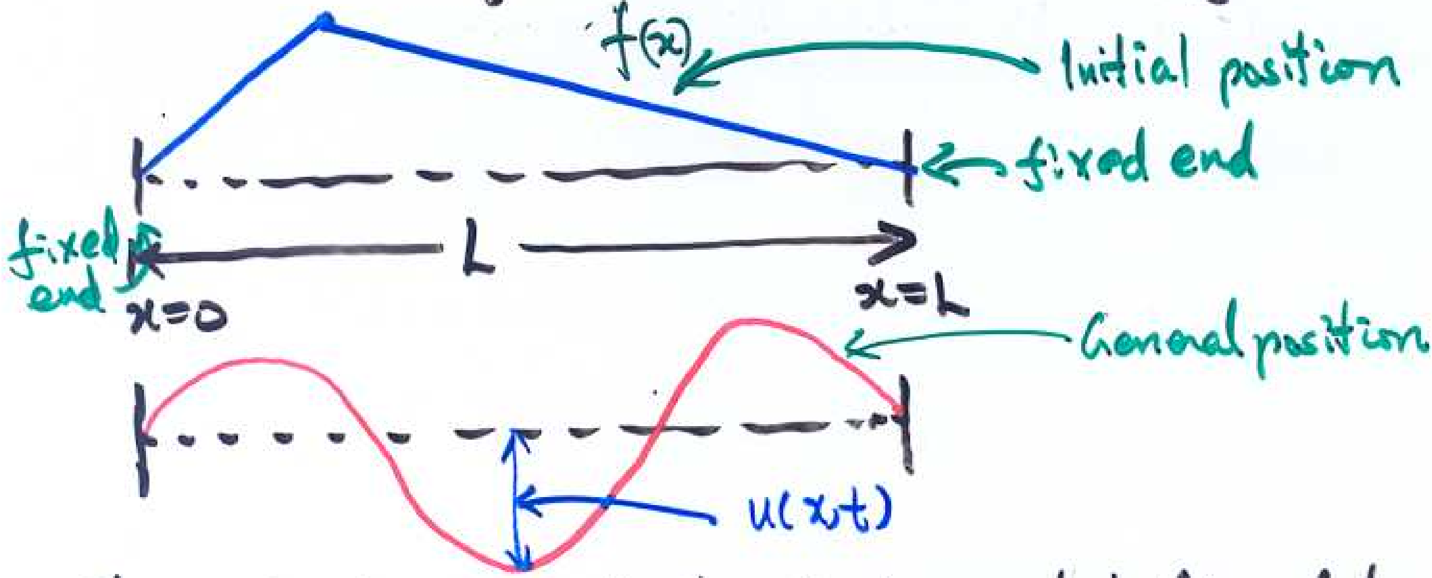
10.7 The Wave Eq$^{\underset{\sim}{n}}$, Vibration of Elastic String



The $u(x,t)$ is called the displacement function of the elastic string at x and at time t.

WAVE Equation
$$
\begin{cases}
a^2 u_{xx} = u_{tt}\,, & 0 < x < L,\ t > 0 \\
u(0,t) = 0 = u(L,t)\,, & t > 0 \\
\underbrace{u(x,0) = f(x)}_{\text{Initial displacement}},\ \underbrace{u_t(x,t) = g(x)}_{\text{Initial pluck}}\,, & 0 \le x \le L
\end{cases}
$$

Both ends fixed $\Rightarrow$ $f(0) = 0 = f(L)$,
$g(0) = 0 = g(L)$.

If released from rest $\Rightarrow$ $g(x) = 0,\quad 0 \le x \le L$.

Separation of variables $\Rightarrow$ $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \underbrace{X''/X}_{(x\text{-only})} = \underbrace{1/a^2 \cdot T''/T}_{(t\text{-only})} = -\lambda \text{ constant}$$

E.g. $\begin{cases} \text{Given initial displacement} \\ \text{No initial pluck} \end{cases}$ $\Longrightarrow$ $\begin{cases} a^2 u_{xx} = u_{tt}, \ 0 < x < L, \ t > 0, \\ u(0,t) = 0 = u(L,t), \ t \geq 0 \\ \left. \begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array} \right\}, \ 0 \leq x \leq L. \end{cases}$

Separation of variables

$$\Longrightarrow \quad u(x,t) = X(x)\,T(t)$$

$$\Longrightarrow \quad X''/X = 1/a^2 \cdot T''/T = -\lambda \text{ constant}$$

$$\Longrightarrow \quad \begin{cases} X'' + \lambda X = 0, \quad X(0) = 0 = X(L) \\ T'' + a^2 \lambda T = 0, \quad T'(0) = 0. \end{cases}$$

$$\Rightarrow \quad \begin{cases} \lambda_n = n^2\pi^2/L^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2,3,\dots \\ T_n(t) = A\cos\left(\frac{n\pi a t}{L}\right) + \cancel{B \sin\left(\frac{n\pi a t}{L}\right)} = A\cos\left(\frac{n\pi a t}{L}\right) \\ \qquad n = 1,2,3,\dots \end{cases}$$

$$\Rightarrow \quad u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right), \quad n = 1,2,3,\dots$$

$$\xRightarrow[\text{position}]{\text{Super-}} \quad u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right).$$

