MATH150 Introduction to Ordinary Differential Equations, 2006-07 Spring
Final Examination Solution: Version A
Part I: Multiple Choice Question.

Question	1	2	3	4	5	6	7	8	**Total**
Answer	c	a	e	b	a	b	d	e	

Part II: Short Questions

2. Suppose that the logistic equation

$$\frac{dx}{dt} = kx(M-x),$$

where k and M are positive constants, models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fishes are removed from the lake at the rate of hx fishes per month, where h is a positive constant.

(a) [*3 pts*] If $0 < h < kM$, show that the new fish population (with fishes removed at the above rate) still satisfy a logistic equation and hence find two equilibrium solutions when fishing occurs.

When fishing occurs, we have the following differential equation:

$$\frac{dx}{dt} = kx(M-x) - hx = kx\left((M-\frac{h}{k}) - x\right)$$

Therefore, the population is still logistic. Moreover, the two equilibrium solutions are $x(t) = 0$ and $x(t) = M - \frac{h}{k}$.

Answer: ______________________________

(b) [*2 pts*] Classify each equilibrium solutions in (a) as asymptotically stable or unstable.

$x(t) = 0$ is asymptotically unstable. $x(t) = M - \frac{h}{k}$ is asymptotically stable.

Answer: ______________________________

(c) [*3 pts*] If $h = kM$, show that $x(t) \to 0$ as $t \to \infty$ i.e. the lake is eventually fished out.

When $h = kM$, we have

$$\frac{dx}{dt} = -kx^2$$

It is a separable equation, so we get

$$\int \frac{1}{x^2}\, dx = -k \int dt$$

Solving, we get the following solution:

$$x(t) = \frac{1}{kt - C}$$

for some constant C. Therefore, as $t \to \infty$, $x(t) \to 0$.

Answer: ______________________________

3. Find the *general solution* of the differential equation

$$y'' - 2y = e^{-x}\sin x.$$

Answer: The answer is The characteristic roots are ± 2. So the complementary functions are $e^{\pm\sqrt{2}x}$. We try

$$Y = Ae^{-x}\sin x + Be^{-x}\cos x.$$

This yields

$$y' = -(A+B)e^{-x}\sin x + (A-B)e^{-x}\cos x$$

and

$$y'' = 2Be^{-x}\sin x - 2Ae^{-x}\cos x.$$

Thus we have

$$2(B-A)e^{-x}\sin x + (-2A-2B)e^{-x}\cos x \equiv e^{-x}\sin x.$$

This yields $A = -1/4 = -B$. So the general solution is given by

$$y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + \frac{1}{4}\big(\cos x - \sin x\big)e^{-x}.$$

4. Consider a piecewise continuous function

$$g(t) = \begin{cases} t - \frac{\pi}{2} & \text{if } 0 \le t < \frac{\pi}{2} \\ \cos t & \text{if } \frac{\pi}{2} \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases}$$

(a) [*3 pts*] Express $g(t)$ in terms of unit step functions $u_c(t)$ and the functions appearing in the definition of $g(t)$.

$$g(t) = (t - \frac{\pi}{2})(1 - u_{\frac{\pi}{2}}(t)) + \cos t(u_{\frac{\pi}{2}}(t) - u_\pi(t))$$

Answer: $g(t)$ = ______________________________

(b) [*3 pts*] Find the Laplace transform of $g(t)$.

