§10.2 Fourier Series

A function $f(x)$ is called periodic of period T if

$$f(x+T) = f(x)$$

whenever $x+T$ lies in the definition of f, and holds for all x.

Clearly

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \cdots$$

T is called the fundamental period of f if there is no smaller T so that $f(x)$ has the above property.

Eg. $\sin\left(\frac{m\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ have fundamental period $T = 2L/m$.

Recall the basic properties:

$$\int_{-L}^{L} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{m\pi x}{L}\right) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n; \\ L, & m = n, \end{cases}$$

and

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \text{ for all } m \text{ and } n.$$

Let u, v be two functions, we define

$$(u,v) = \int_{\alpha}^{\beta} u(x)\,v(x)\,dx$$

to be the inner product of u, v. If $(u,v) = 0$, then the functions are called orthogonal on (α, β).

§10.2 $\cos(a\pm b) = \cos a\cos b \mp \sin a \sin b$, $\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1$

$\Rightarrow \cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$, $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ 2

Verification: $m \neq n$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right) dx$$

$$= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin (m+n)\pi x/L}{m+n} + \frac{\sin (m-n)\pi x/L}{m-n}\right]_{-L}^{L}$$

$$= 0.$$

When $m=n$,

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx = \int_{-L}^{L} \cos^2\frac{m\pi x}{L}\, dx$$

$$= \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(\frac{2m\pi x}{L}\right) dx = \frac{1}{2}\left[x + \frac{L}{2\pi m}\sin\frac{2m\pi x}{L}\right]_{-L}^{L}$$

$$= \frac{1}{2}\left(L - (-L)\right) = L.$$

The case for sine is similar.

We want to express a periodic function $f(x)$ of period $2L$ by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left\{a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right\}$$

where a_m and b_m are coefficients. This is called the Fourier series of $f(x)$. For the moment, there is no guarantee the convergence of the series.

Let us suppose term-by-term integration is permitted:
Multiply the eqⁿ throughout by $\cos\left(\frac{n\pi x}{L}\right)$ and integrate yield

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx +$$

$$+\sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} b_m\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= 0 + a_n \times L + 0 = a_n \cdot L.$$

Hence
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

When $n=0$, then

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2}\int_{-L}^{L} 1\,dx + 0 + 0 = a_0 \cdot L.$$

So the above formula, called the <u>Euler-Fourier formula</u>, holds for $n = 0, 1, 2, 3, \ldots$.

If we multiply the $f(x)$ throughout by $\sin\frac{n\pi x}{L}$ and integrate, then we similarly obtain

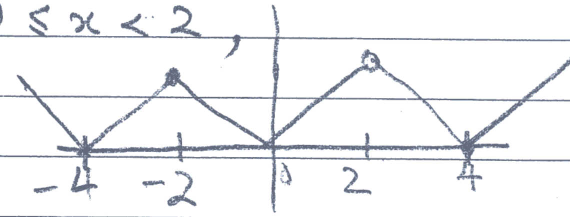
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots$$

This is also called the Euler-Fourier formula.

Example 10.2.1

$$f(x) = \begin{cases} -x, & -2 \le x < 0; \\ x, & 0 \le x < 2, \end{cases}$$

and $f(x+4) = f(x)$.



Since $2L = 4$, so we choose $L = 2$ and write

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{2}\right) + b_m \sin\left(\frac{m\pi x}{2}\right).$$

So
$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left\{\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx\right\}$$
$$= \frac{1}{2}\left(\frac{-x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{2}\right) = \frac{1}{2}(2+2) = 2$$

and
$$a_m = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0} -x\cos\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{m\pi x}{2}$$

$$= \frac{1}{2}\left\{-x\frac{2}{m\pi}\sin\frac{m\pi x}{2}\Big]_{-2}^{0} + \int_{-2}^{0}\frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)dx\right\} +$$

By parts

$$+ \frac{1}{2}\left\{\frac{2x}{m\pi}\sin\frac{m\pi x}{2}\Big]_{0}^{2} - \int_{0}^{2}\frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)dx\right\}.$$

$$= \frac{1}{2}\left\{0 - \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\Big]_{-2}^{0}\right\} + \frac{1}{2}\left\{0 + \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\Big]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left(\frac{2}{m\pi}\right)^2\left\{\cos m\pi - 1\right\} + \frac{1}{2}\left(\frac{2}{m\pi}\right)^2\left\{\cos m\pi - 1\right\}$$

$$= \left(\frac{2}{m\pi}\right)^2(\cos m\pi - 1) = \begin{cases} -8/(m\pi)^2, & m = 1, 3, 5, \cdots \\ 0, & m = 2, 4, 6, \cdots \end{cases}$$

Similarly,

$$b_m = \frac{1}{2}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{2}\right)dx = 0, \quad m = 1, 2, 3, \cdots.$$
(Check!)

