MATH150 Introduction to Ordinary Differential Equations, Spring 2010-11
Solution to Week 12 Worksheet: Systems of ODEs II

1. **(Demonstration)** (§7.6, p. 410, problem 17) Solve the following system in terms of α, determine the *citical values* of α at which the qualitative nature of the the phase diagram changes, and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}.$$

2. **(Demonstration)** (§7.8, p. 428, Q. 3) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}.$$

3. **(Class work)** (§7.6, p. 410, problem 14) Solve the following system in terms of α, determine the *citical values* of α at which the qualitative nature of the the phase diagram changes, and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}.$$

Solution:

$$A = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & 5 \\ -1 & \lambda - \alpha \end{vmatrix} = \lambda^2 - \alpha\lambda + 5 = (\lambda - \lambda_+)(\lambda - \lambda_-) \tag{1}$$

where

$$\lambda_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 20}}{2}$$

(a) $\alpha^2 - 20 < 0 \Leftrightarrow -2\sqrt{5} < \alpha < 2\sqrt{5}$. There are two complex conjugate roots $\lambda_\pm = \frac{\alpha}{2} \pm \frac{\sqrt{20-\alpha^2}}{2}i$. The equilibrium point $(0,0)$ is therefore a spiral point. Solving the equation

$$(\lambda_\pm \mathbf{I} - \mathbf{A})\mathbf{v}_\pm = \mathbf{0}$$

leads to

$$\mathbf{v}_\pm = \begin{pmatrix} -\frac{\alpha}{2} \pm \frac{\sqrt{20-\alpha^2}}{2}i \\ 1 \end{pmatrix} C_\pm.$$

For convinence, we can choose $C_\pm = 1$. The real and the imaginary parts of the solutions $e^{\lambda_\pm t}\mathbf{v}_\pm$ are respectively:

$$\begin{aligned}
\mathbf{x}_{re}(t) &= \mathrm{Re}\{e^{\lambda_+ t}\mathbf{v}_+\} = e^{\frac{\alpha}{2}t} \begin{pmatrix} -\frac{\alpha}{2}\cos\frac{\sqrt{20-\alpha^2}}{2}t - \frac{\sqrt{20-\alpha^2}}{2}\sin\frac{\sqrt{20-\alpha^2}}{2}t \\ \cos\frac{\sqrt{20-\alpha^2}}{2}t \end{pmatrix} \\
\mathbf{x}_{im}(t) &= \mathrm{Im}\{e^{\lambda_+ t}\mathbf{v}_+\} = e^{\frac{\alpha}{2}t} \begin{pmatrix} -\frac{\alpha}{2}\sin\frac{\sqrt{20-\alpha^2}}{2}t + \frac{\sqrt{20-\alpha^2}}{2}\cos\frac{\sqrt{20-\alpha^2}}{2}t \\ \sin\frac{\sqrt{20-\alpha^2}}{2}t \end{pmatrix}
\end{aligned}$$

The general solution is given by

$$\mathbf{x}(t) = C_1\mathbf{x}_{re}(t) + C_2\mathbf{x}_{im}(t)$$

i. $-2\sqrt{5} < \alpha < 0$, the real part of the eigenvalue is negative. Then this spiral point is attractive, and all the solution curves spiral into the origin. To determine the direction of rotation, we need to look into the monotonicity of the argument of $\mathbf{x}_{re}(t)$(or $\mathbf{x}_{im}(t)$). Since

$$\cot\arg\mathbf{x}_{re}(t) = -\frac{\alpha}{2} - \frac{\sqrt{20-\alpha^2}}{2}\tan\frac{\sqrt{20-\alpha^2}}{2}t$$

and

$$\cot\arg\mathbf{x}_{im}(t) = -\frac{\alpha}{2} + \frac{\sqrt{20-\alpha^2}}{2}\cot\frac{\sqrt{20-\alpha^2}}{2}t$$

are both decreasing functions of t, the arguments of $\mathbf{x}_{re}(t)$ and $\mathbf{x}_{im}(t)$ are both increasing functions of t. Thus the direction of rotation is anti-clockwise.