Applying Laplace transform, we get

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{\pi}{2s} - \mathcal{L}\left\{(t - \frac{\pi}{2})u_{\frac{\pi}{2}}(t)\right\} + \mathcal{L}\left\{(\cos t)u_{\frac{\pi}{2}}(t)\right\} - \mathcal{L}\{(\cos t)u_\pi(t)\}$$

Consider the last two terms, we rewrite them as follows:

$$\mathcal{L}\left\{(\cos t)u_{\frac{\pi}{2}}(t)\right\} = -\mathcal{L}\left\{\sin(t - \frac{\pi}{2})u_{\frac{\pi}{2}}(t)\right\}$$

$$\mathcal{L}\{(\cos t)u_\pi(t)\} = -\mathcal{L}\{\cos(t-\pi)u_\pi(t)\}$$

Therefore, we have

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{\pi}{2s} - \frac{e^{-\frac{\pi}{2}s}}{s^2} - \frac{e^{-\frac{\pi}{2}s}}{s^2+1} + \frac{se^{-\pi s}}{s^2+1}$$

Answer: $\mathcal{L}\{g(t)\}$ = ____

5. Use the method of Laplace transform to solve

$$f(t) = \sin 3t + \int_0^t f(s)\sin 3(t-s)\,ds.$$

Answer: The answer is The equation can be written in the convolution form

$$f(t) = \sin 3t + (f * \sin 3x)(t).$$

We apply convolution theorem to give

$$\mathcal{L}(f) = \mathcal{L}(\sin 3t) + \mathcal{L}(f)\cdot\mathcal{L}(\sin(3t)).$$

Hence

$$\mathcal{L}(f) = \frac{\mathcal{L}(\sin(3t))}{1-\mathcal{L}(\sin(3t))} = \frac{3/(s^2+3^2)}{1-3/(s^2+3^2)} = \sqrt{\frac{3}{2}}\frac{\sqrt{6}}{s^2+(\sqrt{6})^2}.$$

So $f(t) = \sqrt{\frac{3}{2}}\sin(\sqrt{6}t)$.

6. [*6 pts*] Let

$$f(x) = \begin{cases} x, & -2 \le x < 0; \\ -x, & 0 \le x \le 2. \end{cases}$$

Find the Fourier series of $f(x)$.

As $f(x)$ is an even function, its Fourier series is a cosine series. We have

$$a_n = \frac{1}{2}\left(\int_{-2}^{0} x\cos(n\pi x/2)dx + \int_0^2 -x\cos(n\pi x/2)dx\right) = -\frac{4}{(n\pi)^2}(\cos(n\pi)-1) \quad \text{when} n \ne 0.$$

For a_0, we find that

$$a_0 = \frac{1}{2}\int f(x)dx = -2.$$

Therefore,

$$f(x) = -1 - \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(\cos(n\pi)-1)\cos(n\pi x/2).$$

Answer: ____ .

7. [*6 pts*] Given that $y(x) = x$ is a solution of

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

Find the general solution of the above differential equation.

We have $p(x) = -\frac{x+2}{x}$. For the second solution, let $y(x) = v(x)x$, we find that $v(x)$ satisfies

$$v'' - v' = 0.$$

Thus, $v(x) = Ae^x + B$ and the general solution is $y(x) = Ax + Bxe^x$ where A and B are constants.

Answer: ______________________________ .

Part III: Long questions.

8. A mass of 2 kg stretches a hanging spring 0.25 metres. The mass is acted on by an external force of $20\cos(8t)$ newtons, and is damped by a force which is proportional and opposite to the motion, and which is 2 newtons when the speed is .25 metres/second. The acceleration due to gravity is 10 metres/sec^2.

(a) [*2 pt*] Find the spring constant k; so, force $= k \cdot$ stretch.
Since force $= k \cdot$ stretch, we have

$$0.25k = 2 \times 10$$

Therefore, $k = 80$.

Answer: $k =$ ______________________________

(b) [*2 pt*] Find the damping constant γ; so, force $= \gamma \cdot$ speed.
Since force $= \gamma \cdot$ speed, we have

$$2 = 0.25\gamma$$

Therefore, $\gamma = 8$.