So

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{2}\right), \qquad (m = \text{odd})$$

$$= 1 + \sum_{n=1}^{\infty}\left(-\frac{8}{(2n-1)^2\pi^2}\right)\cos\frac{(2n-1)\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

Approximation (Eg 10.2.3) We define the mth-partial sum to be

$$S_m(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{m}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}.$$

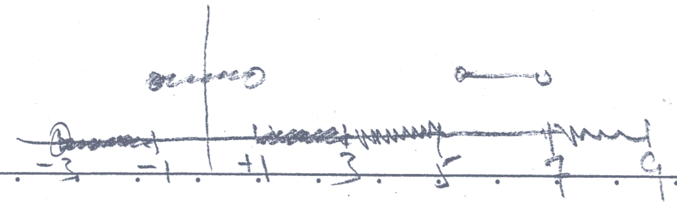
Since

$$\left|\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}\right| \leq \frac{1}{(2n-1)^2}$$

so that "the tail" of the Fourier series tends to zero "relatively fast".

$$e_m = \text{Error} = |f(x) - S_m(x)| \leq \sum_{n=m+1}^{\infty}(|a_n| + |b_n|)$$

The eg above, $|f(x) - S_m| \leq \sum_{n=m+1}^{\infty}\frac{1}{(2n-1)^2}$



(Given)

Eg 10.2.2

$$f(x) = \begin{cases} 0, & -3 < x < -1; \\ 1, & -1 < x < +1; \\ 0, & +1 < x < 3. \end{cases} \qquad f(x+6) = f(x).$$

So $L = 6/2 = 3$, and

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{3} + b_m \sin\frac{m\pi x}{3}.$$

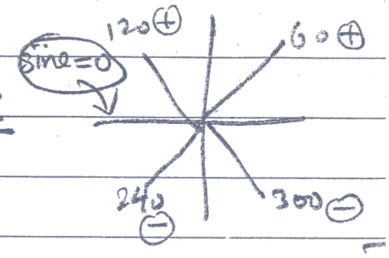
$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_{-1}^{1} 1\,dx = \frac{2}{3}.$$

$$a_m = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{m\pi x}{3}\,dx = \frac{1}{3}\int_{-1}^{1}(1)\cos\frac{m\pi x}{3}\,dx$$

$$= \frac{1}{m\pi}\sin\frac{m\pi x}{3}\Bigg|_{-1}^{1} = \frac{2}{m\pi}\underbrace{\sin\frac{m\pi}{3}}_{=(\pm\sqrt{3}/2)}, \quad m = 1, 2, 3, \ldots.$$

$$b_m = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{m\pi x}{3}\,dx = \frac{1}{3}\int_{-1}^{1}(1)\sin\frac{m\pi x}{3}\,dx$$

$$= \frac{1}{m\pi}\left(-\cos\frac{m\pi x}{3}\right]_{-1}^{1} = 0, \quad m = 1, 2, 3 \ldots.$$



So $$f(x) = \frac{1}{3} + \sum_{m=1}^{\infty}\left(\frac{2}{m\pi}\sin\frac{m\pi}{3}\right)\cos\frac{m\pi x}{3}$$

$$= \frac{1}{3} + \boxed{\frac{\cancel{2}\sqrt{3}}{\pi\cancel{2}}}\left[\cos\frac{\pi x}{3} + \frac{1}{2}\cos\frac{2\pi x}{3} - \frac{1}{4}\cos\frac{4\pi x}{3} - \frac{1}{5}\cos\frac{5\pi x}{3} + \cdots\right]$$

§10.3 Fourier convergence Thm.

<u>Theorem 10.3.1</u> Suppose f and f' are piecewise continuous on $-L \leq x < L$. Then

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

converges to $[f(x+) + f(x-)]/2$ at all points where f is defined.

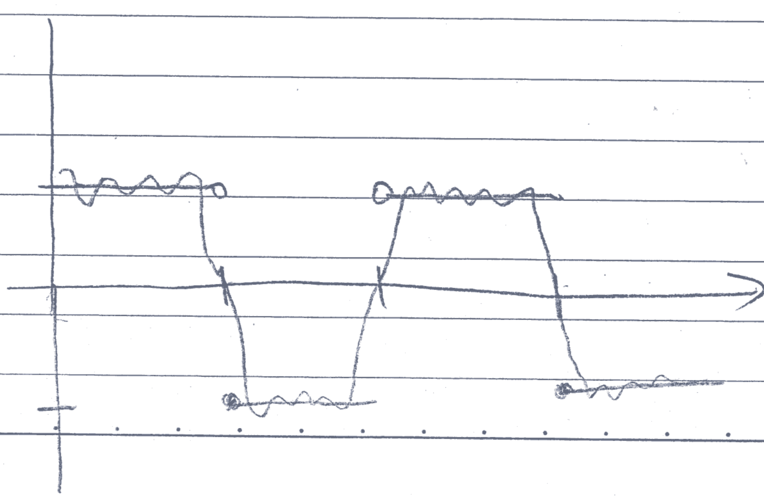
<u>Remark</u> (1) $f(x+) = \lim_{h \to 0} f(x+h)$, $f(x-) = \lim_{h \to 0} f(x-h)$.

