Classical Analysis

Edmund Y. M. Chiang

December 3, 2009

Abstract

This course assumes students to have mastered the knowledge of complex function theory in which the classical analysis is based. Either the reference book by Brown and Churchill [6] or Bak and Newman [4] can provide such a background knowledge. In the all-time classic "A Course of Modern Analysis" written by Whittaker and Watson [23] in 1902, the authors divded the content of their book into part I "The processes of analysis" and part II "The transcendental functions". The main theme of this course is to study some fundamentals of these classical transcendental functions which are used extensively in number theory, physics, engineering and other pure and applied areas.

Contents

Chapter 1

Separation of Variables of Helmholtz Equations

The **Helmholtz equation** named after the German physicist **Hermann von Helmholtz** refers to second order (elliptic) partial differential equations of the form:

$$(\Delta^2 + k^2)\Phi = 0, \tag{1.1}$$

where k is a constant. If $k = 0$, then it reduces to the **Laplace equations**.

In this discussion, we shall restrict ourselves in the Euclidean space $\mathbf{R}^3$. One of the most powerful theories developed in solving linear PDEs is the **the method of separation of variables**. For example, the **wave equation**

$$\Big(\Delta^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\Psi(\mathbf{r},\, t) = 0, \tag{1.2}$$

can be solved by assuming $\Psi(\mathbf{r},\, t) = \Phi(\mathbf{r}) \cdot T(t)$ where $T(t) = e^{i\omega t}$. This yields

$$\Big(\Delta^2 - \frac{\omega^2}{c^2}\Big)\Phi(\mathbf{r}) = 0, \tag{1.3}$$

which is a Helmholtz equation. The questions now is under what $3-$dimensional coordinate system $(u_1,\, u_2,\, u_3)$ do we have a solution that is in the separation of variables form

$$\Phi(\mathbf{r}) = \Phi_1(u_1) \cdot \Phi_2(u_2) \cdot \Phi_3(u_3) \quad ? \tag{1.4}$$

Eisenhart, L. P. (”Separable Systems of Stäckel.” Ann. Math. 35, 284-305, 1934) determines via a certain **Stäckel determinant** is fulfilled (see e.g. Morse, P. M. and Feshbach, H. “Methods of Theoretical Physics, Part I”. New York: McGraw-Hill, pp. 125–126, 271, and 509–510, 1953).

Theorem 1.1 (Eisenhart 1934)**.** *There are a total of* ***eleven*** *curvilinear coordinate systems in which the Helmholtz equation separates.*

Each of the curvilinear coordinate is characterized by **quadrics**. That is, surfaces defined by

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + lz + J = 0. \tag{1.5}$$

One can visit http://en.wikipedia.org/wiki/Quadric for some of the quadric surfaces. Curvilinear coordinate systems are formed by putting relevant orthogonal quadric surfaces. Wikipedia contains quite a few of these pictures. We list the eleven coordinate systems here:

Co-ordinate Systems

Name of system	Transformation formulae	Degenerate Surfaces
(1) Cartesian	$x = x, \quad y = y, \quad z = z$	
(2) Cylindrical	$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$ $\rho \geq 0, \qquad -\pi < \phi \leq \pi$	
(3) Spherical polar	$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,$ $z = r\cos\theta$ $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad -\pi \leq \phi \leq \pi$	
(4) Parabolic cylinder	$x = u^2 - v^2, \quad y = 2uv,$ $z = z$ $u \geq 0, \qquad -\infty < v < +\infty$	Half-plane
(5) Elliptic cylinder	$x = f\cosh\xi\cos\eta, y = f\sinh\xi\sin\eta,$ $z = z$ $\xi \geq 0, -\infty < \eta < +\infty$	Infinite strip; Plane with straight aperture
(6) Rotation paraboloidal	$x = 2uv\cos\phi, \quad 2uv\sin\phi,$ $z = u^2 - v^2$ $u, v \geq 0, \qquad -\pi < \phi < \pi$	Half-line
(7) Prolate spheroidal	$x = \ell\sinh u\sin v\cos\phi,$ $y = \ell\sinh u\sin v\sin\phi,$ $z = \ell\cosh u\cos v,$ $u \geq 0, \ 0 \leq v \leq \pi, \ -\pi < \phi \leq \pi$	Finite line ; segment Two half-lines
(8) Oblate spheroidal	$x = \ell\cosh u\sin v\cos\phi,$ $y = \ell\cosh u\sin v\sin\phi,$ $z = \ell\sinh u\cos v,$ $u \geq 0, \ 0 \leq v \leq \pi, \ -\pi < \phi \leq \pi$	Circular plate (disc); Plane with circular aperture
(9) Paraboloidal	$x = \frac{1}{2}\ell(\cosh 2\alpha + 2\cos 2\beta - \cosh 2\gamma,$ $y = 2\ell\cosh\alpha\cos\beta\sinh\gamma,$ $z = 2\ell\sinh\alpha\sin\beta\cosh\gamma,$ $\alpha, \ \gamma \geq 0, \ -\pi < \beta \leq \pi$	Parabolic plate; Plane with parabolic aperture
(10) Elliptic conal	$x = kr\mathrm{sn}\,\alpha\mathrm{sn}\,\beta;$ $y = (ik/k')r\mathrm{cn}\,\alpha\mathrm{cn}\,\beta,$ $z = (1/k')r\mathrm{dn}\,\alpha\mathrm{dn}\,\beta;$ $r \geq 0, \quad -2K < \alpha \leq 2K,$ $\beta = K + iu, \quad 0 \leq u \leq 2K'$	Plane sector; Including quarter plane
(11) Ellipsoidal	$x = k^2\ell\mathrm{sn}\,\alpha\mathrm{sn}\,\beta\mathrm{sn}\,\gamma,$ $y = (-K^2\ell/k')\mathrm{cn}\,\alpha\mathrm{cn}\,\beta\mathrm{cn}\,\gamma,$ $z = (i\ell/k')\mathrm{dn}\,\alpha\mathrm{dn}\,\beta\mathrm{cn}\,\gamma;$ α, β as in (10), $\gamma = iK' + w, 0 < w \leq K$	Elliptic plate; Plane with elliptic aperture

Laplace & Helmholtz

Coordinate system	Laplace Equation	Helmholtz equation
(1) Cartesian	(Trivial)	(Trivial)
(2) Cylinderical	(Trivial)	Bessel
(3) Spherical polar	Associated Legender	Associated Legender
(4) Parabolic cylinder	(Trivial)	Weber
(5) Elliptic cylinder	(Trivial)	Mathieu
(6) Rotation-paraboloidal	Bessel	Confluent hypergeometric
(7) Prolate spheroidal	Associated Legender	Spheroidal wave
(8) Prolate spheroidal	Associated Legender	Spheroidal wave
(9) Paraboloidal	Mathieu	Whittaker-Hill
(10) Elliptic conal	Lame	Spherical Bessel, Lame
(11) Ellipsoidal	Lame	Ellipsoidal

1. **Associated Legender**:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \Big\{n(n+1) - \frac{m^2}{(1-x^2)}\Big\}y = 0 \tag{1.6}$$

2. **Bessel**:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{1.7}$$

3. **Spherical Bessel**:

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - n(n+1))y = 0 \tag{1.8}$$

4. **Weber**:

$$\frac{d^2y}{dx^2} + (\lambda - \frac{1}{4}x^2)y = 0 \tag{1.9}$$

5. **Confluent hypergeometric**:

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0 \tag{1.10}$$

6. **Mathieu**:

$$\frac{d^2y}{dx^2} + (\lambda - 2q\cos 2x)y = 0 \tag{1.11}$$

7. **Spheroidal wave**:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \Big\{\lambda - \frac{\mu^2}{(1-x^2)} + \gamma^2(1-x^2)\Big\}y = 0 \tag{1.12}$$

8. **Lame**:

$$\frac{d^2y}{dx^2} + (h - n(n+1)k^2\mathrm{sn}^2 x)y = 0 \tag{1.13}$$

9. **Whittaker-Hill**:

$$\frac{d^2y}{dx^2} + (a + b\cos 2x + c\cos 4x)y = 0 \tag{1.14}$$

10. **Ellipsoidal wave**:

$$\frac{d^2y}{dx^2} + (a + bk^2\mathrm{sn}^2 x + qk^4\mathrm{sn}^4 x)y = 0 \tag{1.15}$$

Remark 1.0.1. The spheroidal wave and the Whittaker-Hill do not belong to the hypergeometric equation regime, but to the **Heun equation** regime (which has four regular singular points).

1.1 What is not known?

It is generally regarded that the Bessel functions, Weber functions, Legendre functions are better understood, but the remaining equations/functions are not so well understood.

1. Bessel functions. OK! Still some unknowns.

2. Confluent hypergeometric equations/functions. NOT OK.

3. Spheroidal wave, Mathieu, Lame, Whittaker-Hill, Ellipsoidal wave are poorly understood. Some of them are relatd to the **Heun equation** which has **four regular singular points**. Its research has barely started despite the fact that it has been around since 1910.

4. Mathematicans are separating variables of Laplace/Helmholtz equations, but in more complicated setting (such as in Riemannian spaces, etc)

Chapter 2

Infinite Products

We give a brief introduction to infinite products sufficient for our later applications. The basic theory for infinite products parallels that of infinite series.

2.1 Definitions

Let $\{a_k\}$ be an infinite sequence of complex numbers. Consider

$$P_n = \prod_{k=1}^{n}(1+a_k) = (1+a_1)(1+a_2)\cdots(1+a_n).$$

If $\lim_{n\to\infty} P_n$ exists and is equal to a non-zero P, then we say the infinite product $\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_k)$ exists and its limit is given by

$$\lim_{n\to\infty}\prod_{n=1}^{\infty}(1+a_n) = \lim_{n\to\infty} P_n = P.$$

We may also say that the sequence $\{P_n\}$ *converges to* P. If either finitely or an infinitely many of the factors are equal to zero, and the sequence with those zero-factors deleted converges to a non-zero limit, then we say the infinite product *converges to zero*. An infinite product is said to be *divergent* if it is not convergent. We say an infinite product $\prod_{n=0}^{\infty}(1+a_n)$ *diverges to zero* if it is not due to the failure of the $\lim_{n\to\infty} P_n$ with respect to the non-zero factors. For example, the infinite product

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 1\cdot\frac{1}{2}\cdots\frac{1}{n}$$

diverges to 0. In general, unless otherwise stated, we shall *not* consider infinite products with zero limit in this course.

Theorem 2.1. *If* $\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_k)$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. If the product converges to $P \neq 0$, then

$$1 = \frac{P}{P} = \frac{\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_k)}{\lim_{n\to\infty}\prod_{k=1}^{n-1}(1+a_k)} = \lim_{n\to\infty}(1+a_n)$$

thus showing that $\lim_{n\to\infty} a_n = 0$ as required. If the infinite product converges to the zero limit, then the same argument with these zero-factors deleted still work. $\square$

Example 2.1.1. *Determine the convergence of* $(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots$.
We define

$$\begin{aligned} p_n &= \begin{cases} (1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots(1-\frac{1}{n}), & \textit{if } n \textit{ even} \\ (1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n}), & \textit{if } n \textit{ odd} \end{cases} \\ &= \begin{cases} (1+1)(\frac{1}{2})(\frac{4}{3})(\frac{3}{4})\cdots(\frac{n}{n-1})(\frac{n-1}{n}) = 1, & \textit{if } n \textit{ is even} \\ (1+1)(\frac{1}{2})(\frac{4}{3})(\frac{3}{4})\cdots(\frac{2}{n-1})(\frac{n-1}{n-2})(\frac{n+1}{n}) = 1+\frac{1}{n}, & \textit{if } n \textit{ is odd} \end{cases} \end{aligned}$$

Hence $p_n \to 1$ *as* $n \to \infty$. *We conclude that*

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots = 1. \tag{2.1}$$

We also note that $\prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right)$, *and* $a_n = \frac{(-1)^n}{n} \to 0$ *as* $n \to \infty$.

2.2 Cauchy criterion for infinite products

Theorem 2.2. *The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *is convergent (to a non-zero limit) if and only if given* $\varepsilon > 0$, *there is an integer* $N > 0$ *such that*

$$\left|(1+a_{n+1})\cdots(1+a_m) - 1\right| < \varepsilon \tag{2.2}$$

for all $m > n > N$.

Proof. Let $\prod_{n=1}^{\infty}(1+a_n) = p \neq 0$ and let p_n be the nth partial product of $\prod_{n=1}^{\infty}(1+a_n)$. Then $\{p_n\}$ is a Cauchy sequence in $\mathbb{C}$. Given $\varepsilon > 0$, there is an integer $N > 0$ such that $|p_n| > |p|/2 > 0$ and

$$\left|p_n - p_m\right| < \varepsilon\,\frac{|p|}{2} \tag{2.3}$$

for all $m > n \geq N$. Thus

$$\begin{aligned} \left|(1+a_{n+1})\cdots(1+a_m) - 1\right| &= |p_n|\left|\frac{p_m}{p_n} - 1\right|\frac{1}{|p_n|} \\ &= \left|p_m - p_n\right|\frac{1}{|p_n|} \\ &< \varepsilon\frac{|p|}{2}\,\frac{2}{|p|} = \varepsilon \end{aligned}$$

for all $m > n \geq N$, as required. The case becomes trivial if $p = 0$.