Answer: $\gamma =$ ______________________________

(c) [*3 pt*] Let u denote the distance the mass is from equilibrium. Write the 2nd order differential equation for u.

$$2u'' + 8u' + 80u = 20\cos(8t)$$

Answer: Differential equation is : ______________________________

(d) [*6 pt*] Suppose the mass is initially at rest at its equilibrium position. Formulate the initial value problem for u and hence solve for u.
We need to solve the initial value problem:

$$2u'' + 8u' + 80u = 20\cos(8t), \qquad u(0) = 0, \quad u'(0) = 0$$

Solving the homogeneous part of the solutions, we get

$$e^{-2t}(C_1\cos 6t + C_2\sin 6t)$$

Let the particular solution $u_p(t) = A\cos 8t + B\sin 8t$. Substitute into the differential equation, we get

$$\begin{cases} 24A + 32B & = 0 \\ -24A + 32B & = 10 \end{cases}$$

Solving, we get $A = -\frac{3}{20}$ and $B = \frac{1}{5}$.
Therefore, the general solution is

$$u(t) = e^{-2t}(C_1 \cos 6t + C_2 \sin 6t) + \frac{1}{20}(-3\cos 8t + 4\sin 8t)$$

Applying initial conditions, we get

$$u(t) = \frac{1}{60}e^{-2t}(9\cos 6t - 13\sin 6t) + \frac{1}{20}(-3\cos 8t + 4\sin 8t)$$

Answer: $u =$ ____________________________________

(e) [*4 pt*] Express the steady-state solution in the form $R\cos(\omega t - \delta)$ and hence find the amplitude and phase of the motion.
We need to write $\frac{1}{20}(-3\cos 8t + 4\sin 8t)$ in the form $R\cos(\omega t - \delta)$. Obviously, $\omega = 8$

$$R = \sqrt{(-\frac{3}{20})^2 + (\frac{4}{20})^2} = \frac{1}{4}$$

$$\cos\delta = -\frac{3}{5}, \quad \sin\delta = \frac{4}{5}$$

Therefore, $\delta = 2.21$ rad.
Amplitude $= \frac{1}{4}$ and phase $= 2.21$ rad.

Answer: ____________________________________

9. (A) Suppose an elastic string of length 60cm is being fixed at both ends in a horizontal position. Let $u(x, t)$ be the vertical displacement of the string at x centermeters from one end and at time t. Suppose the displacement function satisfies

$$\begin{aligned} &4u_{xx} = u_{tt}, && t > 0; \\ &u(0, t) = 0 = u(60, t), && t \geq 0; \\ &u_t(x, 0) = 0, \;\; u(x, 0) = f(x), && 0 \leq x \leq 60, \end{aligned} \tag{1}$$

where

$$f(x) = \begin{cases} -2x/50, & 0 \leq x \leq 25; \\ (x - 50)/25, & 25 < x \leq 50. \end{cases}$$

(a) [*2 pts*] Sketch the initial position of the string given by $f(x)$:

Answer: The sketch is ____ .

(b) [*1 pt*] Is the string released initially at rest ____________ (*answer "yes" or "no"*) ? **Yes.**

(c) [*6 pts*] Suppose a solution to (**??**) is in the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) \tag{2}$$

where $u_n(x, t) = X_n(x)T_n(t)$ for each $n = 1, 2, 3, \cdots$. Derive with reason the boundary value problem satisfied by $X_n(x)$. Then solve the problem.

Answers ____ .

Substituting $u_n(x, t) = X_n(x)T_n(t)$ into the wave equation yields $4X_n''T_n = X_nT_n''$. That is, $\frac{X_n''}{X_n} = \frac{1}{4}\frac{T_n''}{T_n} = \lambda_n$. Since $0 = u_n(0, t) = X_n(0)T_n(t) =$ and $0 = u_n(50, t) = X_n(50)T_n(t) =$ hold for all t, so we must have $X_n(0) = X_n(50) = 0$. For otherwise, $T_n \equiv 0$, a contradiction. The condition $(u_n)_t(x, 0) = 0$ translates, in a similar maner, to $T_n'(0) = 0$. Thus, we obtain two boundary value value problems:

$$\begin{aligned} &X_n'' + \lambda X_n = 0, \quad X_n(0) = 0 = X_n(50) \\ &T_n'' + 4\lambda T_n = 0, \quad T_n'(0) = 0. \end{aligned}$$