Initial condition $\Longrightarrow$ $f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

Fourier Sine series (odd extension) $\Longrightarrow$ $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2,3,\dots$

Let us consider

$$\sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{n\pi a t}{L}\right), \quad n=1,2,3,\dots$$

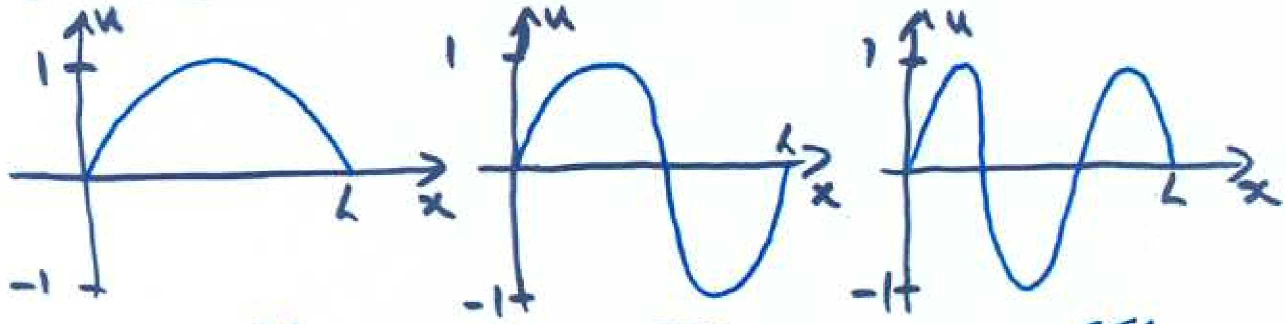
For each fixed n, the above product has the

period $\frac{2L}{na}$, and (natural) **frequency** $\frac{2\pi}{2L/na} = \frac{na\pi}{L}$.

The factor

$\sin\frac{n\pi x}{L}$ (wavelength : $\frac{2L}{n}$)

represents the displacement pattern of a given frequency. Each displacement pattern is called a natural mode of vibration

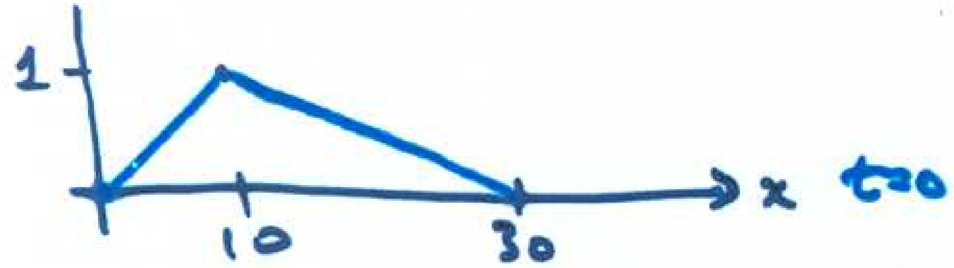


freq. $= \frac{\pi a}{L}$ freq. $= \frac{2\pi a}{L}$ freq. $= \frac{3\pi a}{L}$

wavelength $= 2L$ wavelength $= L$ wavelength $= \frac{2L}{3}$

Example 10.7.1 Suppose a vibrating string of length $L=30$ with both ends fixed, and no initial velocity from the given initial position:

$$u(x,0)=f(x)=\begin{cases} x/10, & 0\le x\le 10,\\ (30-x)/20, & 10\le x\le 30.\end{cases}$$



Given $a=2$, so solve

$$\begin{cases} 2^2 u_{xx}=u_{tt}, & 0\le x\le 30,\ t>0;\\ u(0,t)=0=u(30,t), & t>0;\\ u(x,0)=f(x),\ u_t(x,0)=0, & 0\le x\le 30.\end{cases}$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{30}\right)\cos\left(\frac{n\pi(2)t}{30}\right)$$

where

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx=\frac{2}{30}\int_0^{30} f(x)\sin\frac{n\pi x}{30}\,dx$$

$$=\frac{2}{30}\int_0^{10}\frac{x}{10}\cdot\sin\frac{n\pi x}{30}\,dx+\frac{2}{30}\int_{10}^{30}\frac{30-x}{20}\cdot\sin\frac{n\pi x}{30}\,dx$$

$$=\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3},\quad n=1,2,3,\ldots$$

interior point on the string is motionless for one-third of each period. Figure 10.7.6 shows a three-dimensional plot of u versus both x and t, from which the overall nature of the solution is apparent. Of course, the curves in Figures 10.7.4 and 10.7.5 lie on the surface shown in Figure 10.7.6.