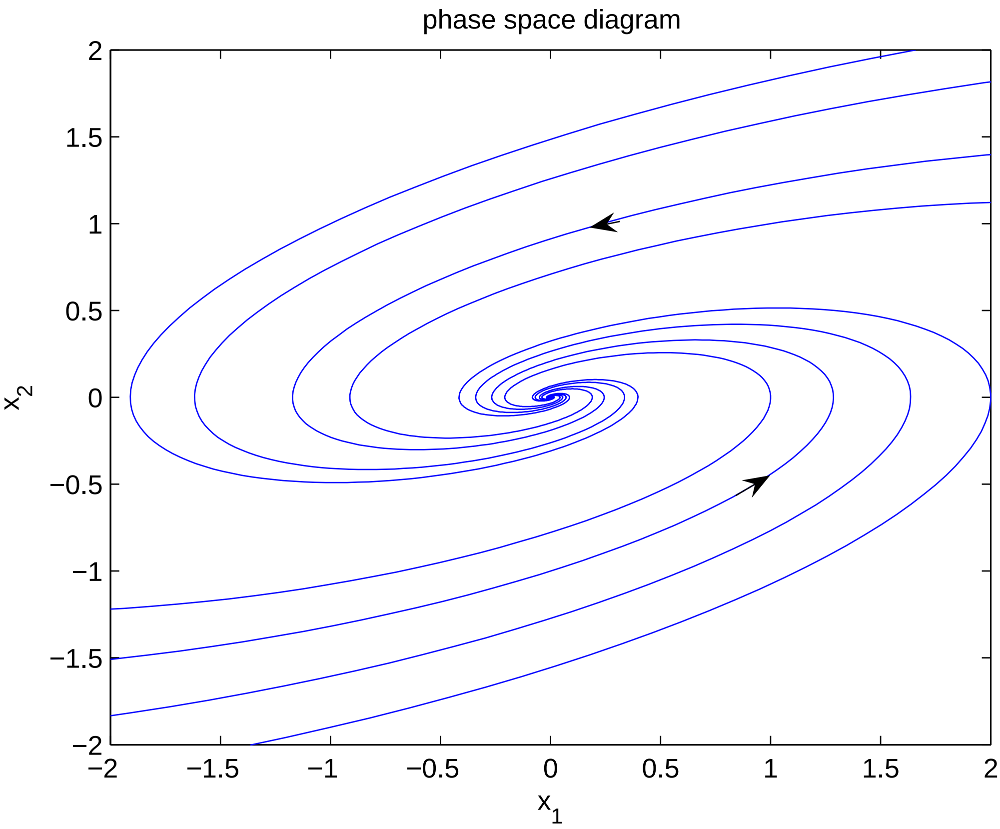

Figure 1: (a)i

ii. If $0 < \alpha < 2\sqrt{5}$, then the real part of the eigenvalue is positive. Then this spiral point is repellent, and all the solution curves spiral out of the origin. The direction of rotation, determined in case i., is anti-clockwise.

iii. If $\alpha = 0$, then the real part is zero. Then this spiral point is neither attractive nor repellent. The solution curve is a family of closed curves. The direction of rotation is anti-clockwise.

(b) If $\alpha^2 - 20 > 0 \Leftrightarrow \alpha < -2\sqrt{5}$ or $\alpha > 2\sqrt{5}$, then there are two different real roots $\lambda_\pm = \frac{\alpha \pm \sqrt{\alpha^2-20}}{2}$. The origin is an equilibrium point. Solving the equation

$$(\lambda_\pm \mathbf{I} - \mathbf{A})\mathbf{v}_\pm = \mathbf{0}$$

leads to

$$\mathbf{v}_\pm = \begin{pmatrix} -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{pmatrix} C_\pm.$$

For convinence, we can choose $C_\pm = 1$. The general solution is given by

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-$$

i. If $\alpha < -2\sqrt{5}$, then the two real roots are both negative. Then the equilibrium point is attractive. Since $\lambda_- < \lambda_+ < 0$, $|\lambda_+| < |\lambda_-|$. As $t \to +\infty$, $e^{\lambda_- t}\mathbf{v}_-$ goes to the origin faster than $e^{\lambda_+ t}\mathbf{v}_+$, $e^{\lambda_+ t}\mathbf{v}_+$ is the dominant term; as $t \to -\infty$, $e^{\lambda_- t}\mathbf{v}_-$ goes to infinity faster than $e^{\lambda_+ t}\mathbf{v}_+$, $e^{\lambda_- t}\mathbf{v}_-$ is the dominant term. Thus, near the origin, the solution curves tend to align with $\mathbf{v}_+$; but when far from the origin, the solution curves tend to be parallel to $\mathbf{v}_-$.

ii. If $\alpha > 2\sqrt{5}$, then the two real roots are both positive. Then the equilibrium point is repellent. Since $\lambda_+ > \lambda_- > 0$, $|\lambda_+| > |\lambda_-|$. As $t \to +\infty$, $e^{\lambda_+ t}\mathbf{v}_+$ goes to infinity faster than $e^{\lambda_- t}\mathbf{v}_-$, $e^{\lambda_+ t}\mathbf{v}_+$ is the dominant term; as $t \to -\infty$, $e^{\lambda_+ t}\mathbf{v}_+$ goes to the origin faster than $e^{\lambda_- t}\mathbf{v}_-$, $e^{\lambda_- t}\mathbf{v}_-$ is the dominant term. Thus, near the origin, the solution curves tend to align with $\mathbf{v}_-$; but when far from the origin, the solution curves tend to be parallel to $\mathbf{v}_+$.

