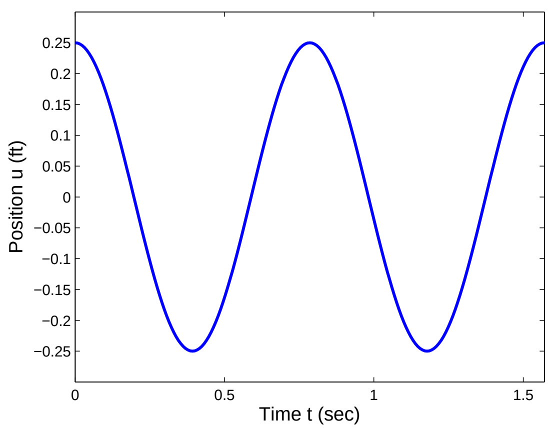
MATH150 Introduction to Ordinary Differential Equations, Spring 2010-11
Hints to Week 07 Worksheet: Mechanical Vibrations

1. **(Demonstration)** (§3.8, page 203, Q. 5) A mass weighing 2 lb streches a spring by 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the position u of the mass at any time t. (Plot u against t) and find the frequence, period, and amplitude of the motion.

   Hints: It is easy to construct the governing equation of this vabration process without damping and external force
   $$m\ddot{u} + ku = 0.$$
   The characteristic equation is $mr^2 + k = 0$ and its solutions is $r_{1,2} = \pm i\sqrt{k/m}$. So the general solution of this equaiton is
   $$u = c_1 \cos(\sqrt{k/m}\ t) + c_2 \sin(\sqrt{k/m}\ t).$$
   Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units system. So we have the gravitational acceleration near earth's surface is about $g = 32$ ft/sec^2. Then the spring constant k is
   $$k = \frac{mg}{u_1} = \frac{2 \times 32}{6/12}\ \text{lb/sec}^2 = 128\ \text{lb/sec}^2,$$
   where u_1 is the stationary stretched length when the spring is in equilibrium position. So $\sqrt{k/m} = 8$ rad/sec. Then we consider the initial condition: $u(0) = \frac{3}{12}$ ft and $\dot{u}(0) = 0$ ft/sec, from which we deduce the value of c_1 and c_2. So:
   $$u = \frac{1}{4}\cos 8t\ \text{ft}$$
   From the expression of u, we see that the frequency is $\omega = \sqrt{k/m} = 8$ rad/sec and the period is $T = 2\pi/\omega = \pi/4$ sec. The amplitude of motion is $R = \frac{1}{4}$ ft.

   The graph of displacement u versus time: We remark that the unit of the spring constant k is in weight



   per lenght, that is, the same in wight per second square.

2. **(Demonstration)** (§3.8, page 203, Q. 10) A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in./sec, find its position u at any time t. (Plot u against t). Determine when the mass first retruns to its equilibrium position. Also find the time τ such that $|u(t)| < 0.01$ in. for all $t > \tau$.

   Hints: Construct the governing equation of this vabration process with damping but without external force:
   $$m\ddot{u} + \gamma\dot{u} + ku = 0.$$
   The characteristic equation is $mr^2 + \gamma r + k = 0$ and its solutions is $r_{1,2} = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4km}$. Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units

system. We easily deduce that the spring constant k and damping constant γ are, respectively,

$$k = \frac{mg}{u_1} = \frac{16 \times 32}{3/12} \text{ lb/sec}^2 = 2048 \text{ lb/sec}^2, \qquad \gamma = 2g \text{ lb/sec} = 64 \text{ lb/sec},$$

