Math150-L2 Final Exam Answers, Spring 06

Part I: MC Questions

Version A

Question	1	2	3	4	5	6	7	8	9	**Total**
Answer	d	b	d	c	e	a	e	b	d	

Version B

Question	1	2	3	4	5	6	7	8	9	**Total**
Answer	c	e	c	a	b	d	b	e	c	

Answers of Version A MC Questions

1. Suppose that the population $p(t)$ of certain field mice after t years from now is described by the initial value problem

$$\frac{dp}{dt} = 0.5p - 500 \ , \qquad p(0) = 800 \ .$$

Find the time when the population just become extinct; i.e., the time T (in years) when $p(T) = 0$.

(a) 1.2476 (b) 1.8637 (c) 2.7645 (d) 3.2189 (e) 4.2537

Answer: (d). $\displaystyle\int \frac{dp}{p-1000} = \int \frac{1}{2}dt \Longleftrightarrow p(t) = 1000 + Ce^{t/2}$. $p(0) = 800$ implies $C = -200$, and hence $p(t) = 1000 - 200e^{t/2}$.

$$p(t) = 0 \Longleftrightarrow t = 2\ln 5 = 3.2189$$

2. For which of the following functions $M(x, y)$ is the equation

$$M(x, y) + (x^3 + 8y - 3x)\frac{dy}{dx} = 0$$

an exact equation?

(a) $x(y^2 + 1)$ (b) $3y(x^2 - 1)$ (c) $x^2 - 2y^2$ (d) $x^2 + y$ (e) xy

Answer: (b). $\dfrac{\partial(3y(x^2-1))}{\partial y} = 3x^2 - 3 = \dfrac{\partial(x^3 + 8y - 3x)}{\partial x}$.

3. By the method of undetermined coefficients, there is a particular solution of the equation

$$y'' - 4y' - 5y = (1 + 2t^2 - t^3)e^{5t}$$

which has the form $u = p(t)e^{5t}$, where $p(t)$ is a polynomial. The degree of $p(t)$ is:

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Answer : (d). 5 is a single root of the characteristic equation $r^2 - 4r - 5 = (r - 5)(r + 1) = 0$. Hence the degree of $p(t)$ is 4. Or by directly putting $u = p(t)e^{5t}$ into the equation to see the degree of $p(t)$:

$$(p''e^{5t} + 10p'e^{5t} + 25pe^{5t}) - 4(p'e^{5t} + 5pe^{5t}) - 5pe^{5t} = (1 + 2t^2 - t^3)e^{5t}$$

$$p'' - 6p' = 1 + 2t^2 - t^3$$

4. A periodic external force $F(t) = 9\cos(\omega t)$ is applied to a undamped spring-mass system so that the equation of motion of the mass is

$$2y'' + 6y = 9\cos(\omega t) .$$

Which of the following values of ω will cause an unbounded oscillation (resonance) of the mass?

(a) $\frac{2}{3}$ (b) 3 (c) $\sqrt{3}$ (d) $\sqrt{\frac{2}{3}}$ (e) none of the above

Answer: (c). The natural frequency is $\sqrt{\frac{6}{2}} = \sqrt{3}$.

5. Which of the following differential equations has $e^{-2t}\cos\sqrt{3}t$ and $e^{-2t}\sin\sqrt{3}t$ as a pair of fundamental solutions?

(a) $y'' + 2y' + 3y = 0$ (b) $y'' - 2y' + 3y = 0$ (c) $y'' + 2y' - 3y = 0$

(d) $y'' + 4y' - 7y = 0$ (e) $y'' + 4y' + 7y = 0$

Answer: (e). $-2 \pm \sqrt{3}\,i$ are the roots of the characteristic equation $r^2 + 4r + 7 = 0$.

6. The Wronskian $W(y_1, y_2)(t)$ of a pair of fundamental solutions $y_1(t)$, $y_2(t)$ of the equation

$$t^2y'' - 3y' + (1 + t^2)y = 0$$

has value $W(y_1, y_2)(3) = 3$ at $t = 3$. Find the value $W(y_1, y_2)(6)$.