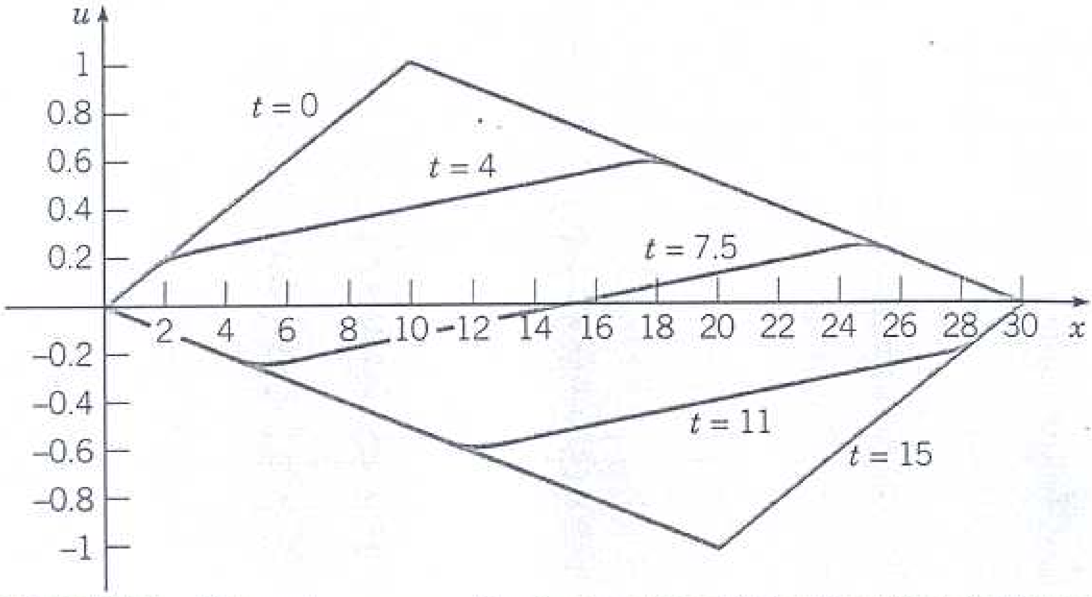


FIGURE 10.7.4 Plots of u versus x for fixed values of t for the string in Example 1.

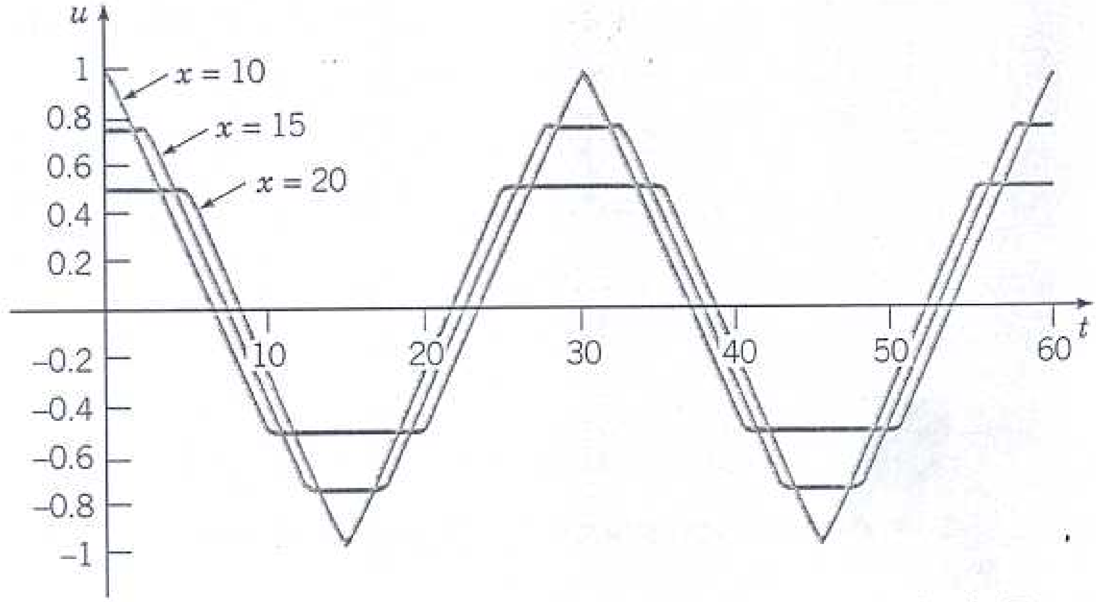


FIGURE 10.7.5 Plots of u versus t for fixed values of x for the string in Example 1.