Since the constant term in the eigenpolynomial (1) is 5, two real eigenvalues cannot ~~be of~~ have different signs and there is no saddle point in this problem.

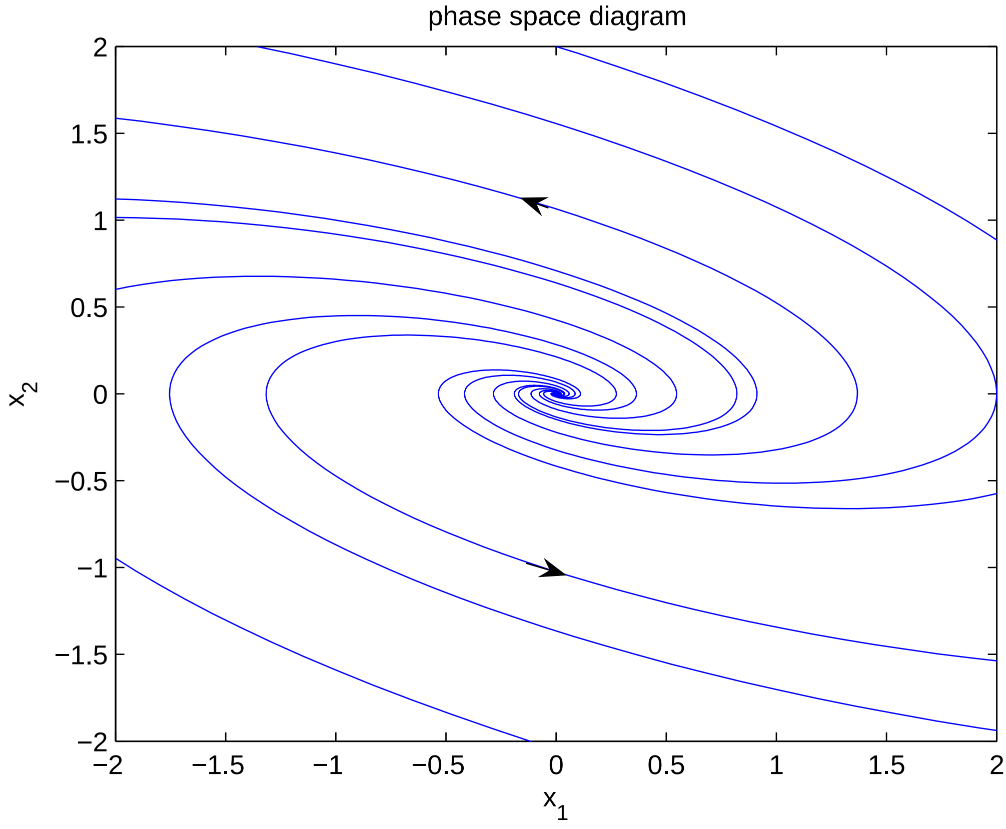

Figure 2: (a)ii

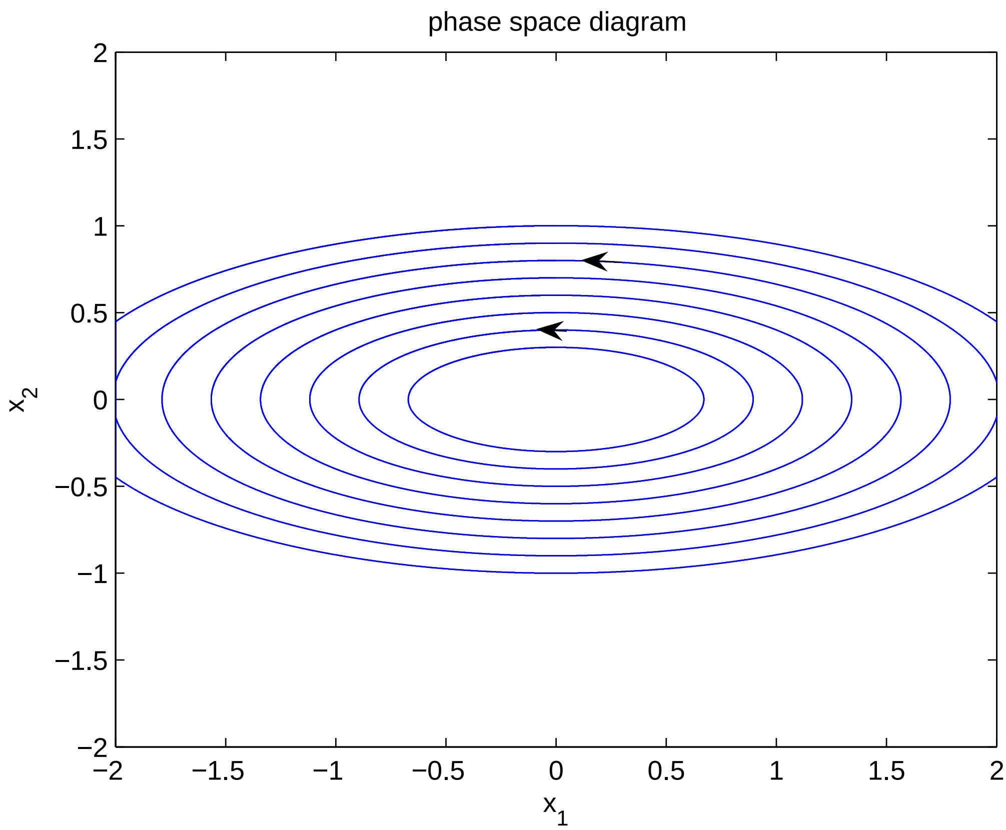

Figure 3: (a)iii

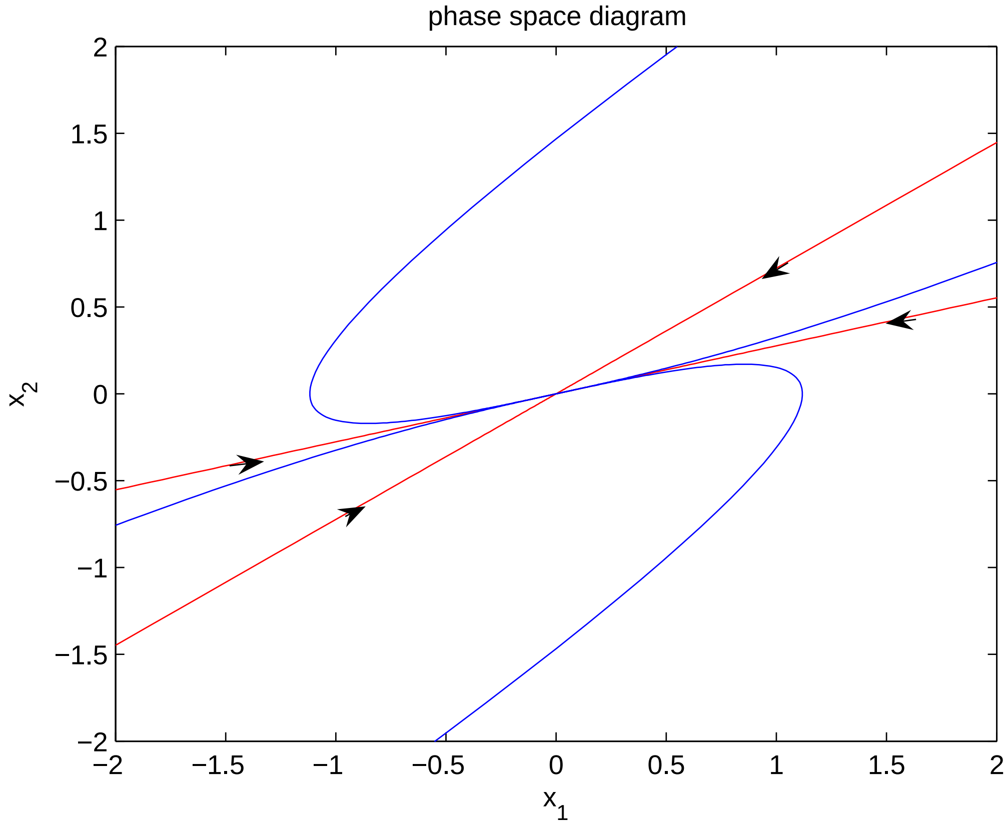

Figure 4: (b)i

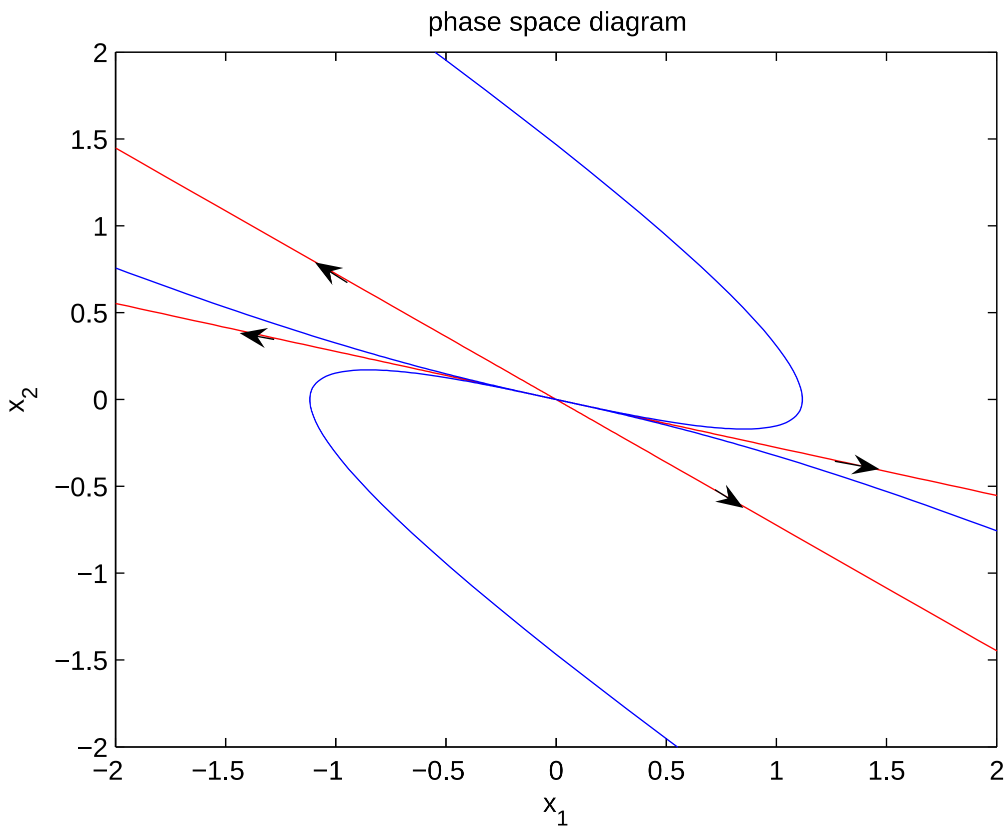

Figure 5: (b)ii

(c) If $\alpha = \pm 2\sqrt{5}$, then there are two repeated eigenvalues $\lambda_+ = \lambda_- = \frac{\alpha}{2} = \pm\sqrt{5}$. Solving the equation

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$

leads to

$$\mathbf{v} = \begin{pmatrix} -\frac{\alpha}{2} \\ 1 \end{pmatrix} C.$$

For convinence, we can choose $C = 1$. In order to find the second independent solution, we solve

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}.$$