Conversely, suppose the inequality (2.8) holds, then given $0 < \varepsilon < 1$, there is N such that

$$\left|\frac{p_m}{p_n} - 1\right| < \varepsilon \tag{2.4}$$

for all $m > n \geq N$. Let $p'_m = \frac{p_m}{p_N}$ for all $m > n \geq N$. So for all $m > N\,(= n)$, we have

$$1 - \varepsilon < |p'_m| < 1 + \varepsilon < 2.$$

Notice that (2.4) is equivalent to the inequality

$$\left|\frac{p'_m}{p'_n} - 1\right| < \varepsilon$$

for all $m > n \geq N$. That is,

$$\left|p'_m - p'_n\right| < \varepsilon |p'_n| < 2\varepsilon,$$

for all $m > n \geq N$. This proves that $\{p'_m\}$ is a Cauchy sequence. Hence $\{p_m\}$ is a Cauchy sequence and so convergent. □

2.3 Absolute and uniform convergence

Proposition 2.3.1. *Suppose all a_n are real and non-negative. Then $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Suppose $\prod_{n=1}^{\infty}(1 + a_n)$ converges. Then the inequality

$$a_1 + \cdots a_n \leq (1 + a_1)\cdots(1 + a_n) \tag{2.5}$$

immediately implies that $\sum_{n=1}^{\infty} a_n$ converges. Conversely, since $1 + a < e^a$ for $a > 0$, so

$$(1 + a_1)\cdots(1 + a_n) < \exp\big(a_1 + \cdots + a_n\big). \tag{2.6}$$

This implies that $\prod_{n=1}^{\infty}(1 + a_n)$ converges. □

Exercise 2.3. *Suppose $a_n \leq 0$ for all n. Write $a_n = -b_n$, and consider $\prod_{k=1}^{\infty}(1 - b_k)$. Show that*

1. *If $b_n \geq 0$, $b_n \neq 1$ for all n and $\sum_{k=1}^{\infty} b_k$, then $\prod_{k=1}^{\infty}(1 - b_k)$ is convergent.*

2. *If $0 \leq b_n < 1$ for all n, and $\sum_{k=1}^{\infty} b_k$ diverges, then $\prod_{k=1}^{\infty}(1-b_k)$ diverges to zero. Hence show that if $0 \leq b_n < 1$ for all n, then $\sum_{k=1}^{\infty} b_k$ and $\prod_{k=1}^{\infty}(1-b_k)$ converge or diverge together.*

We emphasis again that we shall not consider infinite products with any zero-factor in this section. That is, we shall not consider those infinite products that would *converge* to zero here.

Definition 2.3.1. *Let $\{a_n\}$ be an arbitrary sequence of complex numbers (including real numbers) not equal to -1. Then we say the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.*

We easily see from the Proposition 2.3.1 that

Theorem 2.4. *The necessary and sufficient condition for the infinite product $\prod_{n=1}^{\infty}(1+ \ a_n)$ to converge absolutely is the absolute convergence of $\sum_{k=1}^{\infty} a_k$.*

We recall that the infinite product although convergent

$$(1+1)\Big(1-\frac{1}{2}\Big)\Big(1+\frac{1}{3}\Big)\cdots \tag{2.7}$$

is not absolutely convergent.

Theorem 2.5. *If $\prod_{n=1}^{\infty}(1+|a_n|)$ is convergent, then $\prod_{n=1}^{\infty}(1+a_n)$ is convergent.*

Proof. We observe that

$$\big|(1+a_{n+1})\cdots(1+a_m)-1\big| < (1+|a_{n+1}|)\cdots(1+|a_m|)-1, \tag{2.8}$$

holds trivially. For one can expand the left side in the form:

$$\sum a_r + \sum a_r a_s + \sum a_r a_s a_t + \cdots \\ + a_{n+1}a_{n+2}\cdots a_m$$

which is dominated by

$$\sum |a_r| + \sum |a_r a_s| + \sum |a_r a_s a_t| + \cdots \\ + |a_{n+1}a_{n+2}\cdots a_m|.$$

That is the original inequality that we want to show above. Thus the result follows from Theorem 2.2.

Alternatively, let

$$p_n = \prod_{k=1}^{n}(1+a_k), \qquad P_n = \prod_{k=1}^{n}(1+|a_k|). \tag{2.9}$$

Then

$$p_n - p_{n-1} = (1+a_1)\cdots(1+a_{n-1})\,a_n, \tag{2.10}$$

and

$$P_n - P_{n-1} = (1+|a_1|)\cdots(1+|a_{n-1}|)\,|a_n|. \tag{2.11}$$

Hence

$$\big|p_n - p_{n-1}\big| \le P_n - P_{n-1}. \tag{2.12}$$

If $\prod_{k=1}^{\infty}(1+|a_k|)$ is convergent, then P_n tends to a limit, and so $\sum(P_k - P_{k-1})$ is convergent. Hence, by the comparison theorem, $\sum(p_k - p_{k-1})$, and hence p_n also converge.
We can even conclude that the limit cannot be zero. For, since $\sum|a_k|$ is convergent, so $1+a_n \to 1$. Therefore, the series

$$\sum\left|\frac{a_k}{1+a_k}\right|$$

is also convergent. Hence, by what we have just shown above that the infinite product

$$\prod_{k=1}^{n}\Big(1-\frac{a_k}{1+a_k}\Big)$$

is convergent. But this product is equal to $1/p_n$. Hence the $\lim p_n \neq 0$. □

It was shown in Proposition 2.3.1 that if all a_n are real and non-negative. Then $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty}a_n$ converges. This no longer holds when some of the a_n are negative.

Exercise 2.6. *Determine the convergence of the infinite products*

1. $$(1+1)\Big(1-\frac{1}{2}\Big)\Big(1+\frac{1}{3}\Big)\cdots$$

2. $$\Big(1-\frac{1}{\sqrt{2}}\Big)\Big(1+\frac{1}{\sqrt{3}}\Big)\Big(1-\frac{1}{\sqrt{4}}\Big)\cdots.$$

3. $\prod_{k=1}^{\infty}(1+a_k)$ *where* $a_{2k-1}=\frac{-1}{\sqrt{k}}$ *and* $a_{2k}=\frac{1}{\sqrt{k}}+\frac{1}{k}$ *for* $k=1,\ 2\ 3,\ \cdots$.

It turns out that the first problem is known to converge and $\sum_{n=1}^{\infty}a_n$ converges, while even though the second example has $\sum_{n=1}^{\infty}a_n$ convergent but the infinite product actually diverges.

2.4 Associated series of logarithms

We recall that the principal logarithm of a complex number A is defined to be

$$\log A = \log|A| + i\arg A, \qquad -\pi \le \arg A < \pi.$$

Theorem 2.7. *Suppose non of the* $a_n = -1$. *If* $\sum_{n=1}^{\infty} \log(1+a_n)$ *converges, then* $\prod_{n=0}^{\infty}(1+a_n)$ *converges. If* $\prod_{n=0}^{\infty}(1+a_n)$ *converges, then* $\sum_{n=1}^{\infty} \log(1+a_n)$ *converges to a branch of* $\log\prod_{n=0}^{\infty}(1+a_n)$.

Proof. Let the partial product and partial sum be denoted, respectively, by

$$P_n = \prod_{k=1}^{n}(1+a_k), \qquad S_n = \sum_{k=1}^{n}\log(1+a_k). \tag{2.13}$$

Here we assume principal logarithm in the S_n. Then we deduce that $\exp S_n = P_n$. We also know that $\lim_{n\to\infty}\exp S_n = \exp\big(\lim_{n\to\infty} S_n\big)$. Therefore if S_n converges, then P_n also converges. Conversely, if $\prod_{n=0}^{\infty}(1+a_n) = p$ has a limit. Let $p_n = \prod_{k=1}^{n}(1+a_k)$. Then we have $\log p_n/p \to 0$ as $n\to\infty$. We then write

$$\log\frac{p_n}{p} = S_n - \log p + 2\pi i h_n, \tag{2.14}$$

where h_n is an integer for each n. Then

$$\begin{aligned}\log p_{n+1}/p - \log p_n/p &= S_{n+1} - S_n + 2\pi i(h_{n+1}-h_n)\\ &= \log(1+a_{n+1}) + 2\pi i(h_{n+1}-h_n)\end{aligned}$$

It is clear that the left-side of the above equation tends to zero as $n\to\infty$. We know that the term $\log(1+a_{n+1})$ also tends to zero as $n\to\infty$. Hence $h_{n+1}-h_n = 0$ for all n sufficiently large (since they are integers). Let $h_{n+1} = h_n = h$ for all these large n. Then

$$S_n - \log p + 2\pi i h = \log p_n/p \to 0, \tag{2.15}$$

as $n\to\infty$. Hence $S_n \to S := \log p - 2\pi i h$. □

Theorem 2.8. $\prod_{n=0}^{\infty}(1+a_n)$ *converges absolutely if and only if* $\sum_{n=1}^{\infty}\log(1+a_n)$ *converges absolutely.*

Proof. The result follows from Theorem 2.4 and the limit

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1. \tag{2.16}$$

If suffices to show that the two series $\sum_{n=1}^{\infty} |\log(1+a_n)|$ and $\sum_{n=1}^{\infty} |a_n|$ converge and diverge together. The details is left as exercise. $\square$

Exercise 2.9. 1. *Show that if $\sum a_n$ and $\sum |a_n|^2$ are convergent, then $\prod(1+a_n)$ is convergent.*

2. *If a_n are real, and $\sum a_n$ is convergent, the infinite product $\prod(1+a_n)$ converges, diverges to zero according to the convergence or divergence of $\sum a_n^2$ respectively.*

2.5 Uniform convergence

Let $\{p_n(z)\}$ be a sequence of complex-valued functions, each defined in a closed bounded set (*region*) D. We say that the sequence $\{p_n(z)\}$ converges to a limit function $p(z)$, or in short notation, $\lim_{n\to\infty} p_n(z) = p(z)$ if for every z in D and given $\varepsilon > 0$, there is an integer N such that $\big|p_n(\xi) - p(\xi)\big| < \varepsilon$ whenever $n \geq N$. Here the integer function $N = N(\varepsilon, \xi)$ depends on both the ε and the point ξ. We say the convergence is *uniform* if N is independent of ξ. That is, the sequence $\{p_n(z)\}$ *converges uniformly* in the region D if there is a function $p(z)$, such that given $\varepsilon > 0$, there is an integer N and for each $z \in D$,

$$\big|p_n(z) - p(z)\big| < \varepsilon,$$

whenever $n \geq N$.

We recall the

Theorem 2.10. *A necessary and sufficient condition for the uniform convergence of the sequence of functions $p_1(z)$, $p_2(z)$, $\cdots$ in a closed bounded region D is that, corresponding to any positive number ε, there exists an integer $N(\varepsilon)$, depending on ε alone, such that the inequaltiy*

$$\big|p_m(z) - p_n(z)\big| < \varepsilon, \tag{2.17}$$

holds at each point z of D whenever $m > n \geq N$.

The proof is left as an exercise. We note that we have not assumed any additional property on the sequence $\{p_n(z)\}$. If we assume that each function in the sequence is continuous in D, then one can show that the limiting function $p(z)$ of the unformly convergent sequence is also continuous in D. Moreover, if each function in the sequence $\{p_n(z)\}$ is analytic in D, then the limiting function $p(z)$ is analytic in D, and that $\lim p_n'(z) = p'(z)$ for each $z \in D$. This results follows from $\int_C \lim p_n\, dz = \lim \int_C p_n\, dz$ and Morera's theorem when the convergence of $\{p_n(z)\}$ is uniform under any piecewise contour C lying entirely within D. We refer the details to a complex analysis reference.

We can easily extend the above discussion to series of functions $p_n(z) = u_1(z)+u_2(z)+\cdots+u_n(z)$, $n = 1, 2, 3, \cdots$ where each $u_k(z)$ is a function in D. If $p_n(z)$ converges uniformly in D, then the series is said to be *uniformly convergent* in D. It is clear that if each $p_k(z)$ is continuous, and if the series converges uniformly in D, then the $\lim p_n(z)$ is continuous in D. If, in addition, that each $u_k(z)$ is analytic in D, then $\lim p_n(z) = p(z)$ is analytic in D, and that $p'(z) = \sum p_k'(z)$. This also implies that $p^{(n)}(z) = \sum p_k^{(n)}(z)$. Finally, let us recall the Weierstrass $M-$test which gives a sufficient condition for uniform convergence of series.

Theorem 2.11. *The infinite series $\sum p_k(z)$ converges uniformly and absolutely in a closed bounded region D if each $p_k(z)$ satisfies $|p_k(z)| \le M_k$, where M_k is independent of z and $\sum M_k$ is convergent.*

Proof. The series evidently converges absolutely in D. To show that it converges uniformly in D, we notice that, if $S_k(z)$ is the kth-partial sum of the series, then

$$\big|S_{m+r}(z) - S_m(z)\big| = \Big|\sum_{m+1}^{m+r} p_k(z)\Big| \le \sum_{m+1}^{m+r} \big|p_k(z)\big| < \sum_{k=m+1}^{\infty} M_k$$

for every r. Since $\sum M_k$ is convergent, hence given any positive number ε, we can find m so that $\sum_{m+1}^{\infty} M_k < \varepsilon$. It follows that

$$\big|S_{m+r}(z) - S_m(z)\big| < \varepsilon$$

for any z in D and any positive integer r. The required result follows from that of Theorem 2.10. □

Theorem 2.12. *In order for the infinite product $\prod_{k=1}^{\infty}\big(1 + p_k(z)\big)$ to converge uniformly and absolutely in a closed bounded region D it is sufficient to assume that each function $p_k(z)$ satisfies an inequality $|p_k(z)| \le M_k$, where M_k is independent of z and $\sum M_k$ is convergent.*

This is a Weierstrass $M-$test for the infinite product of functions.

Proof. The absolute convergence of the infinite product is clear from the assumption that $\sum |p_k(z)| \le \sum M_k < \infty$ and Theorem 2.4.

