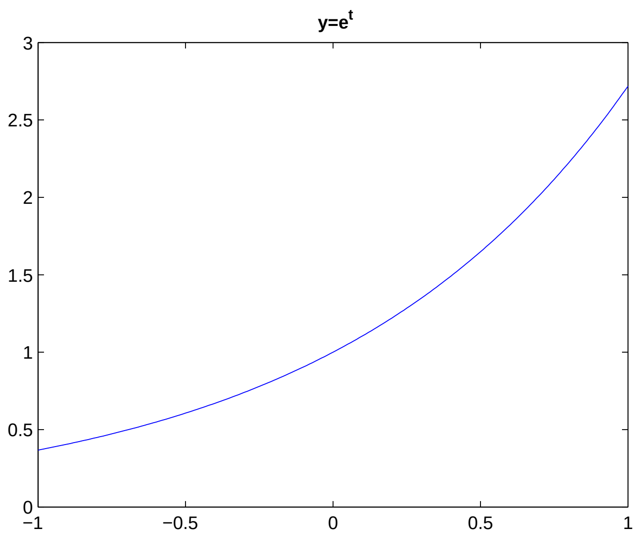
MATH150 Introduction to Ordinary Differential Equations, Spring 2010-11
Week 05 Worksheet: Second order equations **(Ver. T1A)**

Name: ____________________ **ID No.:** ____________________ **Tutorial Section:**______

Complete at least TWO questions from the following questions! The worksheet must be handed in at the end of the tutorial. (The question numbers refer to the main reference book by Boyce & DiPrima (B & D). Partial solution of this worksheet will be available at the course website a week after all the tutorials)

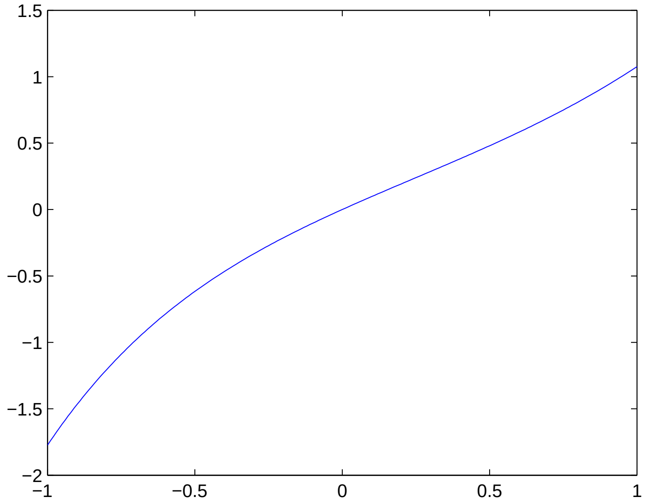
1. **(Demonstration)** (Ex. 3.1, Q. 11 (B & D)) Solve the initial value proble and to sketch the graph and investigate its behaviour when $t \to \infty$: $6y'' - 5y' + y = 0, \quad y(0) = 4,\ y'(0) = 0,$
2. **(Demonstration)** (Ex. 3.1, Q. 21 (B & D)) Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha,\ y'(0) = 2$ and to determine the constant α so that the soluion tends to zero as $t \to \infty$.
3. **(Demonstration)** (Ex. 3.1, Q. 23 (B & D)) Determine (i) the value α below for which all solutions tend to zero as $t \to \infty$ and (ii) the value α below for which all (non-zero) solutions becomes unbounded as $t \to \infty$:
$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$
4. **(Demonstration)** (Ex. 3.1, Q. 25 (B & D)) When time allows.
5. **(Demonstration)** (Ex. 3.3, Q. 8 (B & D)) Solve $y'' - 2y' + 6y = 0$.
6. **(Demonstration)** (Ex. 3.3, Q. 19 (B & D)) Solve $y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 2.$
7. Solve the following initial value problems and sketch their curves:
   (a) **(Class work)** (Ex. 3.1, Q. 9 (B & D)) $y'' + y' - 2y = 0, \quad y(0) = 1, y'(0) = 1;$
   Hints: Since the characteristic equation is $r^2 + r - 2 = 0$, so $r_{1,2} = -2, 1$. Hence the general solution to the ode is $y(t) = c_1e^{-2t} + c_2e^t$. Differentiating y yields $y'(t) = -2c_1e^{-2t} + c_2e^t$. Applying the initial conditions results in two linear equations
$$c_1 + c_2 = 1,$$
$$-2c_1 + c_2 = 1.$$
   Then $c_1 = 0, c_2 = 1$, and $y(t) = e^t$. The curve of the solution is:



