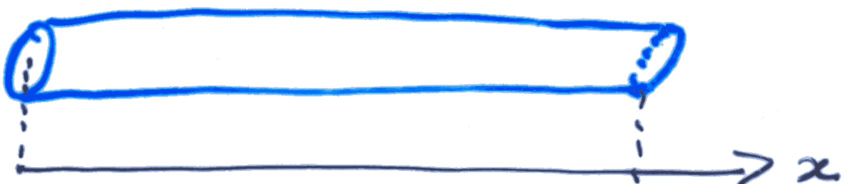
10.6 Other Heat Conduction Problems

Suppose



$u(0,t)=T_1$ $\qquad$ $u(L,t)=T_2, \quad t>0.$

So we need to solve the problem

$$\left\{ \begin{array}{l} \alpha^2 u_{xx} = u_t, \quad u(0,t)=T_1, \; u(L,t)=T_2 \\ u(x,0)=f(x), \; t>0, \; 0\le x\le L \end{array} \right\}.$$

Let

$$u(x,t) \longrightarrow \underbrace{v(x)}_{\text{steady state temp. dist.}} \text{ as } t\to\infty .$$

Then

$$\underset{(0<x<L)}{v''(x)=0}, \quad v(0)=T_1, \quad v(L)=T_2.$$

$$\Longrightarrow \boxed{v(x) = (T_2-T_1)\frac{x}{L} + T_1.}$$

Let us write

$$w(x,t)=u(x,t)-v(x) \quad \text{or} \quad u(x,t)=\underbrace{w(x,t)}_{\text{transient temp. distribution}}+v(x)$$

Then

$$\alpha^2(w+v)_{xx} = (w+v)_t \longrightarrow \alpha^2 w_{xx} = w_t$$

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0,$$

$$w(L,t) = u(L,t) - v(L) = T_2 - T_2 = 0.$$

Thus the problem is being reduced to the problem:

$$\begin{cases} \alpha^2 w_{xx} = w_t, \quad w(0,t) = 0, \quad w(L,t) = 0 \\ \text{and} \\ w(0,x) = u(0,x) - V(x) = f(x) - V(x) \;(= g(x) \text{ say}) \\ \qquad = f(x) - \left[(T_2 - T_1)\frac{x}{L} + T_1\right] = g(x). \end{cases}$$

Hence

$$w(x,t) = \sum_{n=1}^{\infty} c_n w_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L},$$

and

$$c_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

We deduce

$$u(x,t) = V(x) + w(x,t)$$

$$= (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2/L^2 \cdot t} \sin\frac{n\pi x}{L}$$

and

$$c_n = \frac{2}{L}\int_0^L \left[f(x) - (T_2 - T_1)\frac{x}{L} - T_1\right]\sin\frac{n\pi x}{L}\,dx.$$

Eg 10.6.1

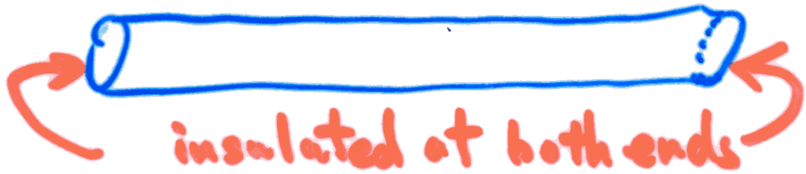
$$u_{xx} = u_t, \quad 0 < x < 30, \; t > 0$$

$$u(0,t) = 20, \quad u(30,t) = 50, \quad t > 0$$

$$u(x,0) = f(x) = 60 - 2x, \quad 0 < x < 30.$$

$$w(x,0) = u(x,0) - V(x) = 60 - 2x - (20 + x) = 40 - 3x.$$

Consider



$\Rightarrow$ No heat flow at both ends

The problem becomes

$$\begin{cases} \alpha^2 u_{xx} = u_t, \quad t>0 \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t>0, \\ u(x,0) = f(x). \end{cases}$$

Separation of variables method implies

$$u(x,t) = X(x)T(x)$$

$$\downarrow$$

$$X'' + \lambda X = 0, \qquad X = k_1 \sin\mu x + k_2 \cos\mu x = k_2 \cos\frac{n\pi x}{L}$$

$$T' + \alpha^2 \lambda T = 0 \qquad T = e^{-n^2\pi^2\alpha^2 t/L^2}, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos\frac{n\pi x}{L}$$

$$u(x,0) = f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L},$$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, \ldots$$

See Eg 10.6.2 and other more general problems.