It follows from Proposition 2.3.1 and the assumption that $\sum M_k < \infty$ implies the convergence of the product sequence $P_n := \prod_{k=1}^{n} \big(1 + M_k\big)$ as $n \to \infty$. One can then obtain a similar inequalty as (2.8) in the form

$$
\begin{aligned}
\big|p_n(z) - p_m(z)\big| &= |p_n(z)|\, \Big|\prod_{n+1}^{m} \{1 + p_k(z)\} - 1\Big| \\
&\le |M_n|\, \Big|\prod_{n+1}^{m} \{1 + M_k\} - 1\Big| \\
&= |P_n - P_m|.
\end{aligned}
$$

Since $\{P_k\}$ converges, so given $\varepsilon > 0$ one can find N such that

$$\big|p_n(z) - p_m(z)\big| \le |P_n - P_m| < \varepsilon$$

whenever $m > n \ge N$. Since the N depends only on ε and not on a particular z in D, so the convergence is uniform. This completes the proof. □

One easily obtain

Theorem 2.13. *If the infinite product* $\prod_{k=1}^{\infty}\{1+p_k(z)\}$ *converges uniformly to* $f(z)$ *in every closed region lying entirely in a closed contour* C*, and if each function* $p_k(z)$ *is analytic, then* $f(z)$ *is also analytic within* C*.*

Proof. The proof essentially follows from the discussion on last page about $\int_C \lim p_n\, dz = \lim \int_C p_n\, dz$ and Morera's theorem when the convergence of $\{p_n(z)\}$ is uniform under any piecewise contour C lying entirely within D □

Exercise 2.14. *Discuss the convergence of the following infinite products*

1. $\Big(1 - \dfrac{z^2}{1^2}\Big)\Big(1 - \dfrac{z^2}{2^2}\Big)\Big(1 - \dfrac{z^2}{3^2}\Big)\cdots;$
2. $\Big(1 - \dfrac{z}{1}\Big)\Big(1 + \dfrac{z}{1}\Big)\Big(1 - \dfrac{z}{2}\Big)\Big(1 + \dfrac{z}{2}\Big)\cdots;$
3. $\Big\{\Big(1 - \dfrac{z}{1}\Big)e^{z}\Big\}\Big\{\Big(1 + \dfrac{z}{1}\Big)e^{-z}\Big\}\Big\{\Big(1 - \dfrac{z}{2}\Big)e^{z/2}\Big\}\Big\{\Big(1 + \dfrac{z}{2}\Big)e^{-z/2}\Big\}\cdots.$

We now discuss the above examples in any closed bounded domain D which does not contain any of the zeros $(\pm 1, \pm 2, \pm 3, \cdots)$. Since D is bounded so there exists a $R > 0$ such that the D lies entirely inside $|z| \le R$.

1. Let n denote an arbitrary zero of the first function above. Then $\left|z^2/n^2\right| \le R^2/n^2$. Clearly the series $\sum R^2/n^2$ is convergent. Thus, the $M-$test implies that the infinite product $\prod \left(1 - z^2/k^2\right)$ converges absolutely and uniformly in D. In fact, it is well-known that the product equals to $\sin \pi z/(\pi z)$, and so is analytic.

2. In the second example above the $M-$test does not apply for the series $2R \sum 1/k$ does not converge. So the infinite product in (2) above does not converge uniformly. Let

$$F_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right), \qquad f_n(z) = nth\text{-partial product of (2)}.$$

Then

$$f_{2n}(z) = F_n(z), \qquad f_{2n+1}(z) = \left(1 - \frac{z}{n+1}\right) F_n(z).$$

This shows that both $f_1(z),\ f_3(z),\ f_5(z),\ \cdots$ and $f_2(z),\ f_4(z),\ f_6(z),\ \cdots$ converge uniformly to $F(z) = \lim F_n(z)$. So the infinite product (2) converges to $F(z)$ uniformly.

3. In the third example, we write

$$\begin{aligned}
\left(1 - \frac{z}{n}\right) e^{z/n} &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k-1)!}\right) \frac{z^k}{n^k} \\
&= \sum_{k=0}^{\infty} \left(\frac{1-k}{k!}\right) \frac{z^k}{n^k} \\
&= 1 - \sum_{k=2}^{\infty} \left(\frac{k-1}{k!}\right) \frac{z^k}{n^k} \\
&:= 1 - u_n(z).
\end{aligned}$$

Hence

$$|u_n(z)| \le \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \frac{R^k}{n^k} = \left(\frac{R}{n}\right)^2 e^{R/n} \le e \left(\frac{R}{n}\right)^2,$$

when $R < n$. A similar formulation exists for the factor

$$\left(1 + \frac{z}{n}\right) e^{z/n} = 1 - v_n(z),$$

and hence $|v_n(z)| \leq e(R/n)^2$ when $z \in D$ and $n > R$. The $M-$test shows that the product (3) converges uniformly and absolutely in D. Notice that since the partial product of order $2n$ equals to $F_n(z)$ in (1), so the third product also converges to $F(z) = \sin \pi z/(\pi z)$.

Chapter 3

Gamma Function

In a letter from the German mathematician Christian Goldbach (1690–1746) to the Swiss mathematician Leonhard Euler (1707–1783) in the year 1729, Goldbach posed an interpolation problem of finding a function $f(x)$ that equals to $n!$ when $x = n$. Euler gave a solution in a paper to this problem in October 1729, and later another paper that addressed a related integration problem in January 1730. Both problems are related to what we now call (Euler) Gamma function. These problems defeated prominent mathematicians such as Daneil Bernoulli and James Stirling at that time, while Euler was only at the age of twenty two. The gamma function turns out to be obiquity in mathematics. Interested reader could consult P. Davis's article [??].

3.1 Infinite product definition

Euler observed that

$$\Big[\Big(\frac{2}{1}\Big)^n\frac{1}{1+n}\Big]\Big[\Big(\frac{3}{2}\Big)^n\frac{2}{2+n}\Big]\Big[\Big(\frac{4}{3}\Big)^n\frac{3}{3+n}\Big]\quad\cdots\quad = n!$$

holds. We notice that the first n terms of the above "infinite product" can be written in the form

$$\Pi(x,\,n) := \frac{n!\,n^x}{(x+1)(x+2)\cdots(x+n)}, \qquad n = 1,\,2,\,3,\,\cdots. \tag{3.1}$$

Hence Euler's original definition of **Gamma function** is given in the form

$$\Pi(x) = \lim_{n\to\infty}\Pi(x,\,n) := \lim_{n\to\infty}\frac{n!\,n^x}{(x+1)(x+2)\cdots(x+n)}, \qquad n = 1,\,2,\,3,\,\cdots. \tag{3.2}$$

Of course, modern convergence idea were not available during Euler's time. Nevertheless, we shall show later that Euler's intuition is actually correct.

That is, $\Pi(m) = \lim_{n\to\infty} \Pi(m, n) = m!$. People later adopted the more popular notation $\Gamma(m+1) = \Pi(m)$ introduced by Legendre[1]. It is perhaps not immediately clear that the infinite product (3.1) converges. To see this, let us write

$$\begin{aligned}
\lim_{n\to\infty} \Pi(x, n) &= \frac{n!\, n^x}{(x+1)(x+2)\cdots(x+n)} \\
&= \lim_{n\to\infty} \frac{n!\,(n+1)^x}{(x+1)(x+2)\cdots(x+n)} \\
&= \lim_{n\to\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n)} \cdot \frac{2^x}{1^x} \cdot \frac{3^x}{2^x} \cdot \frac{4^x}{3^x} \cdots \frac{(n+1)^x}{n^x} \\
&= \lim_{n\to\infty} \Pi_{k=1}^{n} \left[\frac{k}{k+x} \cdot \frac{(k+1)^x}{k^x} \right] \\
&= \lim_{n\to\infty} \Pi_{k=1}^{n} \left[\Big(1+\frac{x}{k}\Big)^{-1} \Big(1+\frac{1}{k}\Big)^{x} \right]. \qquad (3.3)
\end{aligned}$$

The last infinite product is convergent for all real and even complex $x = z$ not equal to a negative integer, as shown in Homework 1, Q. 3. In fact, the convergence is uniform in suitable closed bounded set. So the infinite product is well-defined. What Euler observed was that the function $\Gamma(z)$ not only defined for positive integer m but also for, at least, non-integer.

We next write the infinite product in a different form.

3.2 The Euler (Mascheroni) constant γ

Let

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Theorem 3.1. *Let H_n be as defined above. Then the limit*

$$\gamma = \lim_{n\to\infty} (H_n - \log n) \qquad (3.4)$$

exists which is called the ***Euler constant****. It is not known if the Euler constant is a rational or an irrational number.*

The Euler constant is known to be approximately $\gamma \approx 0.5772$. We shall prove that $0 \le \gamma < 1$.

[1] 1752-1833, French mathematician

Proof. Let $A_n = H_n - \log n$. Then

$$\begin{aligned} A_{n+1} - A_n &= H_{n+1} - H_n - \log(n+1) + \log n \\ &= \frac{1}{n+1} + \log \frac{n}{n+1} \\ &= \frac{1}{n+1} + \log\big(1 - \frac{1}{n+1}\big) \\ &= -\sum_{k=2}^{\infty} \frac{1}{k(n+1)^k} < 0. \end{aligned}$$

Thus the $\{A_n\}$ is a decreasing sequence. We next show that $\{A_n\}$ has a lower bound. We first notice the straightforward inequalities

$$\frac{1}{k} < \int_{k-1}^{k} \frac{dt}{t} < \frac{1}{k-1}$$

holds for each integer $k \geq 2$. Thus

$$H_n - 1 < \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} + \cdots + \int_{n-1}^{n} \frac{dt}{t} < H_{n-1}.$$

That is

$$H_n - 1 < \log n < H_{n-1}.$$

Therefore

$$-1 < -H_n + \log n < -\frac{1}{n},$$

or

$$\frac{1}{n} < A_n < 1,$$

for each n. Since $\{A_n\}$ is decreasing and bounded below by 0, so this completes the proof. We also easily see that $H_n = \log n + \gamma + \varepsilon_n$ and $\varepsilon_n \searrow 0$. □

3.3 Weierstrass's definition

We now give an alternative infinite product definition of the gamma function due to Weierstrass.

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right], \tag{3.5}$$

and γ is again the gamma constant given in (3.4). We need to establish a number of facts before we could do so. In particular, we observe that the zeros of $1/\Gamma(z)$ are at the negative integers and each zero being a simple zero.

Since the infinite product converges uniformly in any compact set in the plane (see Theorem ??), so it is an analytic function in $\mathbb{C}$. As long as we stay away from the zeros we could take logarithm and differentiation on both sides of (3.5) to obtain

$$\begin{aligned}
-\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) \\
&= \frac{1}{z} + \gamma - \sum_{n=1}^{\infty} \frac{z}{n(z+n)},
\end{aligned}$$

where the infinite series on the right converges absolutely and uniformly in any closed subset of $\mathbb{C}$.

We first compute $\Gamma'(1)$. Substituting $z = 1$ into Weierstrass's infinite product gives

$$\begin{aligned}
\frac{1}{\Gamma(1)} &= e^{\gamma} \Pi_{n=1}^{\infty} \left[\left(1 + \frac{1}{n} \right) e^{-1/n} \right], \\
&= e^{\gamma} \lim_{n\to\infty} \Pi_{k=1}^{n} \left[\frac{k+1}{k} \, e^{-1/k} \right] \\
&= e^{\gamma} \lim_{n\to\infty} (n+1) \exp(-H_n) \\
&= e^{\gamma} \lim_{n\to\infty} (n+1) \exp(-\gamma - \log n - \epsilon_n)
\end{aligned} \tag{3.6}$$

for which $\epsilon_n \to 0$ as $n \to \infty$. It follows that

$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty} \frac{(n+1)}{n} \, e^{-\gamma} = 1.$$

Hence $\Gamma(1) = 1$. On the other hand, we have

$$-\frac{\Gamma'(1)}{\Gamma(1)} = 1 + \gamma - \sum_{n=1}^{\infty} \frac{1}{n(1+n)}.$$

That is,

$$\begin{aligned}
\Gamma'(1) &= -1 - \gamma + \sum_{n=1}^{\infty} \frac{1}{n(1+n)} \\
&= -1 - \gamma + \lim_{n\to\infty} \left(1 - \frac{1}{(1+n)} \right) \\
&= -\gamma.
\end{aligned}$$

We next show that the Weierstrass product is Euler's infinite product definition Recall that

$$\begin{aligned}\gamma &= \lim_{n\to\infty}(H_n-\log n) = \lim_{n\to\infty}(H_n-\log(n+1))\\ &= \lim_{n\to\infty}\left(H_n-\sum_{k=1}^{n}\log\frac{k+1}{k}\right).\end{aligned}\tag{3.7}$$

Hence

$$\begin{aligned}\exp(-\gamma z) &= \lim_{n\to\infty}\exp\left[-z\big(H_n-\sum_{k=1}^{n}\log\frac{k+1}{k}\big)\right]\\ &= \lim_{n\to\infty}\Pi_{k=1}^{n}\left[\left(\frac{k+1}{k}\right)^{z}\exp\big(-\frac{z}{k}\big)\right].\end{aligned}$$

Thus

$$\begin{aligned}z\Gamma(z) &= e^{-\gamma z}\lim_{n\to\infty}\Pi_{k=1}^{n}\left[\left(1+\frac{1}{k}\right)^{-1}\exp\big(\frac{z}{k}\big)\right]\\ &= \lim_{n\to\infty}\Pi_{k=1}^{n}\left[\left(\frac{k+1}{k}\right)^{z}\exp\big(-\frac{z}{k}\big)\left(1+\frac{1}{k}\right)^{-1}\exp\big(\frac{z}{k}\big)\right].\end{aligned}$$

That is,

$$\Gamma(z) = \frac{1}{z}\Pi_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right].\tag{3.8}$$

The above product has the following relation with (3.3)

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)}\\ &= \frac{1}{z}\lim_{n\to\infty}\frac{n!\,n^z}{(z+1)(z+2)\cdots(z+n)}\\ &= \frac{1}{z}\cdot\Pi(z) = \frac{1}{z}\cdot\Gamma(z+1)\end{aligned}$$

by Legendre's convention. That is, the z is shifted by one. Hence the two seemingly different infinite product representations are in fact the same.