(a) $3\sqrt{e}$ (b) $\sqrt{3}\,e$ (c) $3e$ (d) $3e^2$ (e) $6e$

Answer: (a). $y'' - \frac{3}{t^2}y' + \frac{1+t^2}{t^2}y = 0$. Thus $W = Ce^{\int \frac{3}{t^2}dt} = Ce^{-3t^{-1}}$.

$$3 = W(3) = Ce^{-1} \Longleftrightarrow C = 3e$$

$$W(6) = 3e \cdot e^{-3/6} = 3e^{1/2} .$$

7. Find the Laplace transform of the piecewise defined function

$$f(t) = \begin{cases} t, & 0 \le t < 3, \\ 2t-3, & t \ge 3 \end{cases}$$

(a) $\dfrac{3e^{-3s}(1-s)}{s^2}$ (b) $\dfrac{3-3s}{s^2}$ (c) $\dfrac{1+2e^{-3s}-3s}{s^2}$

(d) $\dfrac{1+e^{-3s}(2s-3)}{s^2}$ (e) $\dfrac{1+e^{-3s}}{s^2}$

Answer: (e). $f(t) = t + u_3(t)(t-3)$, and hence

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{u_3(t)(t-3)\} = \frac{1}{s^2} + \frac{e^{-3s}}{s^2}$$

8. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the following initial value problem:

$$2y'' - 4y' + 5y = e^{-3t}\cos 2t, \quad y(0) = -1, \quad y'(0) = 2 .$$

(a) $Y(s) = \dfrac{-2s+2}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(b) $Y(s) = \dfrac{-2s+8}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(c) $Y(s) = \dfrac{-2s+4}{2s^2-4s+5} + \dfrac{2}{(2s^2-4s+5)(s^2+6s+13)}$

(d) $Y(s) = \dfrac{-2s}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(e) $Y(s) = \dfrac{-2s+3}{2s^2-4s+5} + \dfrac{2}{(2s^2-4s+5)(s^2+6s+13)}$

Answer: (b). Taking the Laplace transform of the equation,

$$2(s^2Y(s) - s(-1) - 2) - 4(sY(s) - (-1)) + 5Y(s) = \frac{s+3}{(s+3)^2+4}$$

$$(2s^2 - 4s + 5)Y(s) = -2s + 8 + \frac{s+3}{s^2+6s+13}$$

$$Y(s) = \frac{-2s+8}{2s^2-4s+5} + \frac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$$

9. Which of the following convolution integral is a solution of the initial value problem

$$\frac{d^2y}{dt^2} + 4y = 3u_\pi(t)f(t-\pi)\ , \quad y(0)=0,\ \ y'(0)=0$$

where $u_\pi(t)$ is a unit step function.

(a) $y(t) = \int_0^t \frac{3}{2}\sin 2\tau f(t-\tau)d\tau$

(b) $y(t) = \int_0^t \frac{3}{2}\cos 2(t-\tau)f(\tau)d\tau$

(c) $y(t) = \int_0^t \frac{3}{2}\delta(t-\tau+\pi)\sin 2(t-\tau)f(\tau)d\tau$

(d) $y(t) = \int_0^t \frac{3}{2}u_\pi(t-\tau)\sin 2(t-\tau)f(\tau)d\tau$

(e) $y(t) = \int_0^t \frac{3}{2}u_\pi(t)\sin 2t f(t-\tau)d\tau$

Answer: (d). Taking the Laplace transform of the equation,

$$(s^2+4)Y(s) = 3e^{-\pi s}\mathcal{L}\{f(t)\} \Longleftrightarrow Y(s) = \frac{3e^{-\pi s}}{s^2+4}\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{3}{2}u_\pi(t)\sin 2(t-\pi)\right\}\mathcal{L}\{f(t)\}$$

$$Y(s) = \mathcal{L}\left\{\frac{3}{2}u_\pi(t)\sin 2t * f(t)\right\} = \mathcal{L}\left\{\int_0^t \frac{3}{2}u_\pi(t-\tau)\sin 2(t-\tau)f(\tau)d\tau\right\}$$

Part II: Short Questions.