The solution is

$$\mathbf{w} = \tilde{C}\mathbf{v} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Setting $\tilde{C} = 0$ yields

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The general solution is given by

$$\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v} + C_2 e^{\lambda t}(\mathbf{w} + t\mathbf{v}) = e^{\lambda t}\begin{pmatrix} -\frac{\alpha}{2}C_1 + C_2(1 - \frac{\alpha}{2}t) \\ C_1 + C_2 t \end{pmatrix}.$$

i. If $\alpha = -2\sqrt{5}$ and $\lambda_+ = \lambda_- = -\sqrt{5}$, then the equilibrium point $(0,0)$ is attractive. As $t \to +\infty$, the term $C_2 e^{\lambda t} t\mathbf{v}$ decays slower, so it dominates the other solution near the origin and the solution curves tend to align with the vector $\mathbf{v}$; as $t \to -\infty$, the term $C_2 e^{\lambda t} t\mathbf{v}$ grows faster, so it also dominates the other solution far from the origin and the solution curves tend to be parallel with the vector $\mathbf{v}$. The eigenspace $\{(x_1, x_2) : x_1 = -\frac{\alpha}{2}x_2\}$ divides the plane into two half planes. The solution curves locating in the lower half plane satisfy $x_1 > -\frac{\alpha}{2}x_2$. Substituting the solution into this inequality gives $C_2 > 0$. The solution curves locating in the upper half plane satisfy $x_1 > -\frac{\alpha}{2}x_2$. Substituting the solution into this inequality gives $C_2 < 0$. The eigenspace is defined by $C_2 = 0$. As $t \to -\infty$(far from the origin), the solutions with $C_2 > 0$ have two components $x_1, x_2 \to -\infty$(in the third quarter) while both components of the solutions with $C_2 < 0$ have two components $x_1, x_2 \to +\infty$(in the first quarter).