In particular, we notice that $\Gamma(x)>0$ when x is real and positive.

3.4 A first order difference equation

Applying Euler's product representation yields

$$
\begin{aligned}
\frac{\Gamma(z+1)}{\Gamma(z)} &= \frac{z}{z+1}\frac{\frac{1}{z}\Pi_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\,\left(1+\frac{z}{n}\right)^{-1}\right]}{\frac{1}{z}\Pi_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1}\,\left(1+\frac{z+1}{n}\right)^{-1}\right]}\\
&= \frac{z}{z+1}\Pi_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{z}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}\right]\\
&= \frac{z}{z+1}\lim_{n\to\infty}\Pi_{n=1}^{n}\left(\frac{k+1}{k}\cdot\frac{k+z}{k+z+1}\right)\\
&= \frac{z}{z+1}\lim_{n\to\infty}\frac{n+1}{1}\cdot\frac{1+z}{n+z+1}\\
&= z.
\end{aligned}
$$

Therefore

$$\Gamma(z+1) = z\Gamma(z) \tag{3.9}$$

for all finite z that are not the poles of $\Gamma(z)$. Starting with $z = m+1$ and iterating the (3.9) m times yields

$$\Gamma(m+1) = m\cdot\Gamma(m) = \cdots = m\cdots 2\cdot\Gamma(2) = m\cdots 2\cdot 1\cdot\Gamma(1) = m!$$

since we have already computed $\Gamma(1) = 1$. Thus Euler's infinite product is a complex-valued function generalization for the *factorial*.

Exercise 3.2. *Solve the equation* $y(x+1) = \Big(\dfrac{x^2-x-2}{x^2+x-2}\Big)y(x)$.

For the most updated research on generalizations on gamma functions we refer to the 1997 paper of S. Ruijsenaars [**?**]

3.5 Integral representation definition

Euler noticed that there is another way to obtain the above partial product (3.1) by considering the integral

$$\int_0^1 (1-s)^{n-1}s^x\,ds.$$

Integrate this integral by parts yields:

$$\begin{aligned}
\int_0^1 (1-s)^{n-1} s^x \, ds &= \frac{n-1}{x+1} \int_0^1 (1-s)^{n-2} s^{x+1} \, ds \\
&= \frac{(n-1)(n-2)}{(x+1)(x+2)} \int_0^1 (1-s)^{n-3} s^{x+2} \, ds \\
&\cdots \\
&\cdots \\
&= \frac{(n-1)(n-2)\cdots\big(n-(n-1)\big)}{(x+1)(x+2)\cdots(x+n-1)} \int_0^1 (1-s)^{n-n} s^{x+n-1} \, ds \\
&= \frac{n!\, x!}{(x+n)!\, n} \\
&= \frac{\Pi(x;\, n)}{n^{x+1}}.
\end{aligned}$$

Substituting $t = ns$ in the above integral yields

$$\int_0^n \Big(1-\frac{t}{n}\Big)^{n-1} \Big(\frac{t}{n}\Big)^x \frac{dt}{n} = \frac{1}{n^{x+1}} \int_0^n \Big(1-\frac{t}{n}\Big)^{n-1} t^x \, dt = \frac{\Pi(x;\, n)}{n^{x+1}}.$$

Letting $n \to \infty$ yields

$$\Pi(x;\, \infty) = \int_0^\infty e^{-t} t^x \, dt,$$

or

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.$$

Definition 3.5.1. *Let $\Re z > 0$. The complex function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \qquad \Re z > 0 \tag{3.10}$$

is called the **Euler-Gamma function**.

The first question for us is whether this function is well-defined. To do so, we need to establish some basic theory to deal with infinite integrals in the next section.

3.6 Uniform convergence

Let us first review some basic integration theory. If $f(x)$ to be discussed below is a complex-valued function, then we could always write it as a combination two real functions: $f(x) = u(x)+iv(x)$ so that the discussion applies to both real or complex-valued function. Besides, we only require Riemann integration below. Recall that the infinite integral $\int_a^\infty f(t)\,dt$ **converges (exists)** if the limit

$$\lim_{x\to\infty}\int_a^x f(t)\,dt = \ell$$

exists. We say that $\int_a^\infty f(t)\,dt$ **converges absolutely** if the integral $\int_a^\infty |f(t)|\,dt$ converges. Then *an absolutely convergent integral is convergent.* This is because the integrals

$$\int_a^\infty \phi(t)\,dt, \qquad \textbf{and} \qquad \int_a^\infty \psi(t)\,dt$$

both converge, where $\phi(t) = |f(t)| + f(t)$ and $\psi(t) = |f(t)| - f(t)$. One can see this by noting that both functions are positive. Then apply the comparison test. But we could then write $f(t) = \frac{1}{2}(\phi(t) - \psi(t))$. So $\int_a^\infty f(t)\,dt$ converges. An integral is said to be **conditional convergent** if it is not absolutely convergent. It is easy to see that *a necessary and sufficient condition for the convergent of the integal*

$$\int_a^\infty f(t)\,dt$$

is that, given $\varepsilon > 0$, we can find x_0 such that

$$\left|\int_{x_1}^{x_2} f(t)\,dt\right| < \varepsilon,$$

whenever $x_2 > x_1 \geq x_0$.

Definition 3.6.1. *Let $f(x, y)$ be an integrable function with respect to x over the interval $a \leq x \leq b$, for $\alpha \leq y \leq \beta$ and for all values of b. Suppose that the integral*

$$\phi(y) = \int_a^\infty f(x, y)\,dx \tag{3.11}$$

*converges for all values of y in (α, β). Then the integral is said to be **uniformly convergent** if, any given $\varepsilon > 0$, we can find an x_0, depending on ε, but not on y, such that*

$$\left|\phi(y) - \int_a^x f(t, y)\,dt\right| < \varepsilon$$

whenever $x \geq x_0$.

We state a simple uniform convergence criterion which can be regarded as a kind of Weierstrass $M-$test for the convergence of infinite integrals.

Lemma 3.6.1. *The integral (3.11) converges uniformly if there is a positive function $g(x)$, such that $|f(x, y)| \leq g(x)$ over the interval $a \leq x \leq b$, for $\alpha \leq y \leq \beta$ and for all values of b, and such that*

$$\int_a^\infty g(x)\, dx$$

is convergent.

Proof. We notice that

$$\left|\phi(y) - \int_a^x f(t,\, y)\, dt\right| = \left|\int_x^\infty f(t,\, y)\, dt\right| \leq \int_x^\infty g(t)\, dt.$$

Since $\int_a^\infty g(x)\, dx$ is convergent, so there exists an $x_0 \geq a$ such that $\int_x^\infty g(x)\, dx \ < \ \varepsilon$ for $x > x_0$. This completes the proof. $\square$

Theorem 3.3. *Let $f(z,\, t)$ be a continuous function of z and t, where z lies within the closed bounded region D and $a \leq t \leq b$. Suppose that f is an analytic function of z, for every choice of t. Then the function*

$$F(z) = \int_a^b f(z,\, t)\, dt$$

is an analytic function of z, and

$$F'(z) = \int_a^b \frac{\partial f}{\partial z}\, dt.$$

Proof. We divide $[a,\, b]$ into n equal parts by the points

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

We define

$$F_n(z) = \sum_{k=1}^{n} f(z,\, t_k)(t_k - t_{k-1}).$$

Since $f(z,\, t)$ is a continuous function of t, so

$$F_n(z) \to \int_a^b f(z,\, t)\, dt$$

as $n \to \infty$ for each z. We next show that the convergence is uniform. Since $f(z, t)$ is uniformly continuous, so given any $\varepsilon > 0$, we can find an integer m such that

$$\left|f(z, t) - f(z, t')\right| < \varepsilon,$$

whenever $z \in D$ and $|t - t'| < (b - a)/m$. Notice that

$$\begin{aligned}
\left|F_n(z) - F(z)\right| &= \Big|\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(z, t_k) - f(z, t)\, dt\Big| \\
&\leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|f(z, t_k) - f(z, t)\right| dt \\
&< (b - a)\varepsilon
\end{aligned}$$

whenever $n \geq m$. This shows that the covergence is uniform. Hence $F(z)$ is analytic in D. Moreover, since

$$F_n'(z) = \sum_{k=1}^{n} \frac{\partial f(z, t_k)}{\partial z}(t_k - t_{k-1}),$$

so that

$$F'(z) = \int_a^b \frac{\partial f}{\partial z}\, dt,$$

as required. □

Example 3.6.1. *Consider*

$$F(z) = \int_a^b \cos zt\, f(t)\, dt$$

where $f(t)$ is continuous on $[a, b]$. Then $F'(z) = -\int_a^b t \sin zt\, f(t)\, dt$.

The above result can be further extended.

Theorem 3.4. *Suppose in addition to Theorem 3.3 that the integration extends to infinity, and that the infinite integral*

$$F(z) = \int_a^\infty f(z, t)\, dt$$

is uniformly convergent. The $F(z)$ is analytic in D and that

$$F'(z) = \int_a^\infty \frac{\partial f}{\partial z}\, dt.$$

Proof. Let us consider the sequence of functions

$$F_n(z) := \int_a^n f(z,\, t)\, dt,$$

and let $n \to \infty$. The last theorem asserts that each of the $F_n(z)$ is analytic within D, and since the limit function $F(z)$ converges uniformly, so it must be analytic in D. Moreover, since

$$F_n'(z) = \frac{d}{dz} \int_a^n f(z,\, t)\, dt = \int_a^n \frac{\partial f(z,\, t)}{\partial z}\, dt,$$

and so

$$F'(z) = \int_a^\infty \frac{\partial f(z,\, t)}{\partial z}\, dt$$

as required.

□

3.7 Analytic properties of $\Gamma(z)$

To do so, we write

$$\Gamma(z) = \Phi(z) + \Psi(z), \tag{3.12}$$

where

$$\Phi(z) = \int_0^1 e^{-t} t^{z-1}\, dt, \qquad \Psi(z) = \int_1^\infty e^{-t} t^{z-1}\, dt.$$

Proposition 3.7.1. *The function $\Psi(z)$ defined in (3.12) is an entire function.*

Proof. Let z lies in a closed, bounded but otherwise arbitrary domain. So there exists a η such that $\Re z \le \eta$. Hence the integrand of the integral satisfies the following inequality

$$\begin{aligned} |t^{z-1}| &= \big|\exp[(z-1)\log t]\big| = \big|\exp[\Re\{(z-1)\}\log t]\big| \\ &\le \big|\exp[(\eta-1)\log t]\big| = t^{\eta-1}. \end{aligned}$$

holds throughout the bounded region whenever $t \ge 1$. On the other hand, $e^{-\frac{1}{2}t} t^{\eta-1} \to 0$ as $t \to \infty$, so one can find a constant C, depending on η, such that $t^{\eta-1} \le Ce^{\frac{1}{2}t}$ for all $t \ge 1$. In particular,

$$\big|e^{-t} t^{z-1}\big| \le e^{-t} C e^{\frac{1}{2}t} \le Ce^{-\frac{1}{2}t}$$

whenever z lies in the region and $t \ge 1$. Since $\int_1^\infty Ce^{\frac{1}{2}t}\, dt$ is convergent, so Lemma 3.6.1 implies that the infinite integral $\Psi(z)$ converges uniformly, and hence is an analytic function of z by Theorem 3.4. The above argument works for any closed bounded region. So $\Psi(z)$ is an entire function (*i.e.*, analytic in $\mathbb{C}$). $\square$

It remains to consider the $\Phi(z)$.

Proposition 3.7.2. *The function $\Phi(z)$ is analytic in the right half-plane.*

Proof. Let us change the variable in the integration to $t = 1/u$. That is,

$$\Phi(z) = \int_0^1 e^{-t} t^{z-1}\, dt = \int_1^\infty e^{-1/u} u^{-z-1}\, du.$$

Since $|e^{-1/u} u^{-z-1}| \le |u^{-z-1}| = u^{-\Re z - 1}$. Clearly,

$$\left| \int_1^\infty e^{-1/u} u^{-z-1}\, du \right| \le \int_1^\infty u^{-\Re z - 1}\, du$$

converges when $\Re z \ge \delta > 0$. So the infinite integral is uniformly convergent by Lemma 3.6.1. Hence $\Phi(z)$ is analytic by Theorem 3.4. $\square$

This shows that the integral (3.10) is well-defined in the right half-plane.

Let us now consider

$$\begin{aligned}\Phi(z) &= \int_0^1 e^{-t} t^{z-1}\, dt \\ &= \int_0^1 \sum_{k=0}^\infty \frac{(-1)^k t^{k+z-1}}{k!}\, dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{(k+z)k!},\end{aligned} \tag{3.13}$$

where the integration term-by-term is justified for $\Re z > 0$. But the series in (3.13) is uniformly convergent in any closed domain kept away from the negative integers including 0. Thus the infinite series (3.13) provides an **analytic continuation formula** for the gamma function to $\mathbb{C}$. We have proved

Theorem 3.5. *The Gamma function $\Gamma(z)$ is analytic in $\mathbb{C}$ except at $\{0, -1, -2, -3, \cdots\}$. That is, $\Gamma(z)$ is a meromorphic function in $\mathbb{C}$ and*

$$\Gamma(z) = \Psi(z) + \sum_{k=0}^\infty \frac{(-1)^k}{(k+z)k!},$$

with simple poles at $0, -1, -2, -3, \cdots$, with residue being $(-1)^k/k!$ at $z = -k$.