The first equation has general solution

$$X_n = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

and the boundary condition implies $A = 0$ so that $X_n = \sin\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}50 = 0$. Hence $\lambda_n = \frac{n^2\pi^2}{50^2}$ for $n = 1, 2, 3, \cdots$. Thus

$$X_n(x) = \sin\frac{n\pi x}{50}, \quad n = 1, 2, 3, \cdots .$$

The second equation has general solution $T_n = C\cos 2\sqrt{\lambda}t + D\sin 2\sqrt{\lambda}t$. The boundary condition implies $D = 0$ so that

$$T_n(t) = \cos\frac{n\pi t}{25}, \quad n = 1, 2, 3, \cdots .$$

Hence

$$u(x, t) = \sum_{j=1}^{\infty} u_n(x, t) = \sum_{j=1}^{\infty} c_n \sin\frac{n\pi x}{50}\cos\frac{n\pi t}{25}.$$

(e) [*3 pts*] Derive with reason the boundary value problem satisfied by $T_n(t)$. Then solve the problem.

$$T_n'' + 4\lambda T_n = 0, \quad T_n'(0) = 0.$$

The second equation has general solution $T_n = C\cos 2\sqrt{\lambda}t + D\sin 2\sqrt{\lambda}t$. The boundary condition implies $D = 0$ so that

$$T_n(t) = \cos\frac{n\pi t}{25}, \quad n = 1,\, 2,\, 3,\, \cdots.$$

Answers: ____ .

(f) [*3 pts*] Explain how one can determine the unknown coefficients c_n. Write down a formula for c_n without actually evaluating the integrals.

Since

$$u(x,\, t) = \sum_{j=1}^{\infty} u_n(x,\, t) = \sum_{j=1}^{\infty} c_n \sin\frac{n\pi x}{50}\cos\frac{n\pi t}{25}.$$

So the initial condition implies

$$f(x) = u(x,\, 0) = \sum_{j=1}^{\infty} u_n(x,\, 0) = \sum_{j=1}^{\infty} c_n \sin\frac{n\pi x}{50}.$$

We conclude that this is a Fourier sine series for the odd extension of $f(x)$ to $-50 < x < 0$. Thus the Fourier coefficients c_n are given by

$$c_n = \frac{2}{50}\int_0^5 0 \sin\frac{n\pi x}{50} f(x),\, dx$$

for $n = 1,\, 2,\, 3,\, \cdots$. Answer: ____
.

(B) Suppose the problem (A) above is modified so that the displacement function $u(x,\, t)$ still satisfy (**??**) but with new conditions $u(x,\, 0) = 0$ and $u_t(x,\, 0) = g(x), \;\; 0 \le x \le 60$ instead, where $g(x)$ is some given function. We again assume that a solution $u(x,\, t)$ takes the form (**??**) for some different c_n.

(g) [*2 pts*] Is the string released at rest initially ____________ (*answer "yes" or "no"*) ?

No.

(h) [*4 pts*] Determine with details the $u_n(x,\, t),\;\; n = 1,\, 2,\, 3, \cdots$.

Since the boundary condition for X_n remians unchanged so we still have

$$X_n(x) = \sin\frac{n\pi x}{50}, \quad n = 1,\, 2,\, 3,\, \cdots.$$

But the new boundary condition $u(x,\, 0) = 0$ implies $T_n(0) = 0$ so that

$$T_n(t) = \sin\frac{n\pi t}{25}, \quad n = 1,\, 2,\, 3,\, \cdots,$$

and

$$u(x,\, t) = \sum_{j=1}^{\infty} u_n(x,\, t) = \sum_{j=1}^{\infty} d_n \sin\frac{n\pi x}{50}\sin\frac{n\pi t}{25}.$$

One needs to work out $u_t(x,\, 0) = g(x)$ to work out the coefficients by Fouier cosine extension. Answer: ____
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