MATH150 Introduction to Ordinary Differential Equations, Fall 2010
Week 13 Wksht: Normal Modes / Fourier series (TXXX)

1. (**Demonstration**) Find the Fourier series (Fourier sine series) for the sawtooth function

$$f(x) = \frac{x}{\pi}, \quad -\pi < x < \pi.$$

By suitably choosing x, discover the series expansion

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

Hints: We know this function is an odd function with period $2L = 2\pi$, so this one can be written as a Fourier sine series and the Fourier coefficients b_n are

$$\begin{aligned} b_n =& \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{\pi}\int_0^\pi \frac{x}{\pi}\sin nx dx \\ =& \frac{2}{n^2\pi^2}\int_0^{n\pi} X\sin X dX = -\frac{2}{n\pi}\cos n\pi = \frac{2}{n\pi}(-1)^{n+1} \end{aligned}$$

So, the Fourier sine series for this sawtooth function is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin nx$$

Choose $x = \frac{\pi}{2}$, then we have $f(x) = \frac{1}{2} = \sum\limits_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{2}$. When n is an even number, $\sin\frac{n\pi}{2} = 0$; when n is an odd number, $\sin\frac{n\pi}{2} = (-1)^{(n-1)/2}$. So

$$f(\frac{\pi}{2}) = \frac{1}{2} = \sum_{i=1}^{\infty} \frac{2}{(2i-1)\pi}(-1)^{(2i-1)+1}(-1)^{i-1} = \sum_{i=1}^{\infty} \frac{2}{(2i-1)\pi}(-1)^{i-1}$$

So, we have the series expansion:

$$\frac{\pi}{4} = \frac{\pi}{2}f(\frac{\pi}{2}) = \sum_{i=1}^{\infty} \frac{1}{(2i-1)}(-1)^{i-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

2. (**Class work**) Find the Fourier series (Fourier sine series) for the square function

$$f(x) = \begin{cases} \text{-1}, & -\pi < x < 0; \\ 1, & 0 < x < \pi. \end{cases}$$

By suitably choosing x, rediscover the series expansion

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

Hints: We know this function is an odd function with period $2L = 2\pi$, so this one can be written as a Fourier sine series and the Fourier coefficients b_n are

$$\begin{aligned} b_n =& \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{\pi}\int_0^\pi \sin nx dx \\ =& \frac{2}{n^2\pi^2}\int_0^{n\pi} \sin X dX = \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}(1-(-1)^n) \end{aligned}$$

So, the Fourier sine series for this sawtooth function:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin nx$$

Choose $x = \frac{\pi}{2}$, then we have $f(x) = 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin\frac{n\pi}{2}$. When n is an even number, $\sin\frac{n\pi}{2} = 0$; when n is an odd number, $\sin\frac{n\pi}{2} = (-1)^{(n-1)/2}$. So

$$f(\frac{\pi}{2}) = \frac{1}{2} = \sum_{i=1}^{\infty} \frac{2}{(2i-1)\pi}(1-(-1)^{2i-1})(-1)^{i-1} = \sum_{i=1}^{\infty} \frac{4}{(2i-1)\pi}(-1)^{i-1}$$

So, we have the series expansion:

$$\frac{\pi}{4} = \frac{\pi}{4}f(\frac{\pi}{2}) = \sum_{i=1}^{\infty} \frac{1}{(2i-1)}(-1)^{i-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

3. (**Class work**) Consider two equal masses connected by a single spring, with the masses free to move horizontally. Determine the governing equations, and the general solution in terms of normal modes. (Hint: The system will not oscillate if horizontally translated, or if moving horizontally with constant velocity.)

Hints: Let us assume the positions of these two objects at time t be x_{10} and x_{20}, and the original length of this spring is L. Then the governing equations are

$$\begin{aligned} mx_{10}'' &= k(x_{20} - x_{10} - L); \\ mx_{20}'' &= -k(x_{20} - x_{10} - L). \end{aligned}$$

If we set new variables: $x_1 = x_{10} + L/2$ and $x_2 = x_{20} - L/2$, so we have

$$\begin{aligned} mx_1'' &= k(x_2 - x_1); \\ mx_2'' &= -k(x_2 - x_1). \end{aligned}$$

Please note that the change of variables here is purely to simplfy the following calculation. The coefficient matrix is $A = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix}$. So the characteristic equation is

$$0 = \det(\mathbf{A} - m\lambda^2\mathbf{I}) = (-k - \lambda^2)^2 - k^2,$$

and two eigenvalues are

$$\lambda_1^2 = 0, \qquad \lambda_2^2 = -2k/m$$

which means the angular frequencies are $\omega_1 = 0$ and $\omega_2 = \sqrt{2k/m}$.

For eigenvalue $\lambda_1^2 = 0$, solve the equation $\mathbf{A}\mathbf{v}_1 = m\lambda_1^2\mathbf{v}_1$, we can get the eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For eigenvalue $\lambda_2^2 = 0$, solve the equation $\mathbf{A}\mathbf{v}_2 = m\lambda_2^2\mathbf{v}_2$, we can get the eigenvector: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ \text{-1} \end{pmatrix}$.

The final result is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A + Bt) + c_2 \begin{pmatrix} 1 \\ \text{-1} \end{pmatrix} (C\cos\sqrt{2k/m}\ t + D\sin\sqrt{2k/m}\ t)$$