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10. [*8 pts*] The homogeneous equation $(1-t)y''+ty'-y=0$ has two solutions $y_1(t)=t$ and $y_2(t)=e^t$.

(a) Find the Wronskian of y_1 and y_2.

Solution:

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = te^t - e^t = (t-1)e^t$$

Answer: The Wronskian is $W(y_1,y_2)(t) = \underline{(t-1)e^t}$ [*2 pts*]

(b) Find a particular solution of the *nonhomoegeneous equation*

$$(1-t)y''+ty'-y = 2(t-1)^3e^{-t}$$

which has the form $y(t) = tu(t) + e^tv(t)$ for some functions $u(t)$ and $v(t)$. (*You may leave you answers in terms of some integrals.*)

Solution: The equation in standard form: $y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = -2(t-1)^2e^{-t}$.

$$u(t) = -\int \frac{g(t)y_2(t)}{W(t)}dt = \int \frac{2(t-1)^2e^{-t}e^t}{(t-1)e^t}dt = 2\int (t-1)e^{-t}dt$$

$$\left(\ u(t) = -2te^{-t} + C_1\ \right)$$

$$v(t) = \int \frac{g(t)y_1(t)}{W(t)}dt = -\int \frac{2(t-1)^2e^{-t}t}{(t-1)e^t}dt = -2\int t(t-1)e^{-2t}dt$$

$$\left(\ v(t) = t^2e^{-2t} + C_2\ \right)$$

Answer: $y(t) = \underline{2t\int(t-1)e^{-t}dt - 2e^t\int t(t-1)e^{-2t}dt\ (=\ -t^2e^{-t})}$ [*6 pts*]

11. [*8 pts*] Find the solution of the boundary value problem: $y'' + y = 2x, \quad y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 1$.

Solution: The general solution of the homogeneous equation $y'' + y = 0$ is $C_1\cos x + C_2\sin x$. An obvious particular solution of the nonhomogeneous equation $y''+y = 2x$ is $y_p = 2x$. The general solution of the nonhomogeneous equation is thus

$$y = C_1\cos x + C_2\sin x + 2x$$

Putting in the boundary values:

$$2 = y(0) = C_1\cos 0 + C_2\sin 0 + 2(0), \qquad 1 = y\left(\frac{\pi}{2}\right) = C_1\cos\frac{\pi}{2} + C_2\sin\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right)$$

we have $C_1 = 2$, $C_2 = 1 - \pi$.

Answer: The solution is $y(x) = \underline{2\cos x + (1-\pi)\sin x + 2x}$

12. [*8 pts*] After extending the function defined by $f(x) = x^3$, for $-2 \le x \le 2$, to a function of period 4 on the whole real line, the resulting periodic function has a Fourier series expansion, containing only sine terms.

(a) Find this Fourier series. (Hint: use an appropriate integration formula in the formula sheet.)

Solution: The Fourier series is: $\displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$, where

$$b_n = \frac{1}{2}\int_{-2}^{2} x^3\sin\frac{n\pi x}{2}dx \qquad \left(\text{or } \int_0^2 x^3\sin\frac{n\pi x}{2}dx\right)$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi}x^3\cos\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2(3x^2)\sin\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^3(6x)\cos\frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^4(6)\sin\frac{n\pi x}{2}\right]_{-2}^{2}$$

$$= \left(-\frac{16}{n\pi} + \frac{96}{n^3\pi^3}\right)\cos n\pi = (-1)^{n+1}\frac{16}{n\pi}\left(1 - \frac{6}{n^2\pi^2}\right)$$

Answer: The Fourier series is: $\underline{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{16}{n\pi}\left(1-\frac{6}{n^2\pi^2}\right)\sin\frac{n\pi x}{2}}$ [*6 pts*]

(b) At $x = 2$, the Fourier series converges to the value $\underline{\ 0\ }$. [*2 pts*]

13. [*10 pts*] The heat equation problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$u(0,t) = u(2\pi,t) = 0, \quad u(x,0) = 3x, \quad (0 < x < 2\pi)$$

can be solved by considering $u(x,t) = X(x)T(t)$ as a product.

(a) Show that the function $e^{-\lambda^2 t}\sin\lambda x$ satisfies the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ for any constant λ.