   (b) **(Class work)** (Ex. 3.1, Q. 14 (B & D)) $2y'' + y' - 4y = 0, \quad y(0) = 0, y'(0) = 1.$
   Hints: Since the characteristic equation is $2r^2 + r - 4 = 0$, so $r_{1,2} = \frac{1}{4}(-1 \pm \sqrt{33})$. Hence the general solution to the ode is $y(t) = c_1e^{\frac{1}{4}(-1+\sqrt{33})t} + c_2e^{\frac{1}{4}(-1-\sqrt{33})t}$. Differentiating y yields $y'(t) = \frac{1}{4}(-1 + \sqrt{33})c_1e^{\frac{1}{4}(-1+\sqrt{33})t} + \frac{1}{4}(-1 - \sqrt{33})c_2e^{\frac{1}{4}(-1-\sqrt{33})t}$. Applying the initial conditions results in two linear equations

$$c_1 + c_2 = 0,$$
$$\tfrac{1}{4}(-1+\sqrt{33})c_1 + \tfrac{1}{4}(-1-\sqrt{33})c_2 = 1.$$

Then $c_1 = \frac{2\sqrt{33}}{33}, c_2 = -\frac{2\sqrt{33}}{33}$, and $y(t) = \frac{2\sqrt{33}}{33}(e^{\frac{1}{4}(-1+\sqrt{33})t} - e^{\frac{1}{4}(-1-\sqrt{33})t})$. The curve of the solution is:



8. **(Class work)** (Ex. 3.1, Q. 22 (B & D)) Solve the initial value problem $4y''-y=0, \quad y(0)=2,\ y'(0)=\beta$ and to determine the constant β so that the soluion tends to zero as $t \to \infty$.

   Hints: Since the characteristic equation is $4r^2-1=0$, so $r_{1,2} = \pm\frac{1}{2}$. Hence the general solution to the ode is $y = (1+\beta)e^{\frac{1}{2}t} + (1-\beta)e^{-\frac{1}{2}t}$. Since $e^{\frac{1}{2}t} \to \infty$ as $t \to \infty$, $1+\beta = 0$, so $\beta = -1$, $y = 2e^{-\frac{1}{2}t} \to 0$

9. **(Class work)** (Ex. 3.1, Q. 17 (B & D)) Find a differential equation whose general solution is given by $y = c_1e^{2t} + c_2e^{-3t}$, where c_1, c_2 are some constants.

   Hints: Since the roots of the characteristic equation are $r_{1,2} = -3, 2$, the differential equation is

   $$y'' - (r_1+r_2)y' + r_1r_2y = y'' - (-3+2)y' + (-3)\cdot 2y = 0.$$

   Hence the correct differential equation is:

   $$y'' + y' - 6y = 0.$$

10. **(Class work)** (Ex. 3.1, Q. 24 (B & D)) Determine (i) the value α below for which all solutions tend to zero as $t \to \infty$ and (ii) the value α below for which all (non-zero) solutions become unbounded as $t \to \infty$:

    $$y'' + (3-\alpha)y' - 2(\alpha-1)y = 0.$$

    Hints: Since the characteristic equation is $r^2 + (3-\alpha)r - 2(\alpha-1) = 0$, so $r_{1,2} = -2, \alpha-1$. Hence the general solution to the ode is $y(t) = c_1e^{-2t} + c_2e^{(\alpha-1)t}$

    (a) When all solutions tend to zero as $t \to \infty$, must also $e^{(\alpha-1)t}$ tend to zero. But then $\alpha < 1$. Conversely, when $\alpha < 1$, $y(t) = c_1e^{-2t} + c_2e^{\alpha-1}$ tends to zero.

