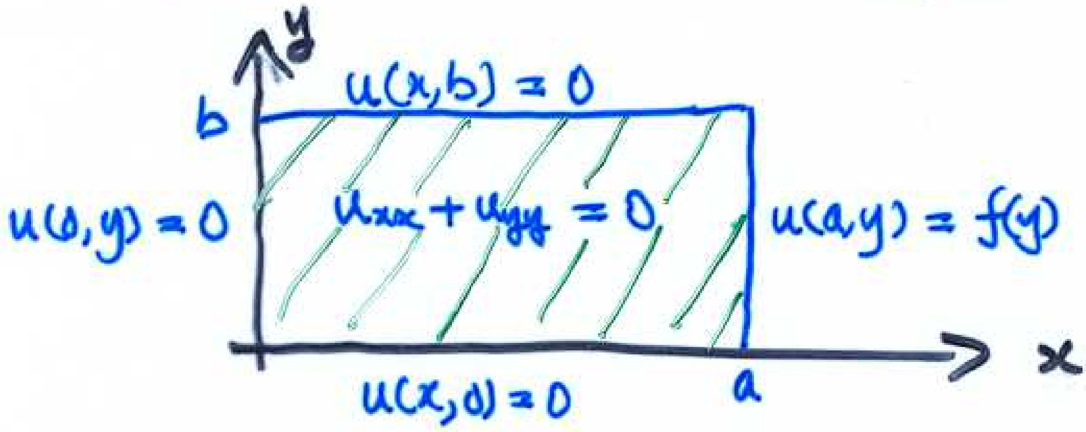
§10.8 Laplace's Equation

Consider

$$u_{xx} + u_{yy} = 0, \quad \text{------} \quad \begin{cases} 0 < x < a; \\ 0 < y < b, \end{cases}$$

$$u(x,0) = 0,\ u(x,b) = 0 \quad \cdots\cdots \quad 0 < x < a,$$

$$u(0,y) = 0,\ u(a,y) = f(y), \ \cdots\cdots \quad 0 \le y \le b.$$



(Direchlet problem for a rectangle.)

Separation of variables: $\quad u(x,y) = X(x)Y(y)$.

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda > 0$$

$$\Rightarrow \quad \begin{cases} X'' - \lambda X = 0, \quad X(0) = 0; & (1) \\ Y'' + \lambda Y = 0 \quad Y(0) = 0,\ Y(b) = 0. & (2) \end{cases}$$

$$(2) \Rightarrow \quad Y = A \sin\sqrt{\lambda}\, y + B \cos\sqrt{\lambda}\, y = A \sin\sqrt{\lambda}\, y$$

$$\lambda = n^2\pi^2/b^2, \quad n = 1, 2, 3, \ldots$$

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right), \quad n = 1, 2, 3, \cdots$$

$$(1) \Rightarrow \quad X(x) = k_1 \cosh\frac{n\pi x}{b} + k_2 \sinh\frac{n\pi x}{b}$$

$$= k_2 \sinh\frac{n\pi x}{b}, \quad n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad u_n(x,y) = u(x,y) = \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$$n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad u(x,y) = \sum_1^{\infty} c_n u_n = \sum c_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

But

$$f(y) = u(a,y) = \sum_1^{\infty} \underbrace{c_n \sinh\frac{n\pi a}{b}}\sin\frac{n\pi y}{b}$$

$$(0 \le y \le b)$$

$$\Rightarrow \quad \underbrace{c_n \sinh\frac{n\pi a}{b}}_{\text{Fourier coeff.}} = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy.$$

$$\Rightarrow \quad c_n = \frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

Remark: coeff: $\dfrac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} \simeq \dfrac{\frac{1}{2}\exp(n\pi x/b)}{\frac{1}{2}\exp(n\pi a/b)}$

$$\simeq \exp[-n\pi(a-x)/b]$$

$\rightarrow 0$ as $n \rightarrow \infty$. Series converges.