(2) We can extend $f(x)$ beyond $[-L, L)$ periodically.

(3) If $f(x)$ is continuous at x_0, then

$$f(x_0) = \frac{1}{2}\left[f(x_0+) + f(x_0-)\right]$$

so that the $S_m \to f(x_0)$.



§10.4 Even and Odd Functions

A function f is called even if

$$f(-x) = f(x) \qquad (x^2, x^4, \cos x, \cdots)$$

and f is called odd if

$$f(-x) = -f(x). \qquad (x, x^3, \sin x, \cdots)$$

Suppose $f(x)$ is an even function and

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

as its Fourier series. Then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{L}\left(\int_{-L}^{0} + \int_{0}^{L}\right) f(x)\cos\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^L f(-t)\cos\left(\frac{-m\pi t}{L}\right)dt + \frac{1}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx, \quad n = 0, 1, 2, \cdots.$$

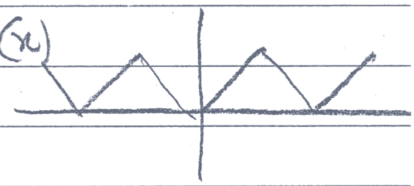
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = \frac{1}{L}\left[\int_0^L f(-t)\sin\left(\frac{-m\pi t}{L}\right)dt + \int_0^L f(t)\sin\frac{m\pi x}{L}\,d\right.$$

$$= \frac{1}{L}\left[\int_0^L -f(x)\sin\left(\frac{m\pi t}{L}\right)dx + \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx\right] = 0, \quad n = 1, 2, 3, \cdots$$

This is called a cosine series (for even functions).

Eg (revisited) 10.2.1 (p. 580) $f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x & 0 \le x < 2 \end{cases}$ Even Extn.

$f(x+4) = f(x)$

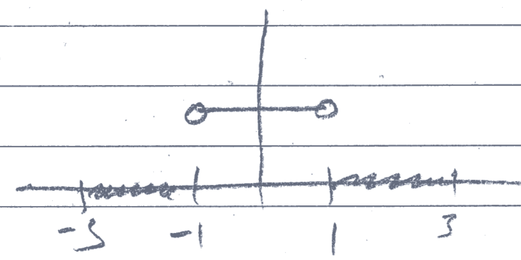


So $f(x)$ is even, and

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos (2n-1)\pi x/2}{(2n-1)^2}.$$ (no sine terms).

Eg (revisited) 10.2.2 (p. 583)

$$f(x) = \begin{cases} 0, & -3 < x < -1; \\ 1 & -1 < x < 1; \\ 0 & 1 < x < 3. \end{cases}$$



$f(x)$ is again even, and

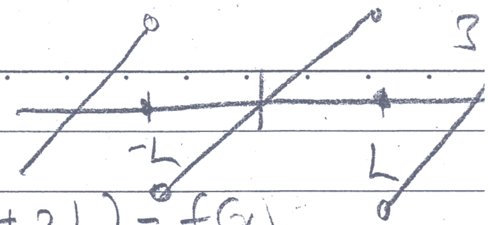
$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{3} \cos \frac{n\pi x}{3}.$$ (no sine)

If $f(x)$ is now odd, then we similarly obtain

$$a_n = 0, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \cdots.$$

This is called a sine series (for odd functions).



Eg. 10.4.1 $f(x) = x, \quad -L < x < L$

$f(-L) = 0 = f(L), \quad f(x+2L) = f(x).$

(Odd extension.)

f is odd. If

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

$$= \sum_1^\infty b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left(-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\Big]_0^L + \int_0^L \frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx \right)$$

$$= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2 \overbrace{\sin\frac{n\pi x}{L}\Big|_0^L}^{=0} - \left(\frac{L}{n\pi}\right) x\cos\frac{n\pi x}{L} \right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{L^2}{n\pi}(-1)^n \right] = (-1)^{n+1}\frac{2L}{n\pi}, \quad n = 1, 2, 3, \ldots$$

Hence $$f(x) = \frac{2L}{\pi}\sum_{n=1}^\infty (-1)^{n+1}\frac{\sin(n\pi x/L)}{n}.$$ (No consine terms

Remark: If we choose $L = 2$ and compair with Eg. 10.2.1. Then one can expand $g(x) = x$, on $(0, 2)$ in consine series (even extension) as

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{\sin(2\pi x/2)}{2} + \frac{\sin(3\pi x/2)}{3} - \cdots \right)$$

on $(0,2)$ and to the f in Eg. 10.4.1 outside $(0, 2)$.