Solution: By computing the partial derivatives,

$$\frac{\partial\left(e^{-\lambda^2 t}\sin\lambda x\right)}{\partial x} = \lambda e^{-\lambda^2 t}\cos\lambda x$$
$$\frac{\partial^2\left(e^{-\lambda^2 t}\sin\lambda x\right)}{\partial x^2} = -\lambda^2 e^{-\lambda^2 t}\sin\lambda x$$

we have

$$\frac{\partial\left(e^{-\lambda^2 t}\sin\lambda x\right)}{\partial t} = -\lambda^2 e^{-\lambda^2 t}\sin\lambda x = \frac{\partial^2\left(e^{-\lambda^2 t}\sin\lambda x\right)}{\partial x^2}$$

[*2 pts*]

(b) Show that there is a positive sequence λ_n so that $u_n(x,t) = e^{-\lambda_n^2 t}\sin\lambda_n x$ satisfies the condition $u(0,t) = u(2\pi,t) = 0$.

Solution: $u(x,t) = e^{-\lambda^2 t}\sin\lambda x$ obviously satisfies $u(0,t) = 0$. By the boundary value at $x = 2\pi$, we have $e^{-\lambda^2 t}\sin 2\lambda\pi = 0$. For postive $\lambda > 0$,

$$\sin 2\lambda\pi = 0 \Longleftrightarrow 2\lambda\pi = n\pi$$

where $n = 1, 2, 3, \ldots$. So the positive sequence λ_n is $\frac{n}{2}$, $n = 1, 2, 3, \ldots$.

Answer: $\lambda_n =$ $\underline{\frac{n}{2}}$, where $n = 1, 2, 3, \ldots$. [*3 pts*]

(c) Using superposition of these basic solutions, i.e., $\sum_{n=1}^{\infty} c_n u_n(x,t)$, and the Fourier sine series of $u(x,0) = 3x$, find the solution of the heat equation problem.

Solution: $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t/4}\sin\frac{nx}{2}$ satisfies $u(0,t) = 0 = u(2\pi,t)$. To satisfy also the boundary condition $u(x,0) = 3x$, just pick c_n to be the corresponding coefficients of the Fourier sine series of the function $3x$:

$$c_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} 3x\sin\frac{nx}{2}dx = \frac{1}{2\pi}\left[-\frac{2}{n}(3x)\cos\frac{nx}{2} + \left(\frac{2}{n}\right)^2(3)\sin\frac{nx}{2}\right]_{-2\pi}^{2\pi}$$
$$c_n = \frac{12(-1)^{n+1}}{n}$$

Answer: the solution is $u(x,t) = \underline{\sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n}e^{-n^2 t/4}\sin\frac{nx}{2}}$ [*5 pts*]

Part III: Long Questions

14. [*12 pts*] A damped forced vibration of a mass is described by the following differential equation:

$$y'' + 4y' + 3y = 3\sin t.$$

(a) Find the general solution of the equation.

Solution: $r^2 + 4r + 3 = (r+3)(r+1) = 0 \Longleftrightarrow r = -1,\ -3$. The general solution of the corresponding *homogeneous equation* is $C_1 e^{-t} + C_2 e^{-3t}$.
To find a particular solution of the form $y_p = A\cos t + B\sin t$, put y_p into the equation:

$$(-A\cos t - B\sin t) + 4(-A\sin t + B\cos t) + 3(A\cos t + B\sin t) = 3\sin t$$

$$(2A + 4B)\cos t + (-4A + 2B)\sin t = 3\sin t$$

$$\begin{cases} A + 2B = 0 \\ -4A + 2B = 3 \end{cases} \Longleftrightarrow \begin{array}{l} A = -\frac{3}{5} \\ B = \frac{3}{10} \end{array}$$

The general solution of the given equation is

$$y = C_1 e^{-t} + C_2 e^{-3t} - \frac{3}{5}\cos t + \frac{3}{10}\sin t$$

Answer: The general solution is $y(t) =$ $\underline{C_1 e^{-t} + C_2 e^{-3} - \frac{3}{5}\cos t + \frac{3}{10}\sin t}$ [*8 pts*]

(b) Explain why the motion of the mass is approximately a damped free vibration as $t \to +\infty$. [*2 pts*]

Solution: As $t \to +\infty$, $e^{-t} \to 0$ and $e^{-3t} \to 0$. Thus $y(t)$ is approximately the periodic vibration $-\frac{3}{5}\cos t + \frac{3}{10}\sin t$ for large t.