3.8 Tannery's theorem

We shall need this theorem to establish the equivalence of the two definitions of the gamma function.

Theorem 3.6 (Tannery's theorem)**.** *Suppose the function $f(t,\, n)$ converges to $g(t)$ uniformly as $n \to \infty$ for any fixed interval of t, $|f(t,\, n)| \le M(t)$ for all n and t for some positive function $M(t)$ such that $\int_a^\infty M(t)\, dt$ converges. If there exists a sequence λ_n such that $\lambda_n \to \infty$, then*

$$\lim_{n\to\infty} \int_a^{\lambda_n} f(t,\, n)\, dt = \int_a^\infty g(t)\, dt.$$

Proof. We choose K so large so that $K \ge a$ and to choose n such that $\lambda_n > K$. Then

$$\begin{aligned} \Big| \int_a^{\lambda_n} f(t,\, n)\, dt - \int_a^\infty g(t)\, dt \Big| &= \Big| \int_a^K (f-g)\, dt + \int_K^{\lambda_n} f\, dt - \int_K^\infty g\, dt \Big| \\ &\le \Big| \int_a^K (f-g)\, dt \Big| + \int_K^{\lambda_n} |f(t,\, n)|\, dt + \int_K^\infty |g(t)|\, dt. \end{aligned}$$

But since $|f(t,\, n)| \le M(t)$ for all n and $f(t,\, n)$ converges $g(t)$, so we also deduce $|g(t)| \le M(t)$. This gives

$$\begin{aligned} &\Big| \int_a^{\lambda_n} f(t,\, n)\, dt - \int_a^\infty g(t)\, dt \Big| \\ &\le \Big| \int_a^K (f-g)\, dt \Big| + 2 \int_K^\infty M(t)\, dt. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \Big| \int_a^{\lambda_n} f(t,\, n)\, dt - \int_a^\infty g(t)\, dt \Big| \le 2 \int_K^\infty M(t)\, dt.$$

Since the right side is independent of K. It follows that the above limit is zero. □

Exercise 3.7. *Let $F(n) = \sum_{k=0}^p v_k(n)$, where $p \to \infty$ as $n \to \infty$. Then if $v_k(n) \to w_k$ as $n \to \infty$ for each fixed k, show that*

$$\lim_{n\to\infty} F(n) = \sum_{k=0}^\infty w_k,$$

provided that $|v_k(n)| \le M_k$, where M_k is independent of n and $\sum M_k$ converges.

3.9 Integral representation

Let

$$\Gamma(z;\, n) := \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}, \qquad n = 1,\, 2,\, 3,\, \cdots .$$

Note that $z\Gamma(z;\, n) = \Pi(z;\, n)$.

As in the case of $\Pi(z;\, n)$, we have

$$\Gamma(z;\, n) = n^z \int_0^1 (1-s)^n s^{z-1}\, ds = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt.$$

We shall show that

$$\Gamma(z;\, n) \longrightarrow \int_0^\infty e^{-t} t^{z-1}\, dt = \Gamma(z).$$

We first establish some inequalities

Lemma 3.9.1. *If* $0 \le \alpha < 1$*, then* $1 + \alpha \le \exp(\alpha) \le (1-\alpha)^{-1}$.

Proof. Omitted. □

Lemma 3.9.2. *If* $0 \le \alpha < 1$*, then* $1 - n\alpha \le (1-\alpha)^n$ *for* $n \in \mathbb{N}$.

Proof. By induction. □

Lemma 3.9.3. *Let* $0 \le t < n$ *where* n *is a positive integer. Then*

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}.$$

Proof. We let $\alpha = t/n$ in Lemma 3.9.1 to obtain

$$1 + \frac{t}{n} \le \exp\left(\frac{t}{n}\right) \le \left(1 - \frac{t}{n}\right)^{-1}$$

from which we get

$$\left(1 + \frac{t}{n}\right)^n \le e^t \le \left(1 - \frac{t}{n}\right)^{-n}$$

or

$$\left(1 + \frac{t}{n}\right)^{-n} \ge e^{-t} \ge \left(1 - \frac{t}{n}\right)^n,$$

so that

$$e^{-t} - \Big(1 - \frac{t}{n}\Big)^n \geq 0.$$

Thus

$$\begin{aligned}
e^{-t} - \Big(1 - \frac{t}{n}\Big)^n &= e^{-t}\Big[1 - e^t\Big(1 - \frac{t}{n}\Big)^n\Big] \\
&\leq e^{-t}\Big[1 - \Big(1 - \frac{t^2}{n^2}\Big)^n\Big] \\
&\leq e^{-t}\Big[1 - \Big(1 - n\frac{t^2}{n^2}\Big)\Big]
\end{aligned}$$

by Lemma 3.9.2. Hence

$$e^{-t} - \Big(1 - \frac{t}{n}\Big)^n \leq e^{-t}\Big[1 - 1 + \frac{t^2}{n}\Big] = \frac{e^{-t}t^2}{n}.$$

□

Theorem 3.8. *Let $\Re z > 0$. Then we have*

$$\begin{aligned}
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt &= \lim_{n\to\infty}\int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\,dt \\
= \lim_{n\to\infty}\Gamma(z;\,n) &= \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)}.
\end{aligned}$$

Proof. We already know that the limit relations from the second and third equalities. So it remains to consider the first equality.

Let $x = \Re z$. Then the Lemma 3.9.3 implies that

$$\begin{aligned}
\Big|e^{-t}t^{z-1} - \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\Big| &\leq t^{x-1}\Big\{e^{-t} - \Big(1 - \frac{t}{n}\Big)^n\Big\} \\
&\leq \frac{t^{x+1}e^{-t}}{n}
\end{aligned}$$

for every positive integer n. Hence the convergence of $f(t,\,n) := \Big(1 - \frac{t}{n}\Big)^n t^{z-1}$ to $e^{-t}t^{z-1}$ is uniform for any t in a fixed interval as n tends to infinity. Besides, we have

$$|f(t,\,n)| = \Big|\Big(1 - \frac{t}{n}\Big)^n t^{z-1}\Big| = \Big(1 - \frac{t}{n}\Big)^n t^{x-1} \leq e^{-t}t^{x-1} := M(t)$$

and $\int_0^\infty e^{-t}t^{x-1} < \infty$ provided $x = \Re z > 1$. Hence the hypothesis of Tannery's theorem are satisfied, and we conclude the limit holds when $\Re z = x > 1$. The case when $\Re z = x > 0$ follows from $\Gamma(z)$ being an analytic function already known from Theorem 3.5 and the principle of analytic continuation. □

Remark 3.9.1. *We note that one could also apply the Lebesgue dominated convergence theorem directly in the above proof.*

3.10 The Eulerian integral of the first kind

Definition 3.10.1. *The* ***beta integral*** *is a function of two complex variables defined by*

$$B(x,\,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\qquad \Re x>0,\,\Re y>0. \tag{3.14}$$

We note that the beta integral is often called the **Eulerian integral of the first kind** while the gamma function $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt$ is called the **Eulerian integral of the second kind**

We observe that the beta integral is symmetric in x and y: a change of variables by $u=1-t$ clearly illustrates this. The complex exponent is understood to be

$$t^{x-1}=\exp[(x-1)\log t],\qquad (1-t)^{y-1}=\exp[(y-1)\log(1-t)]$$

where the logarithms have their principal values (*i.e.,* $\log z=\log r+i\theta$ where $z=re^{i\theta}$ and $-\pi<\arg z=\theta\le\pi$. Clearly the two functions are well defined.

Exercise 3.9. *Prove that the beta integral is an analytic function in x (with $\Re x>0$ and y held fixed) and y (with $\Re y>0$ and x held fixed).*

Definition 3.10.2. *We write, for integer n and any complex number x*

$$(x)_n=x(x+1)(x+1)\cdots(x+n-1).$$

Exercise 3.10. *Show for any x not equal to a negative integer*

$$\Gamma(x)=\lim_{k\to\infty}\frac{k!k^{x-1}}{(x)_k}.$$

Theorem 3.11. *If $\Re x>0$ and $\Re y>0$, then*

$$B(x,\,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.15}$$

Proof. For $\Re x > 0$ and $\Re y > 0$, we have

$$\begin{aligned} B(x,\, y+1) &= \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}\, dt \\ &= B(x,\, y) - B(x+1,\, y). \end{aligned}$$

On the other hand, an integration by parts yields

$$\begin{aligned} B(x,\, y+1) &= \frac{1}{x} t^x (1-t)^{y-1}\Big|_0^1 + \frac{y}{x}\int_0^1 t^x (1-t)^{y-1}\, dt \\ &= \frac{y}{x} B(x+1,\, y). \end{aligned}$$

Combining the above two expressions for $B(x,\, y+1)$ yields the iterative formula for the beta function:

$$B(x,\, y) = \frac{x+y}{y} B(x,\, y+1). \tag{3.16}$$

So iterating this formula several times yields:

$$B(x,\, y) = \frac{(x+y)(x+y+1)}{y(y+1)} B(x,\, y+1) = \cdots = \frac{(x+y)_n}{(y)_n} B(x,\, y+n). \tag{3.17}$$

Rewrite this relation as

$$\begin{aligned} B(x,\, y) &= \frac{(x+y)_n}{n!}\frac{n!}{(y)_n} B(x,\, y+n) \\ &= \frac{(x+y)_n}{n!}\frac{n!}{(y)_n} \int_0^1 s^{x-1}(1-s)^{y+n-1}\, ds \\ &= \frac{(x+y)_n}{n!}\frac{n!}{(y)_n} \int_0^n \Big(\frac{t}{n}\Big)^{x-1}\Big(1-\frac{t}{n}\Big)^{y+n-1}\frac{dt}{n} \\ &= \frac{(x+y)_n}{n!n^{x+y-1}}\frac{n!n^{y-1}}{(y)_n} \int_0^n \Big(\frac{t}{n}\Big)^{x-1}\Big(1-\frac{t}{n}\Big)^{y+n-1} dt. \end{aligned}$$

Letting $n \to \infty$ in the above relation yields

$$B(x,\, y) = \frac{\Gamma(y)}{\Gamma(x+y)} \int_0^\infty t^{x-1} e^{-t}\, dt$$

where we have applied the Tannery theorem or Lebesgue's dominated convergence theorem. □

Exercise 3.12. *Apply $t = \sin^2\theta$ to (3.15) to show*

$$\int_0^{\pi/2} \sin^{2x-1}\theta \, \cos^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}.$$

Hence show that

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}.$$

Exercise 3.13. *Apply $t = s/(s+1)$ to (3.15) to show*

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\, ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.18}$$

3.11 The Euler reflection formula

If $x + y = 1$, then$\Gamma(x+y) = \Gamma(1) = 1$. We show a beautiful formula about gamma function of this sort due to Euler and some of its consequences. We shall use complex variable proof.

Theorem 3.14. *We have when $z \neq \{0, -1, -2, -3, \cdots\}$,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{3.19}$$

Proof. Suppose that z is chosen to lie in the strip $0 < \Re z < 1$. Then both z and $1 - z$ have positive real part. So the Theorem 3.11 applies and we could write

$$\begin{aligned}\Gamma(z)\Gamma(1-z) = B(z, 1-z) &= \int_0^1 t^{z-1}(1-t)^{-z}\, dt\\ &= \int_0^1 \Big(\frac{t}{1-t}\Big)^z \frac{dt}{t}.\end{aligned}$$

Putting $s = t/(1-t)$ this time yields

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{s^{z-1}}{1+s}\, ds, \qquad 0 < \Re z < 1.$$

Let us write $\Re Z = s$ and

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{(\Re Z)^{z-1}}{1+\Re Z}\, d\Re Z, \qquad 0 < \Re z < 1.$$

So we could consider an contour integral in variable Z such that $\Re Z = s$ and whose integrand is given above:

$$\int_C \frac{Z^{z-1}}{1+Z}\, dZ,$$

where the contour C consists of two circles with radii R and r respectively and $R > 1 > r$, connected by a line segment $[r, R]$ lying on the positive real axis. The circle of radius R is around the anti-clockwise direction while that on radius r around the clockwise direction. There are two straight line segments of the contour on opposite direction lying on the segment $[r, R]$ connecting the two circles.

