MATH150 Introduction to Ordinary Differential Equations, Fall 2010

Hints to Week 06 Worksheet: 2nd-order, linear, homogeneous and inhomogeneous odes with constant coefficients

1. (**Demonstration**) (§3.4, Q. 19 B & D) Find the solution of the following initial value problem, sketch the graph of the solution, and describe its behavior for increasing t:

$$\ddot{x} - 2\dot{x} + 5x = 0, \quad x(\pi/2) = 0, \quad \dot{x}(\pi/2) = 2.$$

Hint: First find the characteristic roots from the algebraic equation:

$$r^2 - 2r + 5 = 0.$$

It is easy to find that $r_\pm = 1 \pm 2i$. These are two conjugate complex roots. So it is suggested that the two independent solutions are

$$x_1(t) = e^t \cos 2t, \qquad x_2(t) = e^t \sin 2t.$$

Thus, the general solution can be written as

$$x(t) = e^t (A \cos 2t + B \sin 2t).$$

The two initial conditions read

$$\begin{cases} e^{\frac{\pi}{2}}(-A) = 0, \\ e^{\frac{\pi}{2}}(-A - 2B) = 2. \end{cases}$$

From this system of equations we find that

$$\begin{cases} A = 0, \\ B = -e^{-\frac{\pi}{2}}. \end{cases}$$

Hence, the solution of this IVP is

$$x(t) = -e^{t-\frac{\pi}{2}} \sin 2t.$$

2. (§3.4, Q. 25 B & D) Consider the initial value problem

$$\ddot{x} + 2\dot{x} + 6x = 0, \quad x(0) = 2, \quad \dot{x}(0) = \alpha \geq 0.$$

   (a) Find the solution $x = x(t)$.
   (b) Find α such that $x(1) = 0$.

Hint:

   (a) First find the characteristic roots from the algebraic equation:

$$r^2 + 2r + 6 = 0.$$

It is easy to find that $r_\pm = -1 \pm \sqrt{5}i$. These are two conjugate complex roots. So it is suggested that the two independent solutions are

$$x_1(t) = e^{-t} \cos \sqrt{5}t, \qquad x_2(t) = e^{-t} \sin \sqrt{5}t.$$

Thus, the general solution can be written as

$$x(t) = e^{-t}(A \cos \sqrt{5}t + B \sin \sqrt{5}t).$$

The two initial conditions read

$$\begin{cases} A = 2, \\ -A + \sqrt{5}B = \alpha. \end{cases}$$

From this system of equations we find that

$$\begin{cases} A = 2, \\ B = \dfrac{\alpha + 2}{\sqrt{5}}. \end{cases}$$

Hence, the solution of this IVP is

$$x(t) = e^{-t}\Big(2 \cos \sqrt{5}t + \frac{\alpha + 2}{\sqrt{5}} \sin \sqrt{5}t\Big).$$

(b) Evaluating the solution at $t = 1$, we should have $x(1) = 0$, which is equivalent to

$$2\cos\sqrt{5} + \frac{\alpha + 2}{\sqrt{5}}\sin\sqrt{5} = 0.$$

So $\alpha = -2 - 2\sqrt{5}\cot\sqrt{5}$ is obtained.

3. (§3.5, Q. 16 B & D) (**Demonstration**) Consider the following initial value problem:

$$\ddot{x} - \dot{x} + \frac{1}{4}x = 0, \quad x(0) = 2, \quad \dot{x}(0) = b.$$

Find the solution as a function of b and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

Hint: First find the characteristic roots from the algebraic equation:

$$r^2 - r + \frac{1}{4} = 0.$$

It is easy to find that $r_1 = r_2 = \frac{1}{2}$. These are two repeated roots. So it is suggested that the two independent solutions are

$$x_1(t) = e^{\frac{1}{2}t}, \qquad x_2(t) = te^{\frac{1}{2}t}.$$

Thus, the general solution can be written as

$$x(t) = e^{\frac{1}{2}t}(At + B).$$

The two initial conditions read

$$\begin{cases} B = 2, \\ \frac{1}{2}B + A = b. \end{cases}$$

From this system of equations we find that

$$\begin{cases} A = b - 1, \\ B = 2. \end{cases}$$

Hence, the solution of this IVP is

$$x(t) = e^{\frac{1}{2}t}\big((b-1)t + 2\big).$$

The growing tendency of the solution depends on the coefficient of the linear factor $(b-1)t + 2$. If it is nonnegative, the solution grows unbounded positively; otherwise, it will eventually grow unbounded negatively. Therefore, the critical point separating these two tendencies is obtained from the equality $b - 1 = 0$, which gives that $b = 1$.

4. (§3.5, Q. 18 B & D) Consider the initial value problem

$$9\ddot{x} + 12\dot{x} + 4x = 0, \quad x(0) = a > 0, \quad \dot{x}(0) = -1.$$

(a) Solve the initial value problem.

