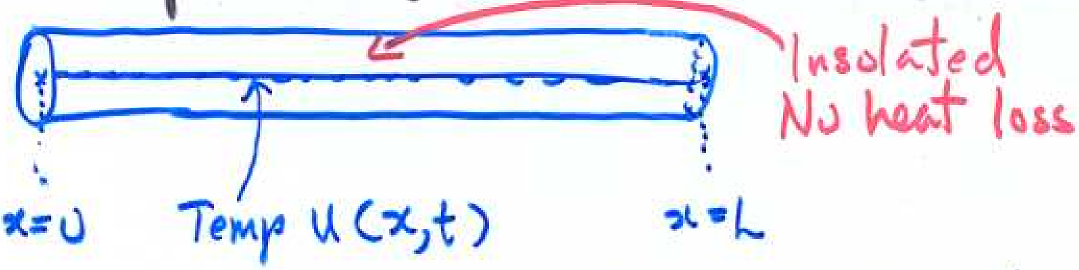
(B&D) §10.5 Separation of Variables: Heat conduction



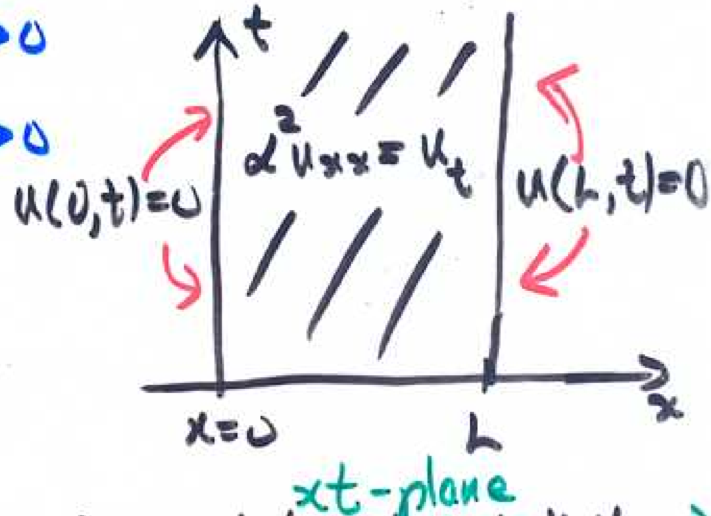
Heat Conduction PDE (see §10.8 for derivation)

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L,\ t > 0 \\ t = 0, & u(x,0) = f(x) \end{cases}$$

Initial temperature distribution

Assume $u(0,t) = T_1$, $t > 0$
$u(L,t) = T_2$, $t > 0$

or simply
$u(0,t) = 0$, $t > 0$
$u(L,t) = 0$, $t > 0$



The α^2 = Thermal diffusivity ($= K/\rho s$, K thermal conductivity, ρ density, s specific heat)

Separation of variables: Try

$$u(x,t) = X(x)\,T(t) \Rightarrow \alpha^2 X'' T = X T',$$

or

$$\underbrace{\frac{X''}{X}}_{(\text{fn of } x)} = \underbrace{\frac{1}{\alpha^2}\frac{T'}{T}}_{(\text{fn of } t)} \Longrightarrow \frac{X''}{X} = \underbrace{\boxed{-\lambda}}_{\text{Constant}} = \frac{1}{\alpha^2}\frac{T'}{T}$$

It remains to solve:

$$X'' + \lambda X = 0, \quad T' + \alpha \lambda T = 0$$

The boundary condition implies for all $t > 0$

$$0 = u(0,t) = X(0)\,T(t) \Longrightarrow X(0) = 0, \quad 0 = u(L,t) = X(L)\,T(t) \Longrightarrow X(L) = 0. \quad \text{since } T(t) \neq 0 \text{ for all } t.$$

So we end up with the boundary value problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0.$$

⌈ Problem of the form $y'' + By = 0$, $y(\alpha) = a$, $y(\beta) = b$ is called a Boundary value problem on $[\alpha, \beta]$. ⌋

Since the general solution is

$$X(x) = A \sin\sqrt{\lambda}\,x + B\cos\sqrt{\lambda}\,x,$$

so the BV $0 = X(0) = B \Longrightarrow B = 0$. But the BV $0 = X(L) = A\sin\sqrt{\lambda}\,L \Longrightarrow \sqrt{\lambda}\,L = n\pi$, $n = 1, 2, 3, \dots$.

We have $\boxed{\lambda = \lambda_n = n^2\pi^2/L^2, \quad n = 1, 2, 3, \dots}$

Thus $X_n(x) = \sin \lambda_n x = \sin\left(\frac{n\pi}{L}x\right)$, $n = 1, 2, 3, \dots$.

The λ_n is called the eigenvalue corresponding to the eigenfunction $X_n(x)$ for each of $n = 1, 2, 3, \dots$ for the BVP.