The function in the integrand has a simple pole at $Z = -1$. Thus the Residue theorem from basic complex function theory yields

$$\begin{aligned}\int_C \frac{Z^{z-1}}{1+Z}\,dZ &= 2\pi i \mathrm{Res}_{Z=-1}\frac{Z^{z-1}}{1+Z}\\ &= 2\pi i \lim_{Z\to -1}(Z-(-1))\frac{Z^{z-1}}{1+Z}\\ &= 2\pi i(-1)^{z-1}\\ &= 2\pi i \exp\big(\pi i(z-1)\big).\end{aligned}$$

On the other hand, applying the contour integration along C splits the integration into four parts in the form

$$\begin{aligned}2\pi i\exp\big(\pi i(z-1)\big) &= \int_C \frac{Z^{z-1}}{1+Z}\,dZ\\ &= \int_0^{2\pi}\frac{(Re^{i\theta})^{z-1}\,d(Re^{i\theta})}{1+Re^{i\theta}} + \int_R^r \frac{(se^{2\pi i})^{z-1}\,d(se^{2\pi i})}{1+se^{2\pi i}}\\ &\quad+\int_{2\pi}^0 \frac{(re^{i\theta})^{z-1}\,d(re^{i\theta})}{1+re^{i\theta}} + \int_r^R \frac{(se^{0\pi i})^{z-1}\,d(se^{0\pi i})}{1+se^{2\pi i}}\\ &= \int_0^{2\pi}\frac{iR^z e^{i\theta z}\,d\theta}{1+Re^{i\theta}} + \int_R^r \frac{s^{z-1}e^{2\pi i z}\,ds}{1+s}\\ &\quad+\int_{2\pi}^0 \frac{ir^{z-1}e^{i\theta z}\,d\theta}{1+re^{i\theta}} + \int_r^R \frac{s^{z-1}e^{0iz}\,ds}{1+s}.\end{aligned}$$

The assumption that $0 < \Re z < 1$ implies that the first and the third integral tend to zero respectively, as $R \to \infty$ and $r \to 0$. Thus we are left with

$$e^{2\pi i z}\int_\infty^0 \frac{s^{z-1}\,ds}{1+s} + \int_0^\infty \frac{s^{z-1}\,ds}{1+s} = -2\pi i e^{\pi i z}.$$

We deduce that

$$\int_0^\infty \frac{s^{z-1}\,ds}{1+s} = \frac{2\pi i e^{i\pi z}}{e^{2\pi i z}-1} = \frac{2\pi i}{e^{i\pi z}-e^{-i\pi z}} = \frac{\pi}{\sin \pi z}.$$

This proves the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

when $0 < \Re z < 1$. On the other hand, each term of this formula is valid on the $\mathbb{C}$ except at 0 and negative integers. The formula must therefore valid throughout $\mathbb{C}$ except at 0 and negative integers by the analytic continuation principle. □

Exercise 3.15. *Use Euler's reflection formula to prove* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

We deduce from the Euler reflection formula the following identities (as listed in Andrews, Askey and Roy [**?**,]

Theorem 3.16. *1.*

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2}\Big), \tag{3.20}$$

2.

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \Big(\frac{1}{x+n} + \frac{1}{x-n}\Big) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{x-k}, \tag{3.21}$$

3.

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{x-k}, \tag{3.22}$$

4.

$$\pi \tan \pi x = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{k + \frac{1}{2} - x}, \tag{3.23}$$

5.

$$\pi \sec \pi x = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{k + x + \frac{1}{2}}, \tag{3.24}$$

6.

$$\frac{\pi^2}{\sin^2 \pi x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2}. \tag{3.25}$$

Proof. We recall the Weierstrass factorization of the gamma function is given by

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \Big[\Big(1 + \frac{x}{n}\Big)\ e^{-x/n}\Big],$$

and γ is the gamma constant. The first identity follows from the factorization of the product (which converges uniformly) and the reflection formula in the form

$$\begin{aligned}
\frac{\sin \pi x}{\pi} &= \Gamma(x)^{-1}\Gamma(1-x)^{-1} = (-x)^{-1}\Gamma(x)^{-1}\Gamma(-x)^{-1} \\
&= \frac{1}{-x} xe^{\gamma x} \prod_{n=1}^{\infty} \Big[\Big(1 + \frac{x}{n}\Big)\ e^{-x/n}\Big] \cdot (-x)\, e^{-\gamma x} \prod_{n=1}^{\infty} \Big[\Big(1 - \frac{x}{n}\Big)\ e^{x/n}\Big] \\
&= x \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2}\Big),
\end{aligned}$$

as required.

The (3.21) is the logarithmic derivative of (3.20) and the convergence is uniform in any compact subset of complex plane with the corresponding simple poles removed. The formula (3.22) follows from the identity

$$\frac{1}{\sin x} = \operatorname{cosec} x = \cot\frac{x}{2} - \cot x.$$

The remaining three identities are left as exercises. □

Exercise 3.17. *Prove the identities (3.23), (3.24) and (3.25) by applying the previous three identities and appropriate trigonometric identities.*

3.12 Stirling's asymptotic formula

According to certain historical record, De Moivre noticed that $n!$ behaves like $n! \approx Cn^{n+1/2}e^{-n}$ when n is large (1730), where C is an unknown constant. Stirling observed that $C = \sqrt{2\pi}$, and de Moivre later proved Stirling's claim. The treatment here mainly follows that of Andrews, Askey and Roy []

Theorem 3.18.

$$\Gamma(x) \approx \sqrt{2\pi}x^{x-1/2}e^{-x}, \qquad \Re x \to \infty. \tag{3.26}$$

Proof. Let

$$\log\Gamma(x+n) := c_n = \sum_{k=1}^{n-1}\log(k+x) + \log\Gamma(x+1).$$

So

$$c_{n+1} - c_n = \log(x+n). \tag{3.27}$$

The idea is that we think of the above summation as certain approximation of an integral representing an area, so that the right side of the equation (3.27) can be interpreted as a derivative and c_n the integral of $\log(x+n)$. Thus integrating $\log t$ from 1 to $n+x$ gives approximately

$$(n+x)\log(x+n) - (n+x)$$

when $n\to\infty$. So we set an error term d_n to be

$$c_n = (n+x)\log(n+x) - (n+x) + d_n,$$

and substitute it into (3.27) to yield

$$d_{n+1} - d_n = (n+x+1)\log(n+1+x) - (n+x)\log(n+x) - (n+x) + 1.$$

Thus

$$\begin{aligned}
d_{n+1} - d_n &= 1 + (1+n+x)\Big[\log(n+x) - \log(n+1+x)\Big] \\
&= 1 - (n+x+1)\Big[1 + \frac{1}{n+x}\Big] \\
&= 1 - (n+x+1)\Big[\frac{1}{n+x} - \frac{1}{2(n+x)^2} + \frac{1}{3(n+x)^3} - \cdots\Big] \\
&= 1 - \Big[\Big(1 + \frac{1}{n+x}\Big) - \frac{n+x+1}{2(n+x)^2} + \frac{n+x+1}{3(n+x)^3} - \cdots\Big] \\
&= \frac{-1}{n+x} + \frac{n+x+1}{2(n+x)^2} - \frac{n+x+1}{3(n+x)^3} + \frac{n+x+1}{4(n+x)^4} \cdots\Big] \\
&= -\frac{1}{2(n+x)} + \frac{1}{6(n+x)^2} - \frac{1}{12(n+x)^3} + \cdots.
\end{aligned}$$

We may similarly think of the d_n in

$$d_{n+1} - d_n = -\frac{1}{2(n+x)} + \frac{1}{6(n+x)^2} - \frac{1}{12(n+x)^3} + \cdots \tag{3.28}$$

as the integral of the leading term of the right side. The area approimation represented by this integral is given in the following exercise.

Exercise 3.19. *Explain why the area representing d_n is approximately*

$$d_n \approx -\frac{1}{2}\log(n+x).$$

Thus we define

$$e_n = d_n - \Big(-\frac{1}{2}\log(n+x)\Big),$$

and substitute into (3.28) to yield

$$\begin{aligned}
e_{n+1} - e_n &= \frac{1}{2}\log\Big(1 + \frac{1}{n+x}\Big) - \frac{1}{2(n+x)} + \frac{1}{6(n+x)^2} - \frac{1}{12(n+x)^3} + \cdots \\
&= -\frac{1}{12(n+x)^2} + O\Big(\frac{1}{(n+x)^3}\Big).
\end{aligned}$$

Thus

$$e_n - e_0 = \sum_{k=0}^{n-1}(e_{k+1} - e_k) = \sum_{k=0}^{n-1}\Big[-\frac{1}{12(k+x)^2} + O\Big(\frac{1}{(k+x)^3}\Big)\Big].$$

We conclude that $\lim_{n\to\infty}(e_n - e_0) := K_1(x)$ must exist. On the other hand, each infinite sum on the right side of the above equation converges too. So we have, provided n is sufficiently large, that

$$e_n = K(x) + \frac{1}{12(n+x)} + O\Big(\frac{1}{(n+x)^2}\Big),$$

where $K(x) = K_1(x) + e_0$. So we can write

$$\begin{aligned} c_n = (n+x)\log(n+x) - (n+x) - \frac{1}{2}\log(n+x) + K(x) \\ + \frac{1}{12(n+x)} + O\Big(\frac{1}{(n+x)^2}\Big). \end{aligned} \tag{3.29}$$

On the other hand, the Euler product representation yields

$$\lim_{n\to\infty}\frac{\Gamma(n+x)n^y}{\Gamma(n+y)n^x} = \frac{\Gamma(x)}{\Gamma(y)}\lim_{n\to\infty}\frac{(x)_n/n^x}{(y)_n/n^y} = \frac{\Gamma(x)}{\Gamma(y)}\frac{\Gamma(y)}{\Gamma(x)} = 1$$

Choosing $y = 0$ in the above limit and together with (3.29) yields

$$\begin{aligned} 1 &= \lim_{n\to\infty} n^{-x}\frac{\Gamma(n+x)}{\Gamma(x)} \\ &= \lim_{n\to\infty}\frac{n^{-x}e^{K(x)}(n+x)^{n+x-1/2}\exp\Big[-(n+x) + \dfrac{1}{12(n+x)} + O\big(1/(n+x)^2\big)\Big]}{e^{K(0)}n^{n-1/2}\exp\Big[-n + \dfrac{1}{12n} + O\big(1/n^2\big)\Big]} \\ &= \lim_{n\to\infty} e^{K(x)-K(0)}\Big(1+\frac{x}{n}\Big)^n\Big(1+\frac{x}{n}\Big)^{x-1/2}e^{-x+O(1/n)} \\ &= e^{K(x)-K(0)}. \end{aligned}$$

Thus proving that $K(x) = K(0) = C$ is a constant function. Hence we deduce

$$\Gamma(x) \approx Cx^{x-\frac{1}{2}}e^{-x}, \qquad \Re x \to \infty.$$

We now quote a classical product of Wallis from 1656 which gives

$$\lim_{n\to\infty}\Big[\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{\sqrt{n}}\Big]^2 = \pi.$$

See Andrews, Askey and Roy [] for the details. Hence we have

$$\begin{aligned} \sqrt{\pi} &= \lim_{n\to\infty}\frac{2^{2n}\,(n!)^2}{(2n)!}\cdot\frac{1}{\sqrt{n}} \\ &= \lim_{n\to\infty}\frac{2^{2n}C^2n^{2n+1}e^{-2n+O(\frac{1}{n})}}{C(2n)^{2n+\frac{1}{2}}e^{-2n+O(\frac{1}{n})}}\cdot\frac{1}{\sqrt{n}} \\ &= \frac{C}{\sqrt{2}}, \end{aligned}$$

thus giving $C = \sqrt{2\pi}$ as required. □

One can obtain a more accurate asymptotic formula. Let us first introduce a new term.

Definition 3.12.1. *The* ***Bernoulli numbers*** B_n *are defined by the power series expansion*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}.$$

The first few Bernouilli numbers are

$$\begin{aligned} B_1 &= -\frac{1}{2}, \\ B_2 &= \frac{1}{6}, \\ B_4 &= -\frac{1}{30}, \\ B_6 &= \frac{1}{42}, \\ \cdots &= \cdots . \end{aligned}$$

Then we have

Theorem 3.20. *For each positive integer* k*,*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} B_{2k} \pi^{2k}.$$

Proof. We deduce from the definition of the Bernoulli numbers that

$$x \cot x = ix \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = ix + \frac{2ix}{e^{2ix} - 1} = 1 - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{2k}}{(2k)!} B_{2k} x^{2k}.$$

On the other hand, the identity (3.21) gives us

$$\begin{aligned} x \cot x = 1 + 2 \sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2 \pi^2} &= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k} \pi^{2k}} \\ &= 1 - 2 \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \Big) \frac{x^{2k}}{\pi^{2k}}. \end{aligned}$$

Equating the coefficient of x^{2k} for each k in the above two infinite sums completes the proof. $\square$

Consider

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},$$

where $B_n(0)=B_n$ for all $n\geq 1$ are the Bernoulli numbers. The $B_n(x)$ are called the **Bernoulli polynomials**.

Theorem 3.21. *Let x be a non-zero and non-negative complex number. Then*

$$\log\Gamma(x)=\frac{1}{2}\log(2\pi)+\Big(x-\frac{1}{2}\Big)\log x-x+\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j-1)2j}\frac{1}{x^{2j-1}}$$
$$-\frac{1}{2m}\int_0^{\infty}\frac{B_{2m}(t-[t])}{(x+t)^{2m}}\,dt.$$

The branch of $\log x$ is chosen so that $\log 1=0$.

We shall not prove this theorem. You are advised to consult either Hardy [14, XIII], Andrews, Askey and Roy [1] or Rainville [18] for a proof.

But we could deduce the Stirling asymptotic formula (3.18) immediately.

Corollary 3.1. *Let $\delta>0$ and $|\arg x|\leq\pi-\delta$. Then*

$$\Gamma(x)\approx\sqrt{2\pi}x^{x-1/2}e^{-x},\qquad |x|\to\infty. \tag{3.30}$$

Exercise 3.22. *Apply the Euler reflection formula for the gamma function to show that*

$$\left|\Gamma\left(\frac{1}{2}+ib\right)\right|\approx\sqrt{2\pi}e^{-\pi|b|/2},\qquad b\to\pm\infty.$$

3.13 Gauss's Multiplication Formula

We first note that

$$(z)_{2n}=2^{2n}\left(\frac{z}{2}\right)_n\left(\frac{z+1}{2}\right)_n.$$

This can be easily verified.

Exercise 3.23. *Prove that*

$$(z)_{kn}=k^{kn}\left(\frac{z}{k}\right)_n\left(\frac{z+1}{k}\right)_n\cdots\left(\frac{z+k-1}{k}\right)_n. \tag{3.31}$$

Thus we deduce

Theorem 3.24 (Legendre's duplication formula)**.**

$$\sqrt{\pi}\,\Gamma(2a) = 2^{2a-1}\Gamma(a)\Gamma(a+\frac{1}{2}),$$

provided that none of the three functions above will encounter a pole.

In fact, it is a special case of Gauss's multiplication theorem below. But first we need the following simple fact.