ii. If $\alpha = 2\sqrt{5}$ and $\lambda_+ = \lambda_- = \sqrt{5}$, then the equilibrium point $(0,0)$ is repellent. The same analysis shows that near the origin the solution curves tend to align with the vector $\mathbf{v}$ while when far from the origin they tend to be parallel with the vector $\mathbf{v}$. The eigenspace $\{(x_1, x_2) : x_1 = -\frac{\alpha}{2}x_2\}$ divides the plane into two half planes. The solution curves locating in the right half plane satisfy $x_1 > -\frac{\alpha}{2}x_2$. Substituting the solution into this inequality gives $C_2 > 0$. The solution curves locating in the left half plane satisfy $x_1 > -\frac{\alpha}{2}x_2$. Substituting the solution into this inequality gives $C_2 < 0$. The eigenspace is defined by $C_2 = 0$. As $t \to +\infty$(far from the origin), the solutions with $C_2 > 0$ have two components $x_1 \to -\infty$ and $x_2 \to +\infty$(in the second quarter) while the solutions with $C_2 < 0$ have two components $x_1 \to +\infty$ and $x_2 \to -\infty$(in the fourth quarter).

4. **(Class work)** (§7.8, p. 428, Q. 4) Solve the following system and sketch a phase diagram

$$\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix}\mathbf{x}.$$

Solution:

$$A = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda + 3 & -\frac{5}{2} \\ \frac{5}{2} & \lambda - 2 \end{vmatrix} = (\lambda + \frac{1}{2})^2$$

There are two repeated real roots $\lambda_1 = \lambda_2 = \lambda = -\frac{1}{2}$. Suppose $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -\frac{1}{2}$. Then

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \begin{pmatrix} \frac{5}{2} & -\frac{5}{2} \\ \frac{5}{2} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad v_1 = v_2$$

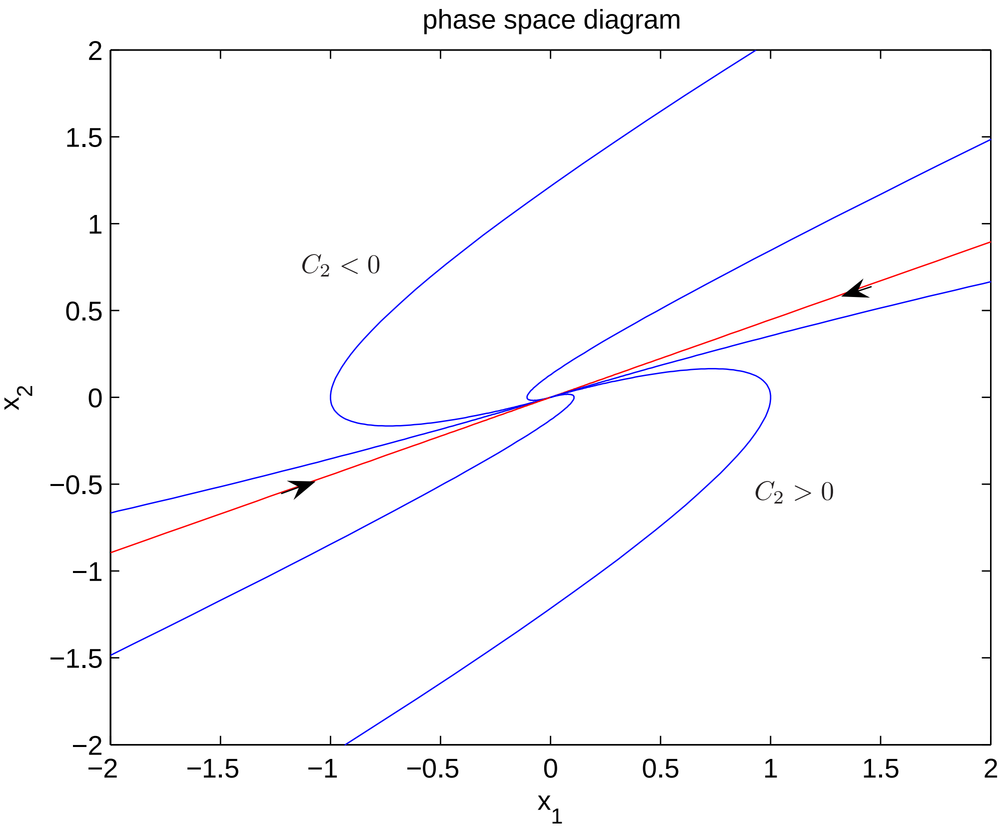

Figure 6: (c)i

Therefore, the eigenspace is 1-dimensional and the eigenvector can be chosen to be