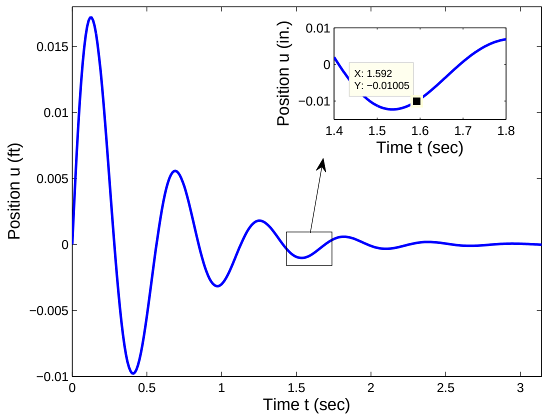
where u_1 is the stationary stretched length when the spring is in equilibrium position. So $-\dfrac{\gamma}{2m} \pm \dfrac{1}{2m}\sqrt{\gamma^2 - 4km} = -2 \pm 2\sqrt{31}i$ and we have the general solution:

$$u = c_1 e^{-2t} \cos 2\sqrt{31}\, t + c_2 e^{-2t} \sin 2\sqrt{31}\, t.$$

Then consider the initial condition: $u(0) = 0$ ft and $\dot{u}(0) = 0.25$ ft/sec, the values of c_1 and c_2 can be solved. So:

$$u = \frac{1}{8\sqrt{31}} e^{-2t} \sin 2\sqrt{31}t \text{ ft}.$$

The graph of displacement u versus time:



Suppose the mass first return to its equilibrium position at $u(t_1) = 0$, so $t_1 = \dfrac{\pi}{2\sqrt{31}}$ sec. From the figure, we know $\tau = 1.59$ sec. We note that the length 0.01 is in inch, that is 1/100 of a foot.

3. **(Demonstration)** (§3.9, page 214, Q. 5/7) A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb.

   (a) Find the position $u(t)$ of the mass from its equilibrium position. Plot a graph.

   (b) If the given external force is replaced by $4 \sin \omega t$, then determine the frequency ω so that resonance occurs.

Hints: (a) The governing equation of this vibration process with external force:

$$m\ddot{u} + ku = F.$$

Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units system. Then the spring constant k is

$$k = \frac{mg}{u_1} = \frac{4 \times 32}{1.5/12} \text{ lb/sec}^2 = 1024 \text{ lb/sec}^2.$$

where u_1 is the stationary stretched length when the spring is in equilibrium position. The external force can be expression as $2g \cos 3t$ lb − ft/sec^2. So the original equation can be transformed into:

$$4\ddot{u} + 1024u = 64 \cos 3t.$$

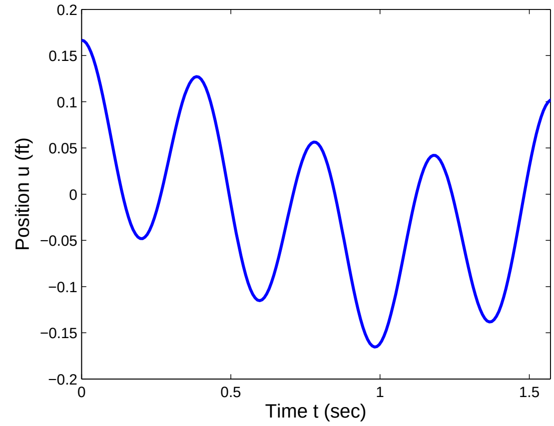
The characteristic equation of homogeneous ODE is $4r^2 + 1024 = 0$. So the general homogeneous solution of this equaiton is

$$u_h = c_1 \cos 16t + c_2 \sin 16t.$$

For the particular solution $u_p = A\cos 3t + B\sin 3t$. Substituting the u_p into the original equation to solve for the coefficients $A = \frac{16}{247}$ and $B = 0$. Then consider the initial condition: $u(0) = \frac{2}{12}$ ft and $\dot{u}(0) = 0$ ft/sec, the value of c_1 and c_2 can be solved. So:

$$u = \frac{151}{1482}\cos 16t + \frac{16}{247}\cos 3t \text{ ft}.$$

The graph of displacement u versus time:



(b) When resonance occurs, $\omega = 16$ rad/sec.

4. **(Class work)** (Ex. 3.8, Q. 1 (B & D)): Write $u = 3\cos 2t + 4\sin 2t$ in the form of $u = R\cos(\omega_0 t - \delta)$.

Hints: Do the transformation $u = 3\cos 2t + 4\sin 2t = R\cos(\omega_0 t - \delta) = R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta$.