Lemma 3.13.1. *Let $k \geq 2$ be an integer. Then* $\displaystyle\prod_{s=1}^{k-1} \sin\frac{\pi s}{k} = \frac{k}{2^{k-1}}$.

Proof. Let $\alpha = e^{2\pi i/k}$. Then

$$x^k - 1 = (x-1)\Pi_{s=1}^{k-1}(x-\alpha^s).$$

Differentiating both sides yields

$$kx^{k-1} = \Pi_{s=1}^{k-1}(x-\alpha^s) + (x-1)g(x),$$

in which $g(x)$ is a polynomial in x. Put $x = 1$ into both sides of the above formula gives us the identity

$$k = \Pi_{s=1}^{k-1}(1-\alpha^s).$$

But

$$1-\alpha^s = 1-e^{2s\pi i/k} = -e^{\pi is/k}\big(e^{\pi is/k} - e^{-\pi is/k}\big) = -2ie^{\pi is/k}\sin\frac{\pi s}{k}.$$

Hence

$$k = (-2\pi i)^{k-1}\exp\big[\frac{1}{2}\pi i(k-1)\big]\Pi_{s=1}^{k-1}\sin\frac{\pi s}{k},$$

thus giving the required result. □

Theorem 3.25 (Gauss's multiplication theorem)**.**

$$(2\pi)^{\frac{k-1}{2}}k^{\frac{1}{2}-ka}\Gamma(ka) = \Gamma(a)\Gamma\Big(a+\frac{1}{k}\Big)\cdots\Gamma\Big(a+\frac{k-1}{k}\Big). \tag{3.32}$$

Proof. We deduce from (3.31) and $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$, that the following identity

$$\frac{\Gamma(z+nk)}{\Gamma(z)} = k^{nk}\prod_{s=1}^{k}\frac{\Gamma\Big(\dfrac{z+s-1}{k}+n\Big)}{\Gamma\Big(\dfrac{z+s-1}{k}\Big)}.$$

Subsitiuting $z = ka$ into the above identity yields

$$\frac{\Gamma(ka+nk)}{\Gamma(z)} = k^{nk}\prod_{s=1}^{k}\frac{\Gamma\Big(a+n+\dfrac{s-1}{k}\Big)}{\Gamma\Big(a+\dfrac{s-1}{k}\Big)}.$$

Rearranging the terms in the above equation such that the left-side is independent of n. This yields

$$\begin{aligned}
\frac{\Gamma(ka)}{\prod_{s=1}^{k}\Gamma\Big(a+\dfrac{s-1}{k}\Big)} &= \frac{\Gamma(ka+nk)}{k^{nk}\prod_{s=1}^{k}\Gamma\Big(a+n+\dfrac{s-1}{k}\Big)}\\
&= \lim_{n\to\infty}\frac{\Gamma(ka+nk)}{k^{nk}\prod_{s=1}^{k}\Gamma\Big(a+n+\dfrac{s-1}{k}\Big)}\\
&= \lim_{n\to\infty}\frac{\Gamma(ka+nk)}{(kn-1)!(nk)^{ka}}\cdot\frac{(kn-1)!(nk)^{ka}}{k^{nk}}\\
&\qquad\cdot\prod_{s=1}^{k}\left[\frac{(n-1)!\,n^{a+\frac{s-1}{k}}}{\Gamma\Big(a+n+\dfrac{s-1}{k}\Big)}\cdot\frac{1}{(n-1)!\,n^{a+\frac{s-1}{k}}}\right]\\
&= 1\cdot\lim_{n\to\infty}\frac{(kn-1)!(nk)^{ka}}{k^{nk}}\cdot\prod_{s=1}^{k}(1)\cdot\frac{1}{(n-1)!\,n^{a+\frac{s-1}{k}}}\\
&= \lim_{n\to\infty}\frac{(kn-1)!(nk)^{ka}}{k^{nk}[(n-1)!]^{k}n^{ka+\frac{1}{2}(k-1)}},
\end{aligned}$$

is a function of a only, since

$$\lim_{n\to\infty}\frac{(n-1)!\,n^{a+\frac{s-1}{k}}}{\Gamma\Big(a+n+\dfrac{s-1}{k}\Big)} = 1,$$

and

$$\lim_{n\to\infty}\frac{(kn-1)!(nk)^{ka}}{\Gamma(ka+nk)} = 1$$

hold. Thus, the limit

$$\lim_{n\to\infty}\frac{\Gamma(ka)}{k^{ka}\prod_{s=1}^{k}\Gamma\Big(a+\dfrac{s-1}{k}\Big)} = c$$

must be a certain constant c to be determined. Let us put $a = 1/k$ and notice that $\Gamma(1) = 1$. Then we have

$$\frac{1}{kc} = \prod_{s=1}^{k-1} \Gamma\Big(\frac{s}{k}\Big) = \prod_{s=1}^{k-1} \Gamma\Big(\frac{k-s}{k}\Big).$$

But then, we deduce from the Gauss reflection formula, that

$$\frac{1}{k^2c^2} = \prod_{s=1}^{k-1} \Gamma\Big(\frac{s}{k}\Big) \cdot \Gamma\Big(\frac{k-s}{k}\Big) = \prod_{s=1}^{k-1} \frac{\pi}{\sin\frac{\pi s}{k}},$$

or symplifying as

$$k^2c^2\pi^{k-1} = \prod_{s=1}^{k-1} \sin\frac{\pi s}{k}.$$

But then the Lemma 3.13.1 gives $\prod_{s=1}^{k-1} \sin\frac{\pi s}{k} = \frac{k}{2^{k-1}}$ and hence

$$c = (2\pi)^{-\frac{1}{2}(k-1)}k^{-\frac{1}{2}}$$

as required. □

Exercise 3.26. *Prove Legendre's duplication formula*

$$\sqrt{\pi}\,\Gamma(2a) = 2^{2a-1}\Gamma(a)\Gamma(a+\frac{1}{2}),$$

by imitating the proof for the Gauss's multiplication theorem.

Chapter 4

Linear Differential Equations

Before we turn to hypergeometric equations, let us first examine differential equations of the form

$$\frac{d^n w}{dz^n} + p_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + p_1(z)\frac{dw}{dz} + p_0(z)w = 0$$

where the coefficients $p_j(z),\ j = 0, 1, \cdots, n-1$ are meromorphic functions in a **domain** (i.e., some open connected set), that is they are analytic except at the isolated poles, and $w(z)$ is a solution of the equation. We will be interested in knowing when will such a solution exist, and how do the properties of the coefficients affect those of the solutions, etc.

The simplest equation of this type is first order equation

$$\frac{dw}{dz} + p(z)w = 0.$$

We can integrate this equation to obtain

$$w = A \exp\big(-\int p(z)\, dz\big),$$

where A is an arbitrary non-zero constant. So relation between the coefficient $p(z)$ and the solution $w(z)$ is explicit. On the other hand, no such explicit fomulation for the above higher order equation is known. We shall restrice ourselves to second order linear differential equations of the form

$$\frac{d^2 w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0, \tag{4.1}$$

where $p(z)$ and $q(z)$ are meromorphic in a domain. This is because the theory of second order equations can easily be extended to higher order equations, and also because many important functions that we shall encounter later satisfy second order equations.

Definition 4.0.1. *A point z_0 is said to be an* ***ordinary point*** *of the equation (4.1) if the functions $p(z)$ and $q(z)$ are both analytic at z_0 (i.e., they are differentiable in a neighbourhood of z_0). Points that are not ordinary points are called* ***singular points****.*

It turns out that the properties of solution at z_0 of (**??**) are influenced by how (or no) "singular" of the coefficients $p(z)$, $q(z)$ there. In general, if the point z_0 is an ordinary point for $p(z)$, $q(z)$, then one can solve any reasonable initial value problem at z_0. But it turns out that the most interesting second order differential equations that people often encounter in all kinds of problems, as we shall see later, have certain "degrees" of singularity at z_0. The most "mild" singularity is called *regular singularity* which will be defined later. Our knowledge about how singularities that are worse than regular singularities is still limited, and yet there are a large number of important second order differential equations fall in this category. We refer to Ince [] and Slavyanov [] for more indepth discussions.

4.1 Ordinary Points

Theorem 4.1 (L. I. Fuchs 1866)**.** *Let z_0 be an ordinary point of the equation (4.1) and let a_0 and a_1 be two arbitrary constants. Then the equation (4.1) has a unique* analytic *solution $w(z)$ that satisfies the* initial condition *$w(z_0) = a_0$ and $w'(z_0) = a_1$.*

Proof. Without loss of generality, we may assume that z_0 in order to simplify the argument. For one can consider $z' = z - z_0$ to recover the general case. Since both $p(z)$, $q(z)$ are analytic at z_0, so let us write their Taylor expansions in the forms

$$p(z) = \sum_{k=0}^{\infty} p_k z^k, \qquad q(z) = \sum_{k=0}^{\infty} q_k z^k,$$

and both converge in $|z| < R$. We substitute formally our "solution"

$$w(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots$$

into the (4.1)

$$\frac{d^2 w}{dz^2} + \Big(\sum_{k=0}^{\infty} p_k z^k\Big) \frac{dw}{dz} + q(z)\Big(\sum_{k=0}^{\infty} q_k z^k\Big)\, w = 0,$$

and equating the coefficients. This yields

$$
\begin{aligned}
-2a_2 &= a_1p_0 + a_0q_0,\\
-2\cdot 3a_3 &= 2a_2p_0 + a_1p_1 + a_1q_0 + a_0q_1,\\
&\cdots\\
&\cdots\\
-(n-1)ka_k &= (k-1)a_{k-1}p_0 + (k-2)a_{k-2}p_1 + \cdots a_1p_{k-2}+\\
&\quad a_{k-2}q_0 + a_{k-3}q_1 + \cdots + a_1q_{k-3} + a_0q_{k-2}
\end{aligned}
$$

for all $n \geq 2$. The above equations show that one can express any a_n, successively, as linear combination of a_0 and a_1. Notice that the above recurrence on a_n is only formal. That is, we still need to justify if the sum $w(z) = \sum_{k=0}^{\infty} a_k z^k$ really converges.

Let

$$M = M_r = \max_{|z|=r} |p(z)|, \qquad N = N_r = \max_{|z|=r} |q(z)|$$

for $r < R$. Then the Cauchy inequality gives

$$|p_k| \leq \frac{M}{r^k}, \qquad |q_k| \leq \frac{N}{r^k}$$

and we may write

$$|p_k| \leq \frac{K}{r^k}, \qquad |q_k| \leq \frac{K}{r^{k+1}},$$

where $K = \max\{M,\, Nr\}$. Writing $b_0 = |a_0|$ and $b_1 = |a_1|$. Then we have

$$2|a_2| \leq b_1|p_0| + b_0|q_0| \leq b_1K + b_0K/r \leq 2b_1K + b_0K/r.$$

We define $2b_2 = 2b_1K + b_0K/r$. Hence $|a_2| \leq b_2$. Similarly, we have

$$
\begin{aligned}
2\cdot 3|a_3| &\leq 2|a_2||p_0| + |a_1||p_1| + |a_1||q_0| + |a_0||q_1|\\
&\leq 2b_2K + b_1K/r + b_1K/r + b_0K/r^2\\
&= 3b_2K + b_1K/r + b_1K/r + b_0K/r^2.
\end{aligned}
$$

We define

$$
\begin{aligned}
2\cdot 3b_3 :&= 3b_2K + b_1K/r + b_1K/r + b_0K/r^2\\
&= 3b_2K + 2b_1K/r + b_0K/r^2
\end{aligned}
$$

Hence $|a_3| \leq b_3$.

Continuing this process yields $|a_n| \leq b_n$ where

$$(k-1)kb_k := kb_{k-1}K + (k-1)b_{k-2}K/r + \cdots b_0K/r^{k-1}.$$

Replacing the k by $k-1$ in the above equation and multiplying both sides of the resulting equation by $\frac{1}{r}$ yield

$$(k-2)(k-1)b_{k-1}/r = (k-1)b_{k-2}K/r + (k-2)b_{k-3}K/r^2 + \cdots b_0K/r^{k-1}.$$

Combining these two equations yields the recurrence relation

$$(k-1)kb_k = kb_{k-1}K + (k-2)(k-1)b_{k-1}/r,$$

or

$$\frac{b_k}{b_{k-1}} = \frac{K}{k-1} + \frac{k-2}{kr} \to \frac{1}{r}$$

as $k \to \infty$. This shows that the radius of convergence of $\sum b_k r^k$ is r. However, since, $|a_k| \le b_k$, so it follows that the radius of convergence of $w(z) = \sum a_k z^k$ cannot be less than r. But $r < R$ is arbitrary, this implies that $w(z) = \sum a_k z^k$ has radius of convergence at least R. It follows that $w(z) = \sum a_k z^k$ is analytic function at the origin. Since the power series $w(z) = \sum a_k z^k$ converges uniformly and absolutely in $|z| < R$, so one may differentiate it term by term and series multiplication and rearrangements are all valid. So one can substitute the series into the equation (4.1) to verify that the series $w(z) = \sum a_k z^k$ is indeed a unique analytic solution in the neighbourhood of the origin. In particular, $w(0) = a_0$ and $w'(0) = a_1$. □

It is known that two vectors $v_1,\ v_2$ in a vector space are linearly independent if $a_1v_1 + a_2v_2 = 0$ necessary implies that $a_1 = a_2 = 0$. This condition is also sufficient. When translated into two functions $f(z),\ g(z)$, say, defned in a neighbourhood of z_0, it becomes a necessary and sufficient condition for the non-vanishing of

$$W(f,\, g)(z) = \begin{vmatrix} f(z) & g(z) \\ f'(z) & g'(z) \end{vmatrix}$$

throughtout the neighbourhood of z_0. The $W(f,\, g)(z)$ is called the **Wronskian** of the two functions.

