MATH150 Introduction to Ordinary Differential Equations, Fall 2010

Week 08 Wksht: Laplace transform methods (Part 1) (T1A)

Name: ____________________ **ID No.:** ____________________ **Tutorial Section:**_______

Your tutor should model Problems 1, 2 and 3. You should try to solve Problems 4, 5 and 7. The remaining inhomogeneous odes can be solved, time permitting, or solved at home. To obtain Worksheet credit, answer sheets showing at least some solution work and your name, id number, and tutorial section (as above) MUST be handed in at the end of the tutorial. (Solution of this worksheet will be available from the tutor's website next week)

1. (**Demonstration:**) Compute the Laplace transform of $f(t) = e^t \sin t$ directly from the definition of the Laplace transform.

2. (**Demonstration:**) (Example 1 on pg 31 of the notes) Use the method of Laplace transform to solve

$$\ddot{x} + 5\dot{x} + 6x = 0, \quad x(0) = 2,\ \dot{x}(0) = 3.$$

3. (**Demonstration:** Example on page 37 of notes) Use the method of Laplace transform to solve

$$\ddot{x} - 3\dot{x} - 4x = 3e^{2t}, \quad x(0) = 1,\ \dot{x}(0) = 0.$$

4. (**Classwork:**) Compute the Laplace transform of $f(t) = e^{at} \sin bt$ directly from the definition of the Laplace transform.

Hints: Since

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}e^{at}\sin bt\,dt = -\frac{1}{b}\int_0^\infty e^{(a-s)t}\,d\big(\cos bt\big) \\
&= -\frac{1}{b}e^{(a-s)t}\cos bt\Big|_0^\infty + \frac{1}{b}\int_0^\infty \cos bt\,d\big(e^{(a-s)t}\big) \\
&= \frac{1}{b} + \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos bt\,dt \\
&= \frac{1}{b} + \frac{a-s}{b^2}\Big[e^{(a-s)t}\sin bt\Big|_0^\infty - \int_0^\infty \sin bt\,d\big(e^{(a-s)t}\big)\Big] \\
&= \frac{1}{b} - \frac{(a-s)^2}{b^2}\int_0^\infty e^{(a-s)t}\sin bt\,dt \\
&= \frac{1}{b} - \frac{(a-s)^2}{b^2}\mathcal{L}\{f(t)\}.
\end{aligned}$$

Hence

$$\mathcal{L}\{f(t)\} = \frac{b}{(s-a)^2 + b^2}.$$

5. (**Classwork:** Example 2 on pg 34 of the notes) Use the method of Laplace transform to solve:

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0.$$

Hints: Since

$$X(s) = \frac{(s+1)x(0) + s\dot{x}(0)}{s^2 + s + 1} = \frac{(s+\frac{1}{2}) + \frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

Applying Table 4.1, we obtain

$$\mathcal{L}^{-1}\Big\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\Big\} = e^{-\frac{1}{2}t}\mathcal{L}^{-1}\{\frac{s}{s^2 + (\frac{\sqrt{3}}{2})^2}\} = e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$$

and

$$\mathcal{L}^{-1}\{\frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\} = \frac{\sqrt{3}}{3}e^{-\frac{1}{2}t}\mathcal{L}^{-1}\{\frac{\frac{\sqrt{3}}{2}}{s^2 + (\frac{\sqrt{3}}{2})^2}\} = \frac{\sqrt{3}}{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t.$$

Hence, the solution to the IVP is

$$x(t) = e^{-\frac{1}{2}t}(\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t).$$

6. (**Classwork:** Example 2 on pg 35 of the notes) Use the method of Laplace transform to solve:

$$\ddot{x} + 2\dot{x} + x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0.$$

Hints: Since

$$X(s) = \frac{(s+2)x(0) + s\dot{x}(0)}{s^2 + 2s + 1} = \frac{1}{(s+1)^2} + \frac{1}{s+1}.$$

Applying inverse Laplace transform yields

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}, \quad \mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{-t}.$$

Hence the solution to the IVP is

$$x(t) = e^{-t}(t+1)$$

7. (**Classwork:** Example on page 38 of notes) Use the method of Laplace transform to solve

$$\ddot{x} - 3\dot{x} - 4x = 2\sin t, \quad x(0) = 0,\ \dot{x}(0) = 1.$$

Hints: Since

$$\begin{aligned} X(s) &= \frac{\mathcal{L}\{2\sin t\} + 1}{s^2 - 3s - 4} = \frac{2\frac{1}{s^2+1} + 1}{s^2 - 3s - 4} \\ &= \frac{s^2 + 3}{(s^2+1)(s-4)(s+1)} = \frac{As + B}{s^2 + 1} + \frac{C}{s-4} + \frac{D}{s+1} \\ &= \frac{(A + C + D)s^3 + (-3A + B + C - 4D)s^2 + (-4A - 3B + C + D)s + (-4B + C - 4D)}{(s^2+1)(s-4)(s+1)} \end{aligned}$$