There are two equations to solve:

$$\begin{cases} \omega_0 = 2, \\ R\cos\delta = 3, \\ R\sin\delta = 4, \end{cases}$$

so, we have $R = 5$ and $\delta = \arctan\frac{4}{3} + n\pi$. Then $u = 5\cos(2t - 0.927 - n\pi)$. Usually we choose $n = 0$ here.

5. **(Class work)** (§3.8, page 203, Q. 7) A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/sec, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

Hints: It is easy to construct the governing equation of this vabration process without damping and external force.

$$m\ddot{u} + ku = 0.$$

The characteristic equation is $mr^2 + k = 0$ and its solutions is $r_{1,2} = \pm i\sqrt{k/m}$. So the general solution of this equaiton is

$$u = c_1\cos\left(\sqrt{k/m}\ t\right) + c_2\sin\left(\sqrt{k/m}\ t\right).$$

Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units system. Then the spring constant k is

$$k = \frac{mg}{u_1} = \frac{3\times 32}{3/12}\text{ lb/sec}^2 = 384\text{ lb/sec}^2.$$

where u_1 is the stationary stretched length when the spring is in equilibrium position. So $\sqrt{k/m} = 8\sqrt{2}$ rad/sec. Then consider the initial condition: $u(0) = -\frac{1}{12}$ ft and $\dot{u}(0) = 2$ ft/sec, the values of c_1 and c_2 can be solved. So:

$$u = -\frac{1}{12}\cos 8\sqrt{2}t + \frac{\sqrt{2}}{8}\sin 8\sqrt{2}t \text{ ft}.$$

From the expression of u, the frequency: $\omega = \sqrt{k/m} = 8\sqrt{2}$ rad/sec. The period: $T = 2\pi/\omega = \sqrt{2}\pi/8$ sec. The amplitude of motion: $R \approx 0.195$ ft.

6. **(Class work)** (§3.9, page 214, Q. 9) If an undamped spring-mass system with a mass that weighs 6 lb and a spring constant 1 lb/in. is suddenly set in motion at $t = 0$ by an external force of $4\cos 7t$ lb, determine the position of the mass at any tiem and draw a graph of the displacement against t. **Hints:** (a) The governing equation of this vabration process with external force is:

$$m\ddot{u} + ku = F.$$

Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units system. Then the spring constant k is

$$k = 1g \text{ lb/sec}^2 = 384 \text{ lb/sec}^2,$$

where u_1 is the stationary stretched length when the spring is in equilibrium position. The external force can be expression as $4g\cos 7t$ lb − ft/sec^2. So the original equation can be transformed into:

$$6\ddot{u} + 384u = 128\cos 7t.$$

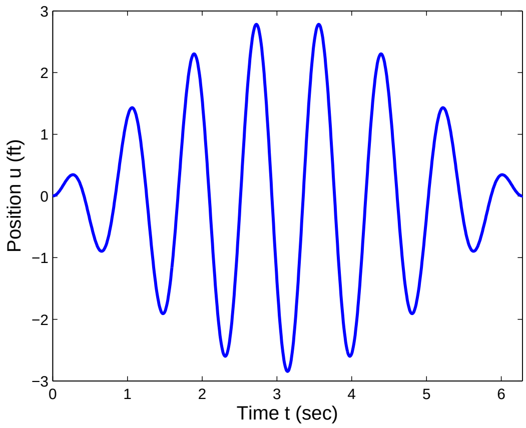
The characteristic equation of homogeneous ODE is $6r^2 + 384 = 0$. So the general homogeneous solution of this equaiton is

$$u_h = c_1 \cos 8t + c_2 \sin 8t.$$

For the particular solution $u_p = A\cos 7t + B\sin 7t$. Substituting the u_p into the original equation to solve the coefficients $A = \frac{64}{45}$ and $B = 0$. Then consider the initial condition: $u(0) = 0$ ft and $\dot{u}(0) = 0$ ft/sec, the value of c_1 and c_2 can be solved. So:

$$u = \frac{64}{45}(-\cos 8t + \cos 7t) \text{ ft}.$$

The graph of displacement u versus time:



7. **(Further work)** (§3.8, page 203, Q. 11.) A spring is stretched 0.1 m by a force of 3 newtons. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 newtons when the velocity of the mass is 5 m/sec.