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solving $(\mathbf{A} - \lambda \mathbf{I})\mathbf{w} = \mathbf{v}$ gives $\mathbf{w} = (C - \frac{2}{5}, C)^{\mathbf{T}}$. Choosing $C = 0$ we obtain $\mathbf{w} = (-\frac{2}{5}, 0)^{\mathbf{T}}$. The general solution is given by

$$\mathbf{x}(t) = C_1 \mathbf{v} e^{\lambda t} + C_2(\mathbf{w} + \mathbf{v}t)e^{\lambda t} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{1}{2}t} + C_2 \left(\begin{pmatrix} -\frac{2}{5} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t \right) e^{-\frac{1}{2}t}$$

Since the repeated eigenvalues are negative, the equilibrium point $(0,0)$ is attractive. There is only one asymptotic line since the eigenspace is 1-D. In the limiting cases that $t \to +\infty$ and $t \to -\infty$, the term $C_2 t e^{-\frac{1}{2}t}(1,1)^{\mathbf{T}}$ dominates in the solution since its components have greater absolute values comparing to other terms. All the solution curves tend to align with the eigenvector near the origin and parallel to the eigenvector far from the origin. The eigenspace $\{(x_1, x_2) : x_1 = x_2\}$ divides the plane into two half planes. The solution curves locating in the lower half plane is labeled by $C_2 < 0$ while the solution curves locating in the upper half plane is labeled by $C_2 > 0$. The eigenspace is labeled by $C_2 = 0$. As $t \to -\infty$, both components of the solutions with $C_2 > 0$(or $C_2 < 0$) go to $-\infty$(or $+\infty$).

5. **(Further work)** (§7.6, modified from Q. 31) Suppose the coupled mass-spring system as on pp. 75-76 of Prof. Chasnov's Notes has $k = 1 = k_{12}$, $m = 1$. Write down the equation in the form $\ddot{\mathbf{x}} = A\mathbf{x}$ and solve for the corresponding eigenvalues/eigenvectors and the general solution.
Solution: Using the parameters given here, the system of differential equations is written in the matrix form $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The ansatz $\mathbf{x} = \mathbf{v}e^{\lambda t}$ leads to the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda^2 \mathbf{v}$. This suggests us to find the eigenvalues and corresponding eigenvectors:

$$\det(\lambda^2 \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda^2 + 2 & -1 \\ -1 & \lambda^2 + 2 \end{vmatrix} = (\lambda^2 + 1)(\lambda^2 + 3).$$