(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

Hint:

(a) First find the characteristic roots from the algebraic equation:

$$9r^2 + 12r + 4 = 0.$$

It is easy to find that $r_1 = r_2 = -\frac{2}{3}$. These are two repeated roots. So it is suggested that the two independent solutions are

$$x_1(t) = e^{-\frac{2}{3}t}, \qquad x_2(t) = te^{-\frac{2}{3}t}.$$

Thus, the general solution can be written as

$$x(t) = e^{-\frac{2}{3}t}(At + B).$$

The two initial conditions read

$$\begin{cases} B = a, \\ -\frac{2}{3}B + A = -1. \end{cases}$$

From this system of equations we find that

$$\begin{cases} A = \frac{2}{3}a - 1, \\ B = a. \end{cases}$$

Hence, the solution of this IVP is

$$x(t) = e^{-\frac{2}{3}t}\Big((\frac{2}{3}a - 1)t + a\Big)$$

(b) It is obvious that when

$$\begin{cases} A \geqslant 0 \\ B \geqslant 0 \end{cases} \quad \text{or} \begin{cases} \frac{2}{3}a - 1 \geqslant 0 \\ a \geqslant 0 \end{cases} \quad \Rightarrow \quad a \geqslant \frac{3}{2},$$

the solutions are always positive for $t \geqslant 0$. Otherwise, we can solve $x(t) < 0$ and find that $t > \frac{a}{1-\frac{2}{3}a}$, which means when t is large enough, say greater than $\frac{a}{1-\frac{2}{3}a}$, $x(t)$ can be negative. Therefore, the critical value of a that separates solutions that become negative from those that are always positive is $a = \frac{2}{3}$.

5. (§3.6, Q. 7 B&D) Find the general solution of the following differential equation:

$$2\ddot{x} + 3\dot{x} + x = t^2 + 3\sin t.$$

Hint: First find the characteristic roots of the homogeneous equation:

$$2r^2 + 3r + 1 = 0 \qquad \Rightarrow \qquad r_1 = -\frac{1}{2}, \ \ r_2 = -1.$$

These are two different real roots. Hence the two independent solutions are

$$x_1(t) = e^{-\frac{1}{2}t}, \qquad x_2(t) = e^{-t}.$$

To find the special solution of the nonhomogeneous equation, we separate the equation to two subequations:

$$2\ddot{x} + 3\dot{x} + x = t^2, \tag{1}$$
$$2\ddot{x} + 3\dot{x} + x = 3\sin t. \tag{2}$$

For (1), since the nonhomogeneous term is t^2, a degree 2 polynomial, we write the special solution in the form

$$x_a(t) = At^2 + Bt + C$$

and substitute this solution into (1). Comparing the coefficients, we ~~can~~ find that

$$A = 1, \qquad B = -6, \qquad C = 14.$$

So

$$x_a(t) = t^2 - 6t + 14.$$

For (2), since the non-homogeneous term is $3\sin t$, a trigonometric function which is the imaginary part of the function $3e^{it}$, and further the two real characteristic roots are not equal to $\pm i$, we write the special solution in the form

$$x_b(t) = D\cos t + E\sin t$$

and substitute this solution into (2). Comparing the coefficients, we can find that

$$D = -\frac{9}{10}, \qquad E = -\frac{3}{10}.$$

So

$$x_b(t) = -\frac{9}{10}\cos t - \frac{3}{10}\sin t.$$

Thus, the general solution can be written as

$$x(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} + (t^2 - 6t + 14) + \Big(-\frac{9}{10}\cos t - \frac{3}{10}\sin t\Big).$$

6. (§3.6, Q. 18 B & D) Find the solution of the following initial value problem:

$$\ddot{x} + 2\dot{x} + 5x = 4e^{-t}\cos 2t, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Hint: First find the characteristic roots of the homogeneous equation:

$$r^2 + 2r + 5 = 0 \qquad \Rightarrow \qquad r_{\pm} = -1 \pm 2i.$$

These are two complex conjugate roots. So we know that ~~the~~ two independent solutions are

$$x_1(t) = e^{-t}\cos 2t, \qquad x_2(t) = e^{-t}\sin 2t.$$

To find the special solution of the nonhomogeneous equation, we should assume that the special solution has similar form as the non-homogeneous term. We ~~can~~ observe that the non-homogeneous term is the product of an exponential function and a trigonometric function, which is the real part of the function $4e^{(-1\pm 2i)t}$. The coefficient in the exponent is exactly equal to one of the characteristic root. This suggests us to assume that the special solution has the form

$$x_s(t) = te^{-t}(A\cos 2t + B\sin 2t)$$

and substitute this solution into (2). Compute that

$$\begin{aligned}
\dot{x} &= (1-t)e^{-t}(A\cos 2t + B\sin 2t) + te^{-t}(-2A\sin 2t + 2B\cos 2t) \\
\ddot{x} &= (t-2)e^{-t}(A\cos 2t + B\sin 2t) + 2(1-t)e^{-t}(-2A\sin 2t + 2B\cos 2t) + te^{-t}(-4A\cos 2t - 4B\sin 2t)
\end{aligned}$$

By comparing the coefficients, we can find that

$$A = 0, \qquad B = 1.$$

So

$$x_s(t) = te^{-t}\sin 2t.$$

Thus, the general solution can be written as

$$x(t) = e^{-t}(C_1\cos 2t + C_2\sin 2t) + te^{-t}\sin 2t.$$

Then the initial condition implies that the general solution is given by

$$x(t) = e^{-t}\big(\cos 2t + \frac{1}{2}\sin 2t\big) + te^{-t}\sin 2t.$$