(c) Find the amplitude of the damped free vibration which approximates the motion of the mass for large t.

Solution: The amplitude of the periodic vibration $-\frac{3}{5}\cos t + \frac{3}{10}\sin t$ is

$$\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{3}{10}\right)^2} = \frac{3}{10}\sqrt{5}$$

[*2 pts*]

15. [*18 pts*] A damped forced vibration of a mass is described by the initial value problem

$$y'' + 3y' + 2y = 4e^{-5t}, \qquad y(0) = 0, \;\; y'(0) = 0 \; .$$

(a) Solve the initial value problem **by the method of Laplace transform**. [*7 pts*]

Solution: Taking Laplace transforms,

$$(s^2 + 3s + 2)Y(s) = \frac{4}{s+5}$$

$$Y(s) = \frac{4}{(s+1)(s+2)(s+5)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+5}$$

where

$$4 = A(s+2)(s+5) + B(s+1)(s+5) + C(s+1)(s+2)$$

Putting in $s = -1$, we have $4 = 4A$; i.e., $A = 1$.
Putting in $s = -2$, we have $4 = -3B$; i.e., $B = -\frac{4}{3}$.
Putting in $s = -5$, we have $4 = 12C$; i.e., $C = \frac{1}{3}$.

$$Y(s) = \frac{1}{s+1} - \frac{4}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s+5} = \mathcal{L}\left\{e^{-t}\right\} - \frac{4}{3}\mathcal{L}\left\{e^{-2t}\right\} + \frac{1}{3}\mathcal{L}\left\{e^{-5t}\right\}$$

$$y(t) = e^{-t} - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-5t}$$

(b) Suppose the external force function $4e^{-5t}$ is cut off at $t = 3$, and an impluse force is applied to the mass at $t = 6$, so that the equation of motion is given by

$$y'' + 3y' + 2y = 4e^{-5t} - 4u_3(t)e^{-5t} + 2\delta(t-6), \qquad y(0) = 0, \;\; y'(0) = 0 \; .$$

(i) Find the solution of the new initial value problem. [*8 pts*]

Solution:

$$y'' + 3y' + 2y = 4e^{-5t} - 4e^{-15}u_3(t)e^{-5(t-3)} + 2\delta(t-6)$$

$$(s+1)(s+2)Y(s) = \frac{4}{s+5} - \frac{4e^{-15}e^{-3s}}{s+5} + 2e^{-6s}$$

$$Y(s) = (1 - e^{-15}e^{-3s})\frac{4}{(s+1)(s+2(s+5)} + 2e^{-6s} \cdot \frac{1}{(s+1)(s+2)}$$

$$Y(s) = (1 - e^{-15}e^{-3s})\left[\frac{1}{s+1} - \frac{4}{3}\cdot\frac{1}{s+2} + \frac{1}{3}\cdot\frac{1}{s+5}\right] + 2e^{-6s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$Y(s) = (1 - e^{-15}e^{-3s})\mathcal{L}\left\{e^{-t} - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-5t}\right\} + 2e^{-6s}\mathcal{L}\left\{e^{-t} - e^{-2t}\right\}$$

$$y(t) = e^{-t} - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-5t} - e^{-15}u_3(t)\left[e^{-(t-3)} - \frac{4}{3}e^{-2(t-3)} + \frac{1}{3}e^{-5(t-3)}\right]$$
$$+2u_6(t)\left[e^{-(t-6)} - e^{-2(t-6)}\right]$$

(ii) Are there jumps in the velocity of the mass at the time $t = 3$ and $t = 6$? If yes, write down the sudden change in velocity respectively. [*3 pts*]

Solution: No velocity jump at $t = 3$; but a jump of $y'(6+) - y'(6-) = 2$ at $t = 6$ caused by the impluse force.