Hence $T' + \alpha^2 n^2\pi^2/L^2\, T = 0$,

and $T_n(t) = \exp\left(-\alpha^2\pi^2 n^2/L^2\, t\right)$, $n = 1, 2, 3, \dots$

Hence

$$u_n(x,t) = X_n T_n = e^{-n^2\pi^2\alpha^2/L^2\, t} \cdot \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \cdots$$

are all solutions of

$$\boxed{\begin{aligned} \alpha^2 u_{xx} &= u_t \\ u(0,t) = 0 &= u(L,t). \end{aligned}}$$

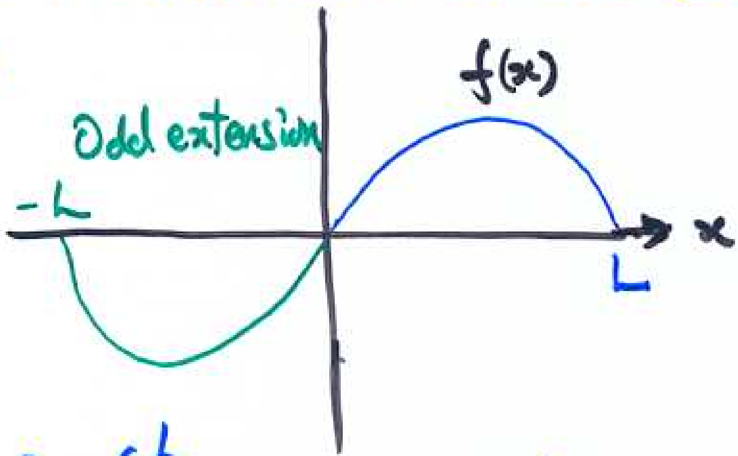
The superposition principle implies that

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n\, e^{-\frac{n^2\alpha^2\pi^2}{L^2} t} \cdot \sin\left(\frac{n\pi x}{L}\right).$$

It remains to find c_n: But the initial temperature distribution gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

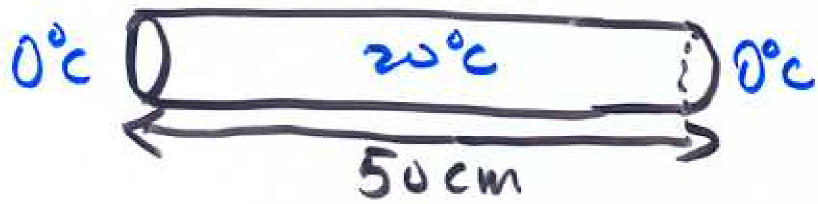
which is a Fourier sine series (odd extension) of $f(x)$. Thus



$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \cdots.$$

We also note that $u(x,t) \to 0$, $t \to \infty$ for each x.

Eg. (B&D 10.5.1) Insulated rod 50 cm, uniform initial temperature distribution at 20 °C with both ends of the rod maintained at 0 °C.



So $u(x,0) = f(x) = 20$ for $0 < x < 50$ $(l = 50)$.

Thus the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\alpha^2\pi^2 t/50^2} \sin\left(\frac{n\pi x}{50}\right)$$

where

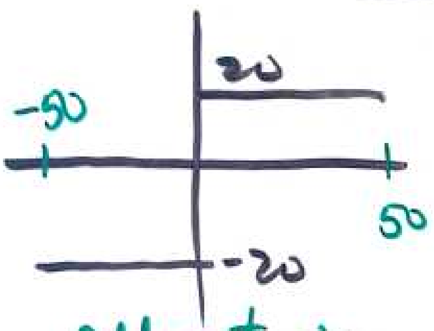
$$c_n = \frac{2}{50}\int_0^{50} 20 \cdot \sin\left(\frac{n\pi x}{50}\right) dx$$

$$= \frac{40}{n\pi}(1-\cos n\pi) = \begin{cases} 80/n\pi, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

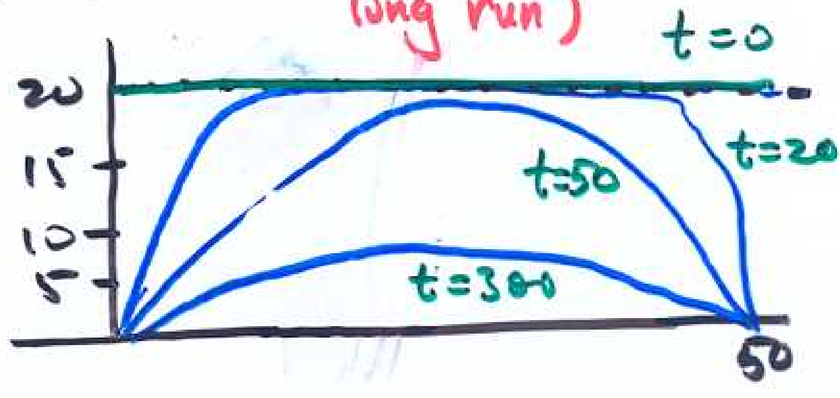
Hence

$$u(x,t) = \frac{80}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1} e^{-(2m+1)^2\alpha^2\pi^2 t/50^2} \sin\frac{(2m+1)\pi x}{50}$$

$\longrightarrow 0, \ t \to \infty$ (loss all heat in the long run)



Odd extension



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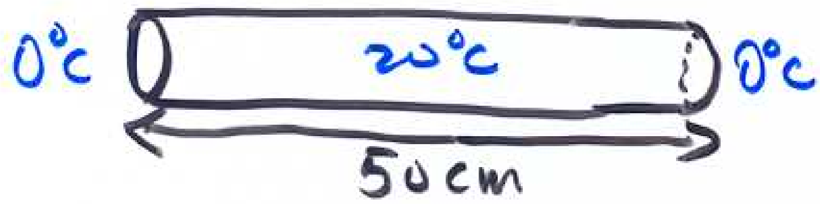

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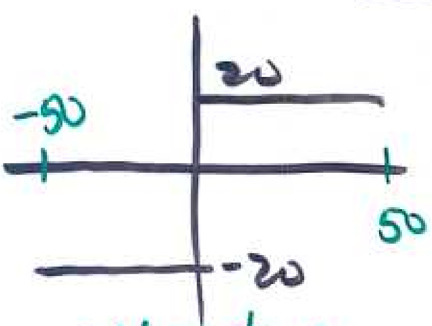

Odd extension