qquad \text{before parachute opens} \qquad (1)$$

   $$m\frac{dv_a}{dt} = mg - k_2 v_a, \qquad \text{after parachute opens} \qquad (2)$$

   The solution of Eq.1 is given by $v_b = \frac{mg}{k_1} - (\frac{mg}{k_1} - v_0)e^{-k_1 t/m}$, and the solution of Eq.2 is given by $v_a = \frac{mg}{k_2} - (\frac{mg}{k_2} - v_1)e^{-k_2(t-t_1)/m}$, where v_0 is the initial velocity and v_1 is the velocity when parachute opens. We know the parachute opens at $t_1 = 10s$ and the initial speed is $v_0 = 0$, so we have

   $$v_1 = \frac{mg}{k_1}(1 - e^{-k_1 t_1/m}) = 88.77 m/s$$

(b) The distance has fallen before the parachute opens is

$$\begin{aligned} s_1 &= \int_0^{t_1} v_b dt \\ &= \int_0^{t_1} \frac{mg}{k_1}(1 - e^{-k_1 t_1/m})dt \\ &= \frac{mg}{k_1}t_1 + \frac{m^2 g}{k_1^2}(e^{-k_1 t_1/m} - 1) \\ &\approx 458.72m \end{aligned}$$

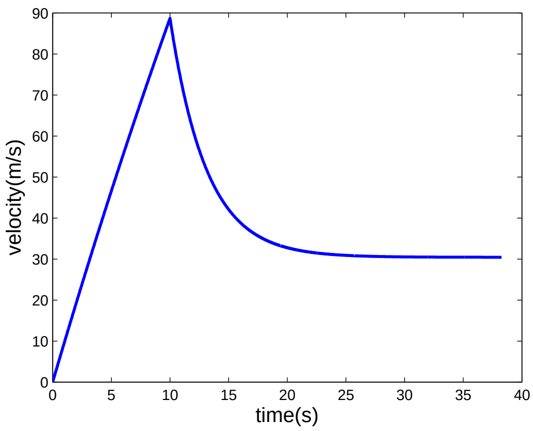
(c) Falling at limiting velocity means $\frac{dv_a}{dt} = 0$, so we have $mg = k_2 v_L$. The limiting velocity $v_L = \frac{mg}{k_2} = 30.44m/s$.

(d) The total distance at time $t (t > 10s)$ is

$$\begin{aligned} s &= \int_0^{t_1} v_b \; dt + \int_{t_1}^{t} v_a \; dt \\ &= \frac{mg}{k_1}t_1 + \frac{m^2 g}{k_1^2}(e^{-k_1 t_1/m} - 1) + \frac{mg(t - t_1)}{k_2} + (\frac{m^2 g}{k_2^2} - \frac{mv_1}{k_2})(e^{-k_2(t-t_1)/m} - 1) \\ &= s_1 + \frac{mg(t - t_1)}{k_2} + (\frac{m^2 g}{k_2^2} - \frac{mv_1}{k_2})(e^{-k_2(t-t_1)/m} - 1) \end{aligned}$$

When sky driver reaches the ground, $s = 1500m$, so we can get, after solving the above equation, $t \approx 38.36s$

(e) The graph of velocity versus time:



3. Find the escape velocity for a body projected upward with an initial velocity v_0 from a point $x_0 = \xi R$ above the center of the earth, where R is the radius of the earth and ξ is a constant greater than unity. Neglect air resistance. Find the initial altitude from which the body must be launched in order to reduce the escape velocity to 85% of its value at the earth's surface.

Hints: Assume that the direction of velocity is opposite to the direction of gravitational acceleration. The differential equation $m\frac{dv}{dt} = -\frac{GMm}{x^2}$, with $\frac{dx}{dt} = v$, where G is the gravitational constant and M is the mass of earth. So we can change the original equation, as indicated in the notes, into

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = -\frac{GM}{x^2} = -\frac{gR^2}{x^2}$$

Solve this equation by integrating the equation after separating the variables from the initial position x_0 to x and from v_0 to v, we get $v^2 - v_0^2 = 2gR^2(\frac{1}{x} - \frac{1}{x_0})$, v_0 and x_0 are the initial speed and initial height. The escape velocity is defined as the minimum initial velocity v_0 such that the mass can escape to infinity. Therefore when $v \to 0$ as $x \to \infty$. So we have

$$v_{\text{escape}} = \sqrt{\lim_{r\to\infty} 2gR^2(\frac{1}{x_0} - \frac{1}{x})} = \sqrt{\frac{2gR^2}{x_0}} = \sqrt{\frac{2gR}{\xi}}$$

When $v_0 = 0.85 v_{\text{escape}}$, we have $x_0 = \dfrac{2gR^2}{v_0^2} = \dfrac{2gR^2}{0.85^2 \times 2gR/\xi} = 1.38\xi R$

4. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

   (a) Find the T required for the original sum to double in value as a function of r.

   (b) Determine T if $r = 7\%$.

   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

   Hints:

   (a) The governing differential equation is $\dfrac{dS}{dt} = rS$ with initial condition $S(0) = S_0$.

   So we have $S = S_0 e^{rt}$, when $S = 2S_0$, the required time T is $T = \dfrac{1}{r}\ln\dfrac{S}{S_0} = \dfrac{1}{r}\ln 2$.

   (b) $T = \dfrac{1}{0.07}\ln 2 \approx 9.90\text{year}$.

   (c) $r = \dfrac{1}{8}\ln 2 \approx 0.087$.

5. A tank initially contains an amount S (liters) of pure water. A mixture containing a concentration γ (grams/liter) of salt enters the tank at a rate r (liters/minute), and the well-stirred mixture leaves the tank at the same rate.

   (a) Determine a differential equation for the amount of salt $M(t)$ (grams) in the tank at any time t by writing an equation for $M(t + \Delta t)$.

   (b) Solve this differential equation using an integrating factor.

   (c) Find the limiting amount of salt in the tank as $t \to \infty$, and show that this corresponds to the solution obtained by setting $dM/dt = 0$.

   Hints:

   (a) The amount of salt at time $t + \Delta t$ is

   $$M(t + \Delta t) = M(t) + \gamma r \Delta t - \frac{M(t)}{S} r \Delta t$$

   This equation holds for very short time when $\Delta t \to \infty$ that the change of concentration of salt in the tank can be neglected. Then we can get the differential equation

   $$\frac{M(t + \Delta t) - M(t)}{\Delta t} = \frac{dM}{dt} = \gamma r - \frac{Mr}{S}$$

   (b) The integration factor is $\nu(t) = e^{\int r/S dt} = C_0 e^{rt/S}$. At the initial time, there are no salt in the tank, so $M(0) = 0$. Then the solution of this equation is $M(t) = \gamma S(1 - e^{-rt/S})$.

   (c) When $t \to \infty$, we know $e^{-rt/S} \to 0$. So the limiting amount of salt is

   $$M_{\text{final}} = \lim_{t\to\infty} \gamma S(1 - e^{-rt/S}) = \gamma S$$