    (b) Even though $e^{(\alpha-1)t}$ tends to infinity if we choose $\alpha > 1$, but since e^{-2t} tends to zero which is bounded as $t \to \infty$, so there's no α such that all non-zero solutions become unbounded as $t \to \infty$.

11. **(Class work)** (Ex. 3.4, Q. 9 (B & D)) Solve $y'' + 2y' - 8y = 0$.

    Hints: Since the characteristic equation is $r^2 + 2r - 8 = 0$, so $r_{1,2} = -4, 2$. Hence the general solution to the ode is $y = c_1e^{-4t} + c_2e^{2t}$.

12. **(Further work)** Consider the second order linear homogeneous differential equation $6y'' + 5y' - 4y = 0$.

(a) Find two (linearly independent) solutions of the equation by determining the two (distinct) roots of the characteristic equation. Hints: Since the characteristic equation is $6r^2 + 5r - 4 = 0$, so $r_{1,2} = \frac{1}{2}, -\frac{4}{3}$. Hence $e^{\frac{1}{2}t}$ and $e^{-\frac{4}{3}t}$ are two linearly independent solutions to the ode.

(b) By taking a suitable linear combination of your two solutions in part (a), find a solution $y_1(t)$ which satisfies the initial condition $y_1(0) = 1$, $y_1'(0) = 0$.
Hints: The general solution to the ode is $y = c_1 e^{1/2t} + c_2 e^{-4/3t}$. Differentiating y yields $y' = \frac{1}{2}c_1 e^{\frac{1}{2}t} - \frac{4}{3}c_2 e^{-\frac{4}{3}t}$. Applying the initial conditions results in

$$c_1 + c_2 = 1,$$
$$\frac{1}{2}c_1 - \frac{4}{3}c_2 = 0.$$

Then $c_1 = \frac{8}{11}, c_2 = \frac{3}{11}$, $y_1(t) = \frac{8}{11}e^{\frac{1}{2}t} + \frac{3}{11}e^{-\frac{4}{3}t}$

(c) By taking a suitable linear combination of your two solutions in part (a), find a solution $y_2(t)$ which satisfies the initial condition $y_2(0) = 0$, $y_2'(0) = 1$.

Hints: Applying the initial conditions results in

$$c_1 + c_2 = 0,$$
$$\frac{1}{2}c_1 - \frac{4}{3}c_2 = 1.$$

Then $c_1 = \frac{6}{11}, c_2 = -\frac{6}{11}, y_2(t) = \frac{6}{11}e^{\frac{1}{2}t} - \frac{6}{11}e^{-\frac{4}{3}t}$

(d) Use the solutions in parts (b) and (c) and the principle of superposition to find a solution to the initial value problem

$$6y'' + 5y' - 4y = 0, \qquad y(0) = B\ ,\ y'(0) = C\ .$$

Hints: The solution is $y = By_1(t) + Cy_2(t)$.

(e) (optional) Explain why the uniqueness part of Theorem 3.2.1 (Existence and Uniqueness Theorem for standard 2nd order linear differential equations) tells us that linear combinations of $y_1(t)$ and $y_2(t)$ give all possible solutions of the equation.

Hints: Since there is exactly one solution of this problem exists through some interval I, so $y = By_1(t) + Cy_2(t)$ is all the possible solution of $6y'' + 5y' - 4y = 0,\ y(0) = B,\ y'(0) = C$.