MATH150 Introduction to Ordinary Differential Equations, Spring 2010-11
Hints to Week 11 Worksheet: Systems of ODEs I (Ver. T1A)

Name: ____________________ **ID No.:** ____________________ **Tutorial Section:**______

1. **(Demonstration)** (§7.5, p. 398, problem 1) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}.$$

2. **(Demonstration)** (§7.5, p. 398, problem 2) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}.$$

3. **(Class work)** (§7.5, p. 398, problem 4) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}.$$

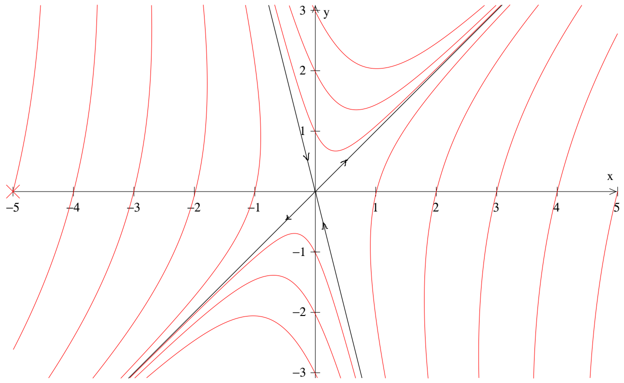
Hints: Setting $\mathbf{x} = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a nonzero solution, we must have $det(\mathbf{A} - r\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = -3$ and $r_2 = 2$. For $r = -3$, the system of equations reduces to $4\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\xi^{(1)} = (1, -4)^T$. Substitution of $r = 2$ results in the single equation $-\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (1, 1)^T$. Since the eigenvalues are distinct, The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

And the phase diagram is



4. **(Class work)** (§7.5, p. 398, problem 6) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}.$$

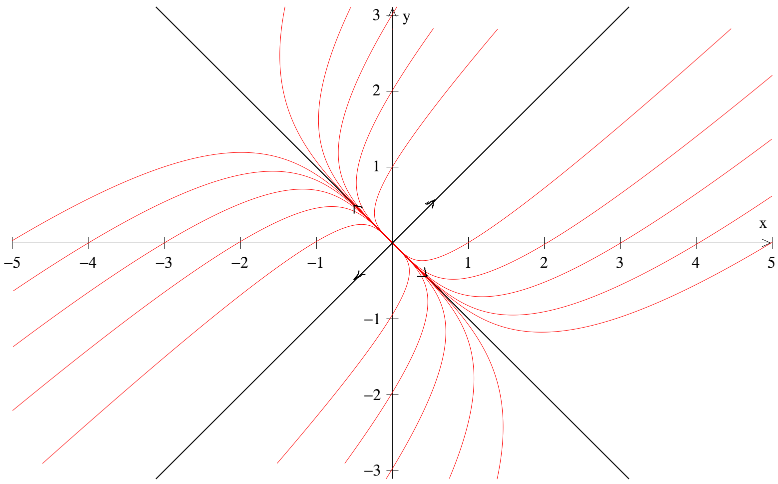
Hints: Setting $x = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} \frac{5}{4} - r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a nonzero solution, we must have $det(\mathbf{A} - r\mathbf{I}) = r^2 - \frac{5}{2}r + 1 = 0$. The roots of the characteristic equation are $r_1 = \frac{1}{2}$ and $r_2 = 2$. For $r = \frac{1}{2}$, the system of equations reduces to $\frac{3}{4}\xi_1 + \frac{3}{4}\xi_2 = 0$. The corresponding eigenvector is $\xi^{(1)} = (1, -1)^T$. Substitution of $r = 2$ results in the single equation $-\frac{3}{4}\xi_1 + \frac{3}{4}\xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (1, 1)^T$. Since the eigenvalues are distinct, The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{1}{2}t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

And the phase diagram is



5. **(Class work)** (§7.5, p. 398, modified from problem 31) Solve the following system

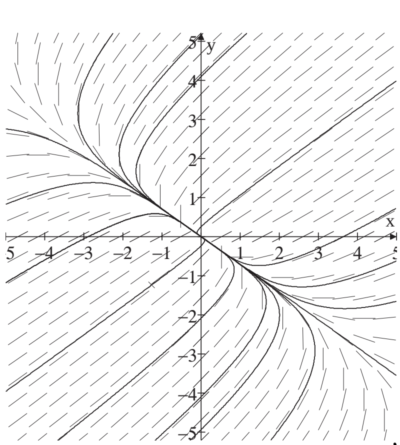
$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x},$$

and determine the value of $\alpha\,(\frac{1}{2} \le \alpha \le 2)$ for which the nature of the node at the origin (equilibrium point) changes.

Hints: Setting $x = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -1-r & 1 \\ -\alpha & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a nonzero solution, we must have $det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r + 1 - \alpha = 0$. The roots of the characteristic equation are $r_1 = -1 - \sqrt{\alpha}$ and $r_2 = -1 + \sqrt{\alpha}$. For $\frac{1}{2} \le \alpha \le 2$, the eigenvalues are distinct. When $\alpha < 1$, $r_2 = -1 + \sqrt{\alpha} < 0$, they are both negative. The equilibrium is a stable node. When $\alpha > 1$, $r_2 = -1 + \sqrt{\alpha} > 0$, they are of opposite sign, hence the equilibrium point is a saddle point. Hence the value of α for which the nature of the node changes is 1. The phase diagrams for the cases when $\alpha = 0.5$ and $\alpha = 2$ are given below respectively



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