MATH150 Introduction to Ordinary Differential Equations, Spring 2010-11
Solution to Week 09 Worksheet: Laplace Transforms (Part II)

1. **(Demonstration)** (§6.3, page 330, problem 8) Find the Laplace transform of the function
$$f(t) = \begin{cases} 0, & t < 1; \\ t^2 - 2t + 2, & t \geq 1. \end{cases}$$

2. **(Demonstration)** (§6.4, p. 337, Q. 3) Use Laplace transform to solve the IVP:
$y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi), \quad y(0) = 0,\ y'(0) = 0.$

3. **(Demonstration)** (§6.5, p. 344, Q. 2) Use Laplace transform to solve the IVP:
$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0,\ y'(0) = 0.$ If time does not allow, then there is no need to do the inverse Laplace transform.

4. **(Class work)** (§6.3, page 330, Q. 9) Find the Laplace transform of the function $f(t) = \begin{cases} 0, & t < \pi; \\ t - \pi & \pi \leq t < 2\pi; \\ 0, & t \geq 2\pi. \end{cases}$

   Sol: The given function can be rewritten in term of Heaviside functions:
$$f(t) = (t - \pi)(u_\pi(t) - u_{2\pi}(t)) = (t - \pi)u_\pi(t) - (t + \pi - 2\pi)u_{2\pi}(t)$$
   From the linearity of the Laplace transform, we can compute that
$$\begin{aligned} \mathcal{L}\{f(t)\} &= e^{-\pi s}\mathcal{L}\{t\} - e^{-2\pi s}\mathcal{L}\{t + \pi\} \\ &= e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\frac{1}{s^2} - \pi e^{-2\pi s}\frac{1}{s} \end{aligned}$$

5. **(Class work)** (§6.4, p. 337, Q. 4) Use Laplace transform to solve the IVP:
$y'' + 4y = \sin t + u_\pi(t)\sin(t - \pi), \quad y(0) = 0,\ y'(0) = 0.$

   Sol: Taking the Laplace transform on both sides of the equation and using the initial values, we can obtain an algebraic equation
$$(s^2 + 4)Y(s) = \frac{1}{s^2 + 1} + e^{-\pi s}\frac{1}{s^2 + 1}$$
   Solving this equation gives
$$Y(s) = \frac{1}{3}(1 + e^{-\pi s})\left(\frac{1}{s^2 + 1} - \frac{1}{2}\frac{2}{s^2 + 4}\right)$$
   Note that here we write the solution in this form so that the inverse transform can be easily seen. Finally, by taking the Laplace inverse transform of $Y(s)$, we can find the solution to the IVP
$$\begin{aligned} y(t) &= \mathcal{L}\{Y(s)\} = \frac{1}{3}(\sin t + u_\pi(t)\sin(t - \pi)) - \frac{1}{6}(\sin 2t + u_\pi(t)\sin 2(t - \pi)) \\ &= \frac{1}{3}(\sin t - u_\pi(t)\sin t) - \frac{1}{6}(\sin 2t + u_\pi(t)\sin 2t) \end{aligned}$$

6. **(Class work)** (§6.5, p. 344, Q. 1) Use Laplace transform to solve the IVP:
$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 0,\ y'(0) = 0.$ If time does not allow, then there is no need to do the inverse Laplace transform.

   Sol: Taking the Laplace transform on both sides of the equation and using the initial values, we can obtain an algebraic equation
$$(s^2 + 2s + 2)Y(s) = e^{-\pi s}$$
   Solving this equation gives
$$Y(s) = e^{-\pi s}\frac{1}{(s + 1)^2 + 1}$$
   Since the inverse transform of the function $\frac{1}{(s+1)^2+1}$ is given by
$$\mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2 + 1}\right\} = e^{-t}\sin t,$$
   by taking the Laplace inverse transform of $Y(s)$, we can find the solution to the IVP
$$y(t) = \mathcal{L}\{Y(s)\} = u_\pi(t)e^{-(t-\pi)}\sin(t - \pi) = -u_\pi(t)e^{-t+\pi}\sin t$$