MATH150 Introduction to Ordinary Differential Equations, Fall 2010
Hints to Week 10 Wksht: Series solutions

For problems 1 and 2, solve the following differential equation by means of a power series about $x = 0$. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions.

1. (**Demonstration**) (§5.2, page 259, problem 2) $y'' - xy' - y = 0$

   Hints: For this equation, each point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting the series of y' and y''

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

into the DE and rearranging yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\text{or} \qquad \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Then we can get two recurrence relations:

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}, \quad \text{for} \quad n = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0 \Rightarrow a_{n+2} = \frac{a_n}{n+2}, \quad \text{for} \quad n \geq 1$$

So we have the general recurrence relation:

$$a_{n+2} = \frac{a_n}{n+2} \quad \text{for} \quad n \geq 0$$

Since a_{n+2} is given in terms of a_n, the a's are determined in steps of two. There exist two sequences depending on the non-zero, but otherwise arbitray, values of a_0 and a_1 separately.

For the sequence a_0, a_2, a_4, ... in the recurrence relation:

$$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4} = \frac{a_0}{8}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6} = \frac{a_0}{48}$$

For the sequence a_1, a_3, a_5, ... in the recurrence relation:

$$a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5} = \frac{a_1}{15}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7} = \frac{a_1}{105}$$

So the solution is:

$$y = a_0 \left[1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots\right] + a_1 \left[x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots\right]$$

2. (**Class work**) (§5.2, page 259, problem 4) $y'' + k^2 x^2 y = 0$; k a constant.

   Hints: For this equation, each point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting the series of y' and y'' into the DE and rearranging yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_{n-2} x^{n+2} = 0$$

or

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} k^2 a_n x^n = 0$$

Then we can get three recurrence relations:

$$2a_2 = 0 \Rightarrow a_2 = 0, \quad \text{for} \quad n = 0$$
$$6a_3 = 0 \Rightarrow a_3 = 0, \quad \text{for} \quad n = 1$$
$$(n+2)(n+1)a_{n+2} + k^2 a_{n-2} = 0 \Rightarrow a_{n+2} = \frac{-k^2 a_{n-2}}{(n+2)(n+1)}, \quad \text{for} \quad n \geq 2$$

So we have the general recurrence relation:

$$a_{n+4} = \frac{-k^2 a_n}{(n+4)(n+3)} \quad \text{for} \quad n \geq 0$$

Since a_{n+4} is given in terms of a_n, the a's are determined in steps of four. There exist two sequences depending on the non zero, but otherwise arbitrary, values of a_0 and a_1 separately. The other two sequences whose values are all zeros.

For the sequence a_0, a_4, a_8, $\ldots$ in the recurrence relation:

$$a_4 = \frac{-k^2 a_0}{4 \cdot 3}, \quad a_8 = \frac{-k^2 a_4}{8 \cdot 7} = \frac{(-k^2)^2 a_0}{8 \cdot 7 \cdot 4 \cdot 3}, \quad a_{12} = \frac{-k^2 a_8}{12 \cdot 11} = \frac{(-k^2)^3 a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$$

For the sequence a_1, a_5, a_9, $\ldots$ in the recurrence relation:

$$a_5 = \frac{-k^2 a_1}{5 \cdot 4}, \quad a_9 = \frac{-k^2 a_5}{9 \cdot 8} = \frac{(-k^2)^2 a_1}{9 \cdot 8 \cdot 5 \cdot 4}, \quad a_{13} = \frac{-k^2 a_9}{13 \cdot 12} = \frac{(-k^2)^3 a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$$

For the sequence a_2, a_6, a_{10}, $\ldots$ and a_3, a_7, a_{11}, $\ldots$ in the recurrence relation:

$$a_2 = a_3 = a_6 = a_7 = a_{10} = a_{11} = 0$$

So the solution is:

$$\begin{aligned} y =& a_0\left[1 + \frac{-k^2}{4 \cdot 3}x^4 + \frac{(-k^2)^2}{8 \cdot 7 \cdot 4 \cdot 3}x^8 + \frac{(-k^2)^3}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}x^{12} + \cdots\right] \\ &+ a_1\left[x + \frac{-k^2}{5 \cdot 4}x^5 + \frac{(-k^2)^2}{9 \cdot 8 \cdot 5 \cdot 4}x^9 + \frac{(-k^2)^3}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}x^{13} + \cdot\right] \end{aligned}$$

3. (§5.2, page 260, problem 21) The equation

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite equation.