Comparing the coefficients for like terms on both sides yield:

$$\begin{cases} A + C + D = 0; \\ -3A + B + C - 4D = 1; \\ -4A - 3B + C + D = 0; \\ -4B + C - 4D = 3 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{17}; \\ B = -\frac{5}{17}; \\ C = \frac{19}{85}; \\ D = -\frac{2}{5}. \end{cases}$$

Thus

$$X(s) = \frac{3}{17}\frac{s}{s^2+1} - \frac{5}{17}\frac{1}{s^2+1} + \frac{19}{85}\frac{1}{s-4} - \frac{2}{5}\frac{1}{s+1}.$$

Hence

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{19}{85}e^{4t} - \frac{2}{5}e^{-t}.$$

8. (**Further work:** Example on page 39 of notes) Use the method of Laplace transform to solve

$$\ddot{x} + \dot{x} - 2x = t^2, \quad x(0) = 1,\ \dot{x}(0) = 0.$$

Hints: Since

$$\begin{aligned} X(s) &= \frac{\mathcal{L}\{t^2\} + (s+1)x(0) + \dot{x}(0)}{s^2 + s - 2} = \frac{\frac{2}{s^3} + s + 1}{s^2 + s - 2} \\ &= \frac{s^4 + s^3 + 2}{s^3(s-1)(s+2)} = \frac{As^2 + Bs + C}{s^3} + \frac{D}{s-1} + \frac{E}{s+2} \\ &= \frac{(A + D + E)s^4 + (A + B + 2D - E)s^3 + (-2A + C + B)s^2 + (-2B + C)s - 2C}{s^3(s-1)(s+2)}. \end{aligned}$$

Comparing the coefficients for like terms on both sides yield:

$$\begin{cases} A + D + E = 1; \\ A + B + 2D - E = 1; \\ -2A + B + C = 0; \\ -2B + C = 0; \\ -2C = 2. \end{cases} \Rightarrow \begin{cases} A = -\frac{3}{4}; \\ B = -\frac{1}{2}; \\ C = -1; \\ D = \frac{4}{3}; \\ E = \frac{5}{12}. \end{cases}$$

Thus

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = -\frac{3}{4} - \frac{1}{2}t - \frac{1}{2}t^2 + \frac{4}{3}e^t + \frac{5}{12}e^{-2t}.$$

9. (**Further work:** Example on page 42 of notes) Use the method of Laplace transform to solve

$$\ddot{x} - 3\dot{x} - 4x = 5e^{-t}, \quad x(0) = 1, \ \dot{x}(0) = 0.$$

Hints: Since

$$\begin{aligned} X(s) &= \frac{\mathcal{L}\{5e^{-t}\} + (s-3)x(0) + \dot{x}(0)}{s^2 - 3s - 4} = \frac{\frac{5}{s+1} + s - 3}{(s+1)(s-4)} = \frac{s^2 - 2s + 2}{(s+1)^2(s-4)}; \\ &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}; \\ &= \frac{(A+C)s^2 + (-3A + B + 2C)s + (-4A - 4B + C)}{(s+1)^2(s-4)}. \end{aligned}$$

Comparing the coefficients for like terms on both sides yield:

$$\begin{cases} A + C = 1; \\ -3A + B + 2C = -2; \\ -4A - 4B + C = 2. \end{cases} \Rightarrow \begin{cases} A = \frac{3}{5}; \\ B = -1; \\ C = \frac{2}{5}. \end{cases}$$

Thus

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}e^{-t} - te^{-t} + \frac{2}{5}e^{4t}.$$

10. (**Further work:** §6.2, B & Dpage 323, problem 28.) Let $F(s)$ be the Laplace transform of $f(t)$. You may assume the convergence of all integrals. Show that (i) $F'(s) = \mathcal{L}[-tf(t)]$ and (ii) $F^{(n)}(s) = \mathcal{L}[(-t)^n f(t)]$.

Hints: (i) We have

$$F'(s) = \left(\int_0^\infty e^{-st} f(t)\, dt\right)' = \int_0^\infty -te^{-st} f(t)\, dt = \mathcal{L}\{-tf(t)\}.$$

(ii) By induction, suppose we already have

$$F^{(n-1)}(s) = \mathcal{L}\{(-t)^{n-1} f(t)\}.$$

Then

$$\begin{aligned} F^{(n)}(s) &= (F^{(n-1)}(s))' = (\mathcal{L}\{(-t)^{n-1} f(t)\})' \\ &= (\int_0^\infty e^{-st}(-t)^{n-1} f(t) dt)' \\ &= \int_0^\infty -te^{-st}(-t)^{n-1} f(t) dt \\ &= \int_0^\infty e^{-st}(-t)^n f(t) dt \\ &= \mathcal{L}\{(-t)^n f(t)\} \end{aligned}$$