HKUST

MATH150 Introduction to Differential Equations

Final Examination (Version A) Name:____________________

25th May 2006 Student I.D.:____________________

8:30am–10:30am Tutorial Section:____________________

Directions:

- Write your name, ID number, and tutorial section in the space provided above.
- DO NOT open the exam until instructed to do so.
- When instructed to open the exam, check that you have, in addition to this cover page, 8 pages of questions, printed on both sides of each page.
- Turn off all mobile phones and pagers during the examination.
- This is a closed book examination.
- You are advised to try the problems you feel more comfortable with first.
- There are 9 multiple choice questions. **DO NOT guess wildly! If you do not have confidence in your answer leave the answer box blank. Each incorrectly answered question will result in a 0.5 point deduction**.
- For the short and long questions, you must show the working steps of your answers in order to receive all points.
- Unless stated otherwise, you may assume that all units are in SI system.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Question No.	**Points**	**Out of**
Q. 1-9		36
Q. 10-13		34
Q. 14		12
Q. 15		18
Total Points		100

Part I: Each correct answer *in the answer box* for the following 9 multiple choice questions is worth 4 points. DO NOT guess wildly! If you do not have confidence in your answer leave the answer box blank. Each incorrectly answered question will result in a 0.5 point deduction.

Question	1	2	3	4	5	6	7	8	9	**Total**
Answer										

1. Suppose that the population $p(t)$ of certain field mice after t years from now is described by the initial value problem
$$\frac{dp}{dt} = 0.5p - 500 , \qquad p(0) = 800 .$$
Find the time when the population just become extinct; i.e., the time T (in years) when $p(T) = 0$.

   (a) 1.2476 (b) 1.8637 (c) 2.7645 (d) 3.2189 (e) 4.2537

2. For which of the following functions $M(x, y)$ is the equation
$$M(x, y) + (x^3 + 8y - 3x)\frac{dy}{dx} = 0$$
an exact equation?

   (a) $x(y^2 + 1)$ (b) $3y(x^2 - 1)$ (c) $x^2 - 2y^2$ (d) $x^2 + y$ (e) xy

3. By the method of undetermined coefficients, there is a particular solution of the equation
$$y'' - 4y' - 5y = (1 + 2t^2 - t^3)e^{5t}$$
which has the form $u = p(t)e^{5t}$, where $p(t)$ is a polynomial. The degree of $p(t)$ is:

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

4. A periodic external force $F(t) = 9\cos(\omega t)$ is applied to a undamped spring-mass system so that the equation of motion of the mass is

$$2y'' + 6y = 9\cos(\omega t) \ .$$

Which of the following values of ω will cause an unbounded oscillation (resonance) of the mass?

(a) $\frac{2}{3}$ (b) 3 (c) $\sqrt{3}$ (d) $\sqrt{\frac{2}{3}}$ (e) none of the above

5. Which of the following differential equations has $e^{-2t}\cos\sqrt{3}t$ and $e^{-2t}\sin\sqrt{3}t$ as a pair of fundamental solutions?

(a) $y'' + 2y' + 3y = 0$ (b) $y'' - 2y' + 3y = 0$ (c) $y'' + 2y' - 3y = 0$
(d) $y'' + 4y' - 7y = 0$ (e) $y'' + 4y' + 7y = 0$

6. The Wronskian $W(y_1, y_2)(t)$ of a pair of fundamental solutions $y_1(t)$, $y_2(t)$ of the equation

$$t^2y'' - 3y' + (1+t^2)y = 0$$

has value $W(y_1, y_2)(3) = 3$ at $t = 3$. Find the value $W(y_1, y_2)(6)$.

(a) $3\sqrt{e}$ (b) $\sqrt{3}\,e$ (c) $3e$ (d) $3e^2$ (e) $6e$

7. Find the Laplace transform of the piecewise defined function

$$f(t) = \begin{cases} t, & 0 \le t < 3, \\ 2t - 3, & t \ge 3 \end{cases}$$

(a) $\dfrac{3e^{-3s}(1-s)}{s^2}$ (b) $\dfrac{3-3s}{s^2}$ (c) $\dfrac{1+2e^{-3s}-3s}{s^2}$
(d) $\dfrac{1+e^{-3s}(2s-3)}{s^2}$ (e) $\dfrac{1+e^{-3s}}{s^2}$

8. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the following initial value problem:

$$2y'' - 4y' + 5y = e^{-3t}\cos 2t, \quad y(0) = -1, \;\; y'(0) = 2\ .$$

(a) $Y(s) = \dfrac{-2s+2}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(b) $Y(s) = \dfrac{-2s+8}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(c) $Y(s) = \dfrac{-2s+4}{2s^2-4s+5} + \dfrac{2}{(2s^2-4s+5)(s^2+6s+13)}$

(d) $Y(s) = \dfrac{-2s}{2s^2-4s+5} + \dfrac{s+3}{(2s^2-4s+5)(s^2+6s+13)}$

(e) $Y(s) = \dfrac{-2s+3}{2s^2-4s+5} + \dfrac{2}{(2s^2-4s+5)(s^2+6s+13)}$

9. Which of the following convolution integral is a solution of the initial value problem

$$\frac{d^2y}{dt^2} + 4y = 3u_\pi(t)f(t-\pi)\ , \quad y(0) = 0, \;\; y'(0) = 0$$

where $u_\pi(t)$ is a unit step function.