(a) Find the first four terms in each of two linearly independent solutions about $x = 0$.

(b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial . Find the polynomial solutions for $\lambda = 0, 2, 4, 6$. Note that each polynomial is determined only up to a multiplicative constant.

(c) The Hermite polynomial $H_n(x)$ is defines as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), H_1(x), H_2(x), H_3(x)$.

Hints: For this equation, each point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting the series of y' and y'' into the DE and rearranging yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

$$\text{or} \qquad \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

Then we can get two recurrence relations:

$$2a_2 + \lambda a_0 = 0 \Rightarrow a_2 = -\frac{\lambda a_0}{2}, \quad \text{for} \quad n = 0$$

$$(n+2)(n+1)a_{n+2} - 2na_n + \lambda a_n = 0 \Rightarrow a_{n+2} = \frac{(2n-\lambda)a_n}{(n+2)(n+1)}, \quad \text{for} \quad n \geq 1$$

So we have the general recurrence relation:

$$a_{n+2} = \frac{(2n-\lambda)a_n}{(n+2)(n+1)} \quad \text{for} \quad n \geq 0$$

Since a_{n+2} is given in terms of a_n, the a's are determined in steps of four. There exist two sequences depending on the non zero, but otherwise arbitrary, values of a_0 and a_1 separately.

For the sequence a_0, a_2, a_4, ... in the recurrence relation:

$$a_2 = \frac{-\lambda a_0}{2}, \quad a_4 = \frac{(4-\lambda)a_2}{4 \cdot 3} = \frac{-\lambda(4-\lambda)a_0}{4!}, \quad a_6 = \frac{(8-\lambda)a_4}{6 \cdot 5} = \frac{-\lambda(4-\lambda)(8-\lambda)a_0}{6!}$$

For the sequence a_1, a_3, a_5, ... in the recurrence relation:

$$a_3 = \frac{(2-\lambda)a_1}{3!}, \quad a_5 = \frac{(6-\lambda)a_3}{5 \cdot 4} = \frac{(2-\lambda)(6-\lambda)a_1}{5!}, \quad a_7 = \frac{(10-\lambda)a_5}{7 \cdot 6} = \frac{(2-\lambda)(6-\lambda)(10-\lambda)a_1}{7!}$$

(a) So the solution is:

$$\begin{aligned} y =& a_0\Big[1 + \frac{-\lambda}{2}x^2 + \frac{-\lambda(4-\lambda)}{4!}x^4 + \frac{-\lambda(4-\lambda)(8-\lambda)}{6!}x^6 + \cdots\Big] \\ &+ a_1\Big[x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(6-\lambda)}{5!}x^5 + \frac{(2-\lambda)(6-\lambda)(10-\lambda)}{7!}x^7 + \cdots\Big] \end{aligned}$$

(b) If $\lambda = 0$, 4 or a double of an even integer, then the series solutions $y_1(x)$ associates with a_0 becomes a polynomial. From the expression of the series solutions associates with a_0, the first two polynomial solutions are $y_1(x) = a_0$ for $\lambda = 0$ and $y_1(x) = a_0(1 - 2x^2)$ for $\lambda = 4$.

If $\lambda = 2$, 6 or double of an odd integer, the series solutions $y_2(x)$ associates with a_1 becomes a polynomial. From the expression of series solutions associates with a_1, the first two polynomial solutions are $y_2(x) = a_1 x$ for $\lambda = 2$ and $y_1(x) = a_1(x - \frac{2}{3}x^3)$ for $\lambda = 6$.

(c) The coefficient of x^n is 2^n for Hermite polynomial $H_n(x)$. So we can get the values of a_0 or a_1 for each λ respectively.

$$H_0(x) = 1, \qquad H_1(x) = 2x$$

$$H_2(x) = -2(1 - 2x^2), \qquad H_3(x) = -12(x - \frac{2}{3}x^3)$$

4. (§5.3, page 265, problem 10) The Chebyshev differential equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0, \qquad \alpha \text{ a constant.}$$

(a) Determine two linearly independent solution in powers of x for $|x| < 1$.