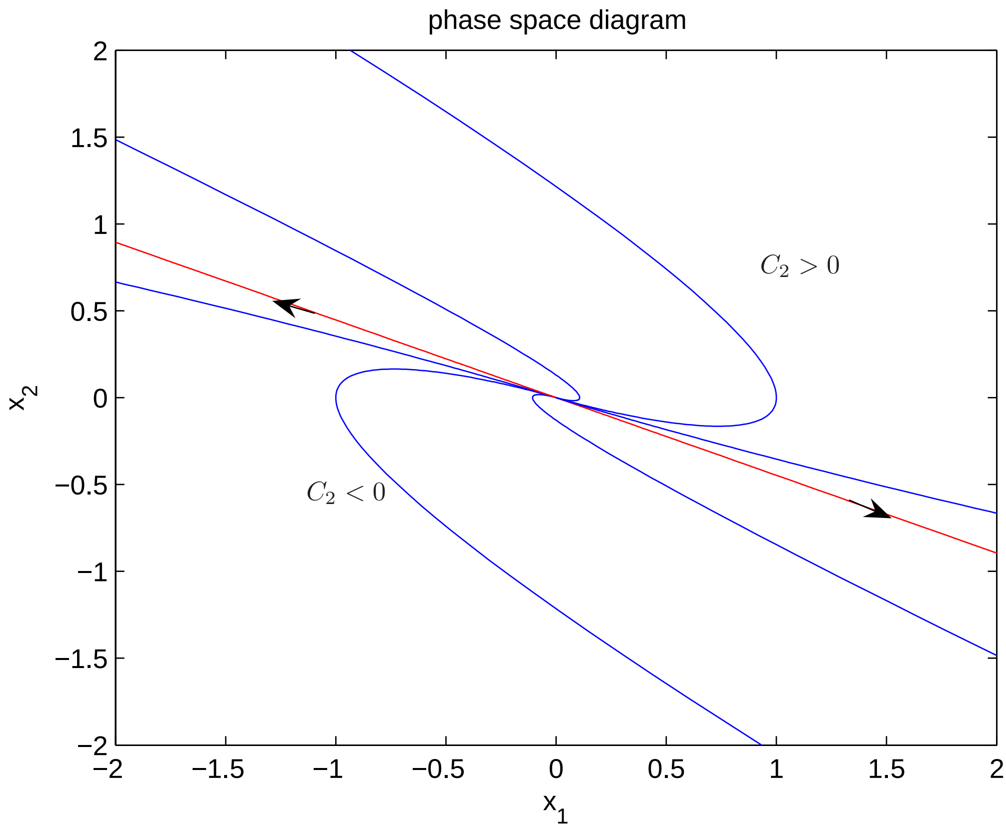

Figure 7: (c)ii

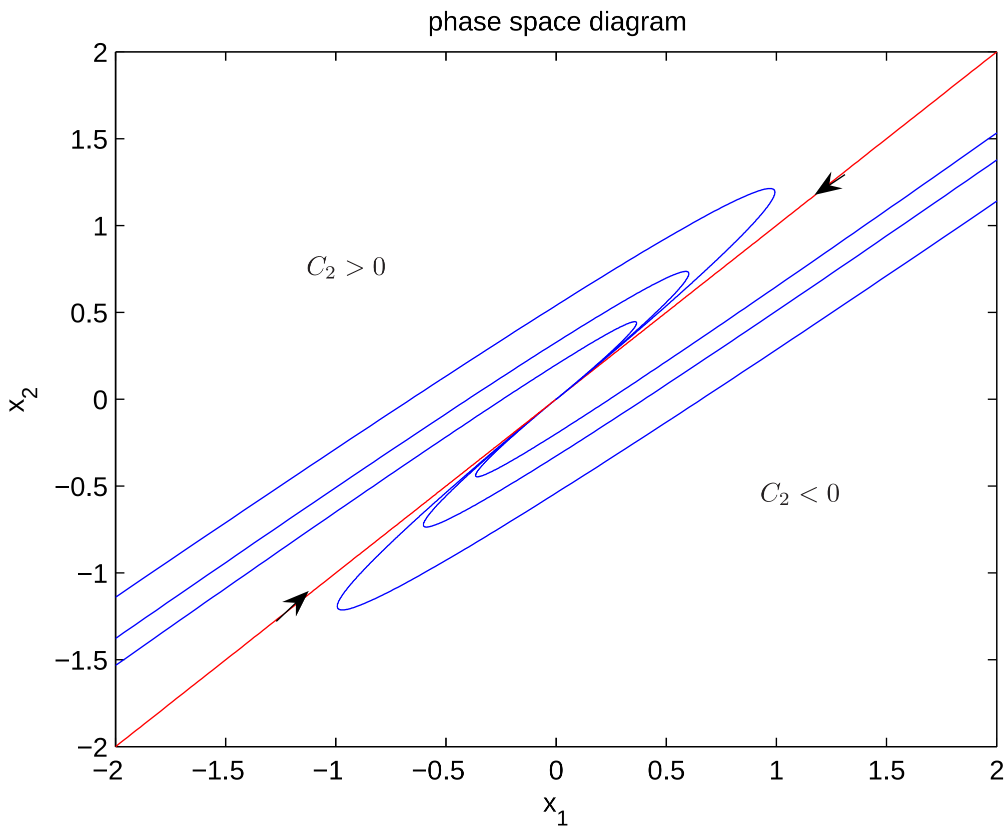

Figure 8: The red line indicates the eigenspace while the blue lines show the orbits of the solution curves.

The two different real eigenvalues are $\mu_1 = -1$ and $\mu_2 = -3$. The corresponding eigenvectors are obtained by solving $(\mu_i \mathbf{I} - \mathbf{A})\mathbf{v}_i = \mathbf{0}$ with $i = 1, 2$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The values of λ are $\lambda_{\pm}^{(1)} = \pm i$, $\lambda_{\pm}^{(2)} = \pm\sqrt{3}i$. Separating the real part and imaginary part of the solutions $\mathbf{x}_i = \mathbf{v}_i e^{\lambda_{\pm}^{(i)} t}$ with $i = 1, 2$, we can obtain four independent solutions. Thus the general solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{v}_1(A\mathrm{Re}\{e^{\lambda_+^{(1)} t}\} + B\mathrm{Im}\{e^{\lambda_+^{(1)} t}\}) + \mathbf{v}_2(C\mathrm{Re}\{e^{\lambda_+^{(2)} t}\} + D\mathrm{Im}\{e^{\lambda_+^{(2)} t}\}) \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(A\cos t + B\sin t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(C\cos\sqrt{3}t + D\sin\sqrt{3}t)
\end{aligned}$$