(a) $y(t) = \displaystyle\int_0^t \frac{3}{2}\sin 2\tau f(t-\tau)d\tau$

(b) $y(t) = \displaystyle\int_0^t \frac{3}{2}\cos 2(t-\tau)f(\tau)d\tau$

(c) $y(t) = \displaystyle\int_0^t \frac{3}{2}\delta(t-\tau+\pi)\sin 2(t-\tau)f(\tau)d\tau$

(d) $y(t) = \displaystyle\int_0^t \frac{3}{2}u_\pi(t-\tau)\sin 2(t-\tau)f(\tau)d\tau$

(e) $y(t) = \displaystyle\int_0^t \frac{3}{2}u_\pi(t)\sin 2tf(t-\tau)d\tau$

Part II: Answer each of the following 4 short questions. ***Show all your work for full credit.***

Question	10	11	12	13	**Total**
Points	/8	/8	/8	/10	/34

10. [*8 pts*] The homogeneous equation $(1-t)y''+ty'-y=0$ has two solutions $y_1(t)=t$ and $y_2(t)=e^t$.

(a) Find the Wronskian of y_1 and y_2.

Answer: The Wronskian is $W(y_1,y_2)(t) =$ ____________________________ [*2 pts*]

(b) Find a particular solution of the *nonhomoegeneous equation*

$$(1-t)y''+ty'-y=2(t-1)^3e^{-t}$$

which has the form $y(t)=tu(t)+e^tv(t)$ for some functions $u(t)$ and $v(t)$. (*You may leave you answers in terms of some integrals.*)

Answer: $y(t) =$ ____________________________________ [*6 pts*]

11. [*8 pts*] Find the solution of the boundary value problem: $y'' + y = 2x, \quad y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 1$.

Answer: The solution is $y(x) =$ ______________________

12. [*8 pts*] After extending the function defined by $f(x) = x^3$, for $-2 \leq x \leq 2$, to a function of period 4 on the whole real line, the resulting periodic function has a Fourier series expansion, containing only sine terms.

(a) Find this Fourier series. (Hint: use an appropriate integration formula in the formula sheet.)

Answer: The Fourier series is: ______________________

[*6 pts*]

(b) At $x = 2$, the Fourier series converges to the value __________. [*2 pts*]

13. [*10 pts*] The heat equation problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$u(0,t) = u(2\pi,t) = 0, \quad u(x,0) = 3x, \quad (0 < x < 2\pi)$$

can be solved by considering $u(x,t) = X(x)T(t)$ as a product.

(a) Show that the function $e^{-\lambda^2 t} \sin \lambda x$ satisfies the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for any constant λ.

[*2 pts*]

(b) Show that there is a positive sequence λ_n so that $u_n(x,t) = e^{-\lambda_n^2 t} \sin \lambda_n x$ satisfies the condition $u(0,t) = u(2\pi,t) = 0$.

Answer: $\lambda_n =$ ________________ , where $n = 1, 2, 3, \ldots$. [*3 pts*]

(c) Using superposition of these basic solutions, i.e., $\sum_{n=1}^{\infty} c_n u_n(x,t)$, and the Fourier sine series of $u(x,0) = 3x$, find the solution of the heat equation problem.

Answer: the solution is $u(x,t) =$ ____ [*5 pts*]

Part III: Answer the following two long questions.

14. [*12 pts*] A damped forced vibration of a mass is described by the following differential equation:

$$y'' + 4y' + 3y = 3\sin t.$$

(a) Find the general solution of the equation.

Answer: The general solution is $y(t) =$ ________________________________ [*8 pts*]

(b) Explain why the motion of the mass is approximately a damped free vibration as $t \to +\infty$. [*2 pts*]

(c) Find the amplitude of the damped free vibration which approximates the motion of the mass for large t. [*2 pts*]

15. [*18 pts*] A damped forced vibration of a mass is described by the initial value problem

$$y'' + 3y' + 2y = 4e^{-5t}, \qquad y(0) = 0, \;\; y'(0) = 0 \;.$$

(a) Solve the initial value problem **by the method of Laplace transform**. [*7 pts*]

(b) Suppose the external force function $4e^{-5t}$ is cut off at $t = 3$, and an impluse force is applied to the mass at $t = 6$, so that the equation of motion is given by

$$y'' + 3y' + 2y = 4e^{-5t} - 4u_3(t)e^{-5t} + 2\delta(t-6), \qquad y(0) = 0, \;\; y'(0) = 0 \;.$$

(i) Find the solution of the new initial value problem. [*8 pts*]

(ii) Are there jumps in the velocity of the mass at the time $t = 3$ and $t = 6$? If yes, write down the sudden change in velocity respectively. [*3 pts*]

Math150-L2 Formula Sheet

Integration Formulas

For any polynomial $p(x)$,

$$\int p(x)e^{ax}\,dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \cdots \quad \text{(signs alternate: } + - + - \cdots.\text{)}$$

$$\int p(x)\sin ax\,dx = -\frac{1}{a}p(x)\cos ax + \frac{1}{a^2}p'(x)\sin ax + \frac{1}{a^3}p''(x)\cos ax - \cdots \quad \text{(signs alternate in : } - + + - - + + - - \cdots.\text{)}$$

$$\int p(x)\cos ax\,dx = \frac{1}{a}p(x)\sin ax + \frac{1}{a^2}p'(x)\cos ax - \frac{1}{a^3}p''(x)\sin ax - \cdots \quad \text{(signs alternate in pairs: } + + - - + + - - \cdots.\text{)}$$

Laplace Transform Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}, \quad s > 0$
2	e^{at}	$\frac{1}{s-a}, \quad s > a$
3	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > 0$
12	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
14	$e^{ct}\,f(t)$	$F(s-c)$
15	$f(ct)$	$\frac{1}{c}\,F\left(\frac{s}{c}\right), \quad c > 0$
16	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17	$\delta(t-c)$	e^{-cs}
18	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19	$(-t)^n f(t)$	$F^{(n)}(s)$