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.

(c) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

Hints: For this equation, the points in domain $|x| < 1$ are an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting the series of y' and y'' into the DE and rearranging yields

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

$$\text{or} \qquad \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

Then we can get three recurrence relations:

$$2a_2 + \alpha^2 a_0 = 0 \Rightarrow a_2 = \frac{-\alpha^2 a_0}{2}, \quad \text{for} \quad n = 0$$

$$6a_3 - a_1 + \alpha^2 a_1 = 0 \Rightarrow a_3 = \frac{(1-\alpha^2)a_1}{6}, \quad \text{for} \quad n = 1$$

$$(n+2)(n+1)a_{n+2} - (n(n-1) + n - \alpha^2)a_n = 0 \Rightarrow a_{n+2} = \frac{(n^2 - \alpha^2)a_n}{(n+2)(n+1)}, \quad \text{for} \quad n \geq 2$$

So we have the general recurrence relation:

$$a_{n+2} = a_{n+2} = \frac{(n^2 - \alpha^2)a_n}{(n+2)(n+1)} \quad \text{for} \quad n \geq 0$$

Since a_{n+2} is given in terms of a_n, the a's are determined in steps of four. There exist two sequences which depending on th non zero, but otherwise arbitrary, values of a_0 and a_1 separately.

For the sequence a_0, a_2, a_4, ... in the recurrence relation:

$$a_2 = \frac{-\alpha^2 a_0}{2}, \quad a_4 = \frac{(4-\alpha^2)a_2}{4 \cdot 3} = \frac{-\alpha^2(2^2 - \alpha^2)a_0}{4!}, \quad a_6 = \frac{(16-\alpha^2)a_4}{6 \cdot 5} = \frac{-\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)a_0}{6!}, \cdots$$

so the general expression of even sequence is $a_{2n} = \dfrac{\prod_{i=1}^{n}((2i-2)^2 - \alpha^2))}{(2n)!} a_0$ for $n \geq 1$.

For the sequence a_1, a_3, a_5, ... in the recurrence relation:

$$a_3 = \frac{(1-\alpha^2)a_1}{3!}, \quad a_5 = \frac{(3^2-\alpha^2)a_3}{5 \cdot 4} = \frac{(1-\alpha^2)(3^2-\alpha^2)a_1}{5!}, \quad a_7 = \frac{(5^2-\alpha^2)a_5}{7 \cdot 6} = \frac{(1-\alpha^2)(3^2-\alpha^2)(5^2-\alpha^2)a_1}{7!}, \cdots$$

so the general expression of odd sequence is $a_{2n+1} = \dfrac{\prod_{i=1}^{n}((2i-1)^2 - \alpha^2))}{(2n+1)!} a_1$ for $n \geq 1$.

(a) So the general solution is:

$$\begin{aligned} y =& a_0\Big[1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}((2i-2)^2 - \alpha^2))}{(2n)!} x^{2n}\Big] \\ &+ a_1\Big[x + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}((2i-1)^2 - \alpha^2))}{(2n+1)!} x^{2n+1}\Big] \end{aligned}$$

(b) If α is an nonnegative even integer $2n$, then there always exist a$i = n+1$ to make $((2i-2)^2 - \alpha^2) = 0$, so that the a_0 series solutions $y_1(x)$ becomes a polynomial with all $a_{2n+2} = 0$, $a_{2n+4} = 0$, $\cdots$, which means the degress is $2n$.

Similarly, if α is an nonnegative odd integer $2n + 1$, then there always exist a $i = n + 1$ to make $((2i - 1)^2 - \alpha^2) = 0$, so that the a_1 series solutions $y_2(x)$ becomes a polynomial with all $a_{2n+3} = 0$, $a_{2n+5} = 0$, $\cdots$, which means the degress is $2n + 1$.

(c) It is easy to write the polynomial solutions

$$P_0(x) = a_0 \text{ for } \alpha = 0, \qquad P_2(x) = a_0(1 - 2x^2) \text{ for } \alpha = 2$$

$$P_1(x) = a_1 x \text{ for } \alpha = 1, \qquad P_3(x) = a_1(x - \frac{4}{3}x^3) \text{ for } \alpha = 3$$