   (a) If the mass is pulled down 0.05 m below its equilibrium position and given an initial downward velocity of 0.1 m/sec, determine its position u at any time t.

   (b) Find the quasi frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

   Hints: (a) Construct the governing equation of this vabration process with damping but without external force.

$$m\ddot{u} + \gamma\dot{u} + ku = 0.$$

The characteristic equation is $mr^2+\gamma r+k=0$ and its solutions is $r_{1,2}=-\frac{\gamma}{2m}\pm\frac{1}{2m}\sqrt{\gamma^2-4km}$. Then let us consider the quantitative solution of this problem, all the quantities here are in *lb-ft-sec* units system. Then the spring constant k and damping constant γ are

$$k=\frac{mg}{u_1}=\frac{3}{0.1}\ \mathrm{N/m}=30\ \mathrm{N/m},\qquad \gamma=\frac{f}{v}=0.6\ \mathrm{N-sec/m}=0.6\ \mathrm{kg/sec},$$

where u_1 is the stationary stretched length when the spring is in equilibrium position. So $-\frac{\gamma}{2m}\pm\frac{1}{2m}\sqrt{\gamma^2-4km}=-0.15\pm 3.8701i$ and we have the general solution:

$$u=c_1e^{-0.15t}\cos 3.8701t+c_2e^{-0.15t}\sin 3.8701t.$$

Then consider the initial condition: $u(0)=0.05$ m and $\dot{u}(0)=0.1$ m/sec, the values of c_1 and c_2 can be solved. So:

$$u=0.05e^{-2t}\cos 3.8701t+0.028e^{-0.15t}\sin 3.8701t\ \mathrm{m}.$$

(b) The quasi frequency: $\nu=3.8701$ rad/sec. The governing equation of this vibration process without damping effect:

$$m\ddot{u}+ku=0.$$

The natural frequency: $\omega_0=\sqrt{k/m}=\sqrt{15}\approx 3.8730$ rad/sec. And the ratio: $\nu/\omega_0=0.9993$.

8. **(Further work)** (§3.9, page 214, Q. 6.) A mass of 5 kg stretches a hanging spring 0.1 metres. The mass is acted on by an external force of $10\sin(t/2)$ newtons, and is damped by a force which is proportional and opposite to the motion, and which is 2 newtons when the speed is .04 metres/second.

   (a) Find the spring constant k; so, force $=k\cdot$ stretch.

   (b) Find the damping constant γ; so, force $=\gamma\cdot$ speed.

   (c) Formulate the initial value problem if the mass is set in motion from its equilibrium position with a velocity of 0.03 metres/second.

   Hints: (a) All the quantities here are in *kg-m-sec* units system(SI unit system), where the gravitational acceleration near earth's surface is about $g=9.8$ m/sec^2. Then the spring constant k is

$$k=\frac{mg}{u_1}=\frac{5\times 9.8}{0.1}\ \mathrm{kg/sec^2}=490\ \mathrm{kg/sec^2}=490\ \mathrm{N/m},$$

   where u_1 is the stationary stretched length when the spring is in equilibrium position.

   (b) The damping constant can be expressed as:

$$\gamma=\frac{f}{v}=\frac{2}{0.04}\ \mathrm{N-sec/m}=50\ \mathrm{kg/sec}.$$

   (c) The governing equation of this vabration process with damping effect:

$$m\ddot{u}+\gamma\dot{u}+ku=F.$$

   Then substituting all the quantitative values, the original equation can be transformed into:

$$5\ddot{u}+50\dot{u}+490u=10\sin(t/2),\quad \text{or}\quad \ddot{u}+10\dot{u}+98u=2\sin(t/2)$$

   with the initial condition: $u(0)=0$ m and $\dot{u}(0)=0.03$ m/sec.