Exercise 4.2 (Copson)**.** *Let $f(z)$ and $g(z)$ be two linearly independent solutions of (4.1) in a neighbourhood of z_0. Show that*

$$W(f,\, g)'(z) + p(z)W(f,\, g)(z) = 0.$$

Suppose the $p(z)$ in (4.1) has a pole at z_0, and that $f_1(z)$ and $g_1(z)$ are two linearly independent solutions of (4.1) obtained after the original pair $f(z)$ and $g(z)$ being analytically continued around z_0 in a single anti-clockwise revolution. Deduce that

$$W(f_1,\, g_1)'(z) = e^{-2\pi i R} W(f,\, g)(z)$$

where R is the residue of $p(z)$ at z_0.

Exercise 4.3 (Whittaker & Watson)**.** *Show that the equation*

$$(1-z^2)u'' - 2zu' + \frac{3}{4}u = 0,$$

has the origin as an ordinary point and that both the series

$$u_1 = 1 - \frac{3}{8}z^2 - \frac{21}{128}z^4 - \cdots,$$

and

$$u_2 = z + \frac{5}{24}z^3 + \frac{15}{128}z^5 + \cdots,$$

are two linearly independent solutions of the above equation. Find the general coefficient in each series and show that their radii of convergence are both 1.

Exercise 4.4.

Find two solutions of the equation

$$\frac{d^2w}{dz^2} - zw = 0$$

about $z = 0$ *such that* $w_1 = z + \frac{1}{12}z^4 + \cdot$ *and* $w_2 = 1 + \frac{1}{6}z^3 + \cdots$. *You need to obtain the general coefficients for the expansions. What kind of singularity does the origin of the differential equation has? Investigate the region of convergence of the series solutions.*

4.2 Regular Singular Points

Definition 4.2.1. *A point* z_0 *is said to be a* ***regular singular point*** *of the equation (4.1) if the functions* $(z-z_0)p(z)$ *and* $(z-z_0)^2q(z)$ *are both analytic at* z_0 *. Points that are not regular singular points are called* ***irregular singular points****.*

Again without loss of generality, we may assume that $z_0 = 0$. That is, $zp(z)$ and $z^2q(z)$ have Taylor expansions

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k, \qquad z^2q(z) = \sum_{k=0}^{\infty} q_k z^k, \tag{4.2}$$

where the coefficients p_0, q_0 and q_1 are not all zero.

Suppose we write the solution of (4.1) in the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{\alpha+k}, \tag{4.3}$$

where α is a constant to be determined. Multiply z^2 on both sides of the equation (4.1) and substitute this expansion $f(z)$ into (4.1) yields

$$\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k z^{\alpha+k} + \Big(\sum_{k=0}^{\infty} p_k z^k\Big)\Big(\sum_{k=0}^{\infty}(\alpha+k)a_k z^{\alpha+k}\Big)$$
$$+\Big(\sum_{k=0}^{\infty} q_k z^k\Big)\Big(\sum_{k=0}^{\infty} a_k z^{\alpha+k}\Big) = 0.$$

Collecting the coefficients of $z^{k+\alpha}$ yields:

$$\begin{aligned}(\alpha+k)(\alpha+k-1)a_k &+ \sum_{j=0}^{k}(\alpha+k-j)p_{k-j}a_j \\ &+ \sum_{j=0}^{k} q_{k-j}a_j = 0,\end{aligned} \tag{4.4}$$

or

$$F(\alpha+k)a_k + \sum_{j=0}^{k-1} a_j\big((\alpha+j)p_{k-j} + q_{k-j}\big) = 0, \qquad k \geq 1, \tag{4.5}$$

where we have set

$$F(\alpha+k) = (\alpha+k)(\alpha+k-1) + p_0(\alpha+k) + q_0 \tag{4.6}$$

and

$$F(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0 \tag{4.7}$$

when $k = 0$. The equation $F(\alpha) = 0$ is called the **indicial equation** and its roots the **(characteristic) exponents** of the regular singularity z_0. We see that if we choose α to be one of the two roots of the equation $\alpha(\alpha-1)+p_0\alpha+q_0 = 0$, then a_0 above can be arbitrarily chosen. If we assume that this quadratic equation has two distinct roots which do not differ by an integer, then the coefficient $F(\alpha+k)$ from (4.6) is non zero for any $k \geq 1$, implying that one can determine the coefficient a_k in terms of a_s for $0 \leq s \leq k-1$, and hence in terms of a_0 after applying the recurrence relation (4.5) successively. We will then obtain two different formal expansions with respect to two different exponents. Thus we have

Proposition 4.2.1. *Suppose that z_0 is a regular singular point for the equation (4.1) and that the indicial equation has two distinct roots. Then the equation (4.1) has two different formal power series expansions solutions.*

If we havea double root, then it is clear that we will find only one formal series expansion solution for (4.1). If the two roots are differed by an integer instead, then the coefficient $F(\alpha+k)$ will vanish at $k=n$, say (assuming that α is the smaller root). This means that the coefficient a_n and indeed the subsequent a_k could be arbitrarily chosen. So we do not obtain a formal series expansion in both these cases discussed. We shall discuss the cases when the indicial equation has double root or two roots that are differed by an integer at a later stage.

Theorem 4.5. *Let the equation (4.1) to have a regular singular point at $z=0$, and that $zp(z)$ and $z^2q(z)$ are analytic in $|z|<R$. Suppose that the indicial equation of (4.1) at the regular singular point $z=0$ to have two distinct roots $\alpha,\ \alpha'$ such that their difference is not zero or an integer. Then the radius of convergence of the solution (4.3) $w(z)=\sum_{k=0}^{\infty} a_k z^{\alpha+k}$ at $z=0$ is at least R.*

Proof. We may assume without loss of generality that the series for $w(z)$ does not terminate. Let $\alpha,\ \alpha'$ be two distinct roots of (4.1) that are not equal and also do not differ by an integer. Substituting the formal solution $w(z)$ into the equation (4.1), then it is easy to show that the recurrence formula (E:recurrence-0) can be written in the form

$$k(k+\alpha-\alpha')a_k=-\sum_{j=0}^{k-1} a_j[(\alpha+j)p_{k-j}+q_{k-j}].$$

Let $b_k=|a_k|$, $0\le k<\delta=|\alpha-\alpha'|$. Let $m=[\delta]+1$ and $|\alpha|=\tau$. Then

$$\begin{aligned} m(m-\delta)|a_m| &\le |m(m+\alpha-\alpha')a_m| \\ &=\Big|\sum_{j=0}^{m-1} a_j[(\alpha+j)p_{m-j}+q_{m-j}]\Big| \\ &\le \sum_{j=0}^{m-1} b_j[(\tau+j)|p_{m-j}|+|q_{m-j}|], \end{aligned}$$

where $b_j=|a_j|$. Let $M,\ N$ be the maximum values of $|zp(z)|$ and $|z^2q(z)|$ respectively, on $|z|=r=|\alpha|$. Then

$$|p_k|\le\frac{M}{r^n},\qquad |q_k|\le\frac{N}{r^n}$$

for $r<R$ and so

$$|p_k|\le\frac{K}{r^n},\qquad |q_k|\le\frac{K}{r^n}$$

where $K = \max\{M,\, N\}$. Substituting these bounds for $|p_k|$ and $|q_k|$ implies that

$$|a_m| \le b_m,$$

where

$$m(m-\delta)b_m := \sum_{j=0}^{m-1} K b_j[(\tau+j+1)/r^{m-j}].$$

Similarly, we can show that $|a_k| \le b_k$ when $k \ge m$, where

$$k(k-\delta)b_k := \sum_{j=0}^{k-1} K b_j[(\tau+j+1)/r^{k-j}].$$

Combining this and a similar one with k replaced by $k-1$ show that b_k satisfies a recurrence formula given by

$$k(k-\delta)b_k - (k-1)(k-1-\delta)b_{k-1}/r = K(k+\tau)b_{k-1}/r.$$

Thus, we have

$$\frac{b_k}{b_{k-1}} = \frac{(k-1)(k-1-\delta)}{k(k-\delta)r} + \frac{K(k+\tau)}{k(k-\delta)r},$$

and so

$$\lim_{k\to\infty} \frac{b_k}{b_{k-1}} = \frac{1}{r},$$

proving that the series $\sum b_k z^k$ has the radius of convergence r. Since $|a_k| \le b_k$ so the comparison test implies that $\sum a_k z^k$ has a radius of convergence at least r. But r is arbitrary but less than R, so the series must be convergent in $|z| < R$. Hence the series $z^\alpha \sum a_k z^k$ converges uniformly and absolutely in $|z| \le R$. Similar argument can be applied to $z^{\alpha'} \sum a_k z^k$ to show this is the second linearly independent solution of (4.1). □

We note that at least one of the two linearly independent solutions considered above has the origin to be a branch point.

We now discuss when the regular singular for the equation (4.1) is at infinity. We have

Proposition 4.2.2. *The differential equation*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0$$

has a regular singular point at ∞ if and only if $zp(z)$ and $z^2q(z)$ are analytic at ∞.

Proof. Let $w(z)$ be a solution of the differential equation. Then the behaviour of $w(1/t)$ near $t = 0$ is equivalent to that of $w(z)$ at ∞. The same applies to $p(z),\ q(z)$. Without loss of generality, we may write $\tilde{w}(t) = w\big(\frac{1}{t}\big)$. Then one can easily verify that $\tilde{w}(t)$ satisfies the differential equation

Exercise 4.6.

$$\frac{d^2\tilde{w}}{dt^2} + \Big(\frac{2}{t} - \frac{1}{t^2}p\big(\frac{1}{t}\big)\Big)\frac{d\tilde{w}}{dt} + \frac{1}{t^4}q\big(\frac{1}{t}\big)\tilde{w}(t) = 0. \tag{4.8}$$

Thus (4.8) has a regular singular point at $t = 0$ if and only if

$$t\Big(\frac{2}{t} - \frac{1}{t^2}p\big(\frac{1}{t}\big)\Big) = 2 - \frac{1}{t}p\big(\frac{1}{t}\big), \qquad t^2 \cdot \frac{1}{t^4}q\big(\frac{1}{t}\big) = \frac{1}{t^2}\,q\big(\frac{1}{t}\big),$$

are analytic at $t = 0$. That is, if and only if $zp(z)$ and $z^2q(z)$ are analytic at ∞. □

This shows that

$$p(z) = \frac{p_0}{z} + \frac{p_1}{z^2} + \frac{p_3}{z^3} + \cdots, \qquad q(z) = \frac{q_0}{z^2} + \frac{q_1}{z^3} + \frac{q_4}{z^3} + \cdots.$$

We note in this case, and assuming that the difference of the two roots of the indicial equation do not differ by an integer or 0, that the two solutions can be written as

$$w(z) = z^{-\alpha}\sum_{k=0}^{\infty} a_k z^{-k}, \qquad w(z) = z^{-\alpha'}\sum_{k=0}^{\infty} a'_k z^{-k},$$

where $\alpha,\ \alpha'$ are solutions of

$$\alpha^2 - (p_0 - 1)\alpha + q_0 = 0.$$

Exercise 4.7. *Show that the equation*

$$zw'' + w' + zw = 0$$

has a regular singular point at $z = 0$ and an irregular singular point at ∞.

4.3 Reduction of Order

We shall deal with the cases when the difference of the two roots of the indicial equation is an integer (including zero). If the two roots are identical, then we see from the last section that we could obtain at most one series expansion. If, however, that the roots differ by an integer, then the recurrence relation (4.5) will not give useful information of any second linearly independent solution. The main idea for getting a second solution is by the well-known method of **reduction of order** as explained below.

Theorem 4.8 (L. I. Fuchs (1866))**.** *Let z_0 be a regular singular point of the differential equation*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

Suppose the exponents of the indicial equation, denoted by α and α', are such that $\alpha - \alpha' = s \in \mathbb{N} \cup \{0\}$. Then the equation possesses a fundamental set of solutions in the form

$$w_1(z) = \sum_{k=0}^{\infty} a_k z^{k+\alpha}, \tag{4.9}$$

and

$$w_2(z) = g_0 w_0(z) \log z + z^{\alpha+1} \sum_{k=0}^{\infty} b_k z^k \tag{4.10}$$

if $s = 0$, and

$$w_2(z) = g_s w_0(z) \log z + z^{\alpha'} \sum_{k=0}^{\infty} c_k z^k, \tag{4.11}$$

if $s \neq 0$.

Proof. Let $w_0(z)$ be the first solution (4.9) of the differential equation. The idea of *reduction of order* works as follows. First we introduce a new dependent variable v by setting $w = w_0(z)v$. Then it is easy to verify that v satisfies the differential equation

$$\frac{d^2v}{dz^2} + \Big(\frac{2w_0'}{w_0} + p(z)\Big)\frac{dv}{dz} = 0.$$

Simple integrations of this equation give

$$v(z) = A + B\int^z \frac{1}{w_0(z)^2} \exp\Big(-\int^z p(z)\,dz\Big)\,dz,$$

where A, B are arbitrary constants for which $B \neq 0$. This shows that a second solution around $z = 0$ is given by

$$w(z) = w_0(z)v = w_0(z) \int^z \frac{1}{w_0(z)^2} \exp\Big(- \int^z p(z)\, dz \Big)\, dz.$$

Since α and $\alpha' = \alpha - s$ are roots of the indicial equation

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

so that $p_0 = 1 + s - 2\alpha$. Hence

$$\begin{aligned}
&\frac{1}{w_0(z)^2} \exp\Big(- \int^z p(z)\, dz \Big) = \\
&\qquad \frac{1}{z^{2\alpha}\big(\sum_{k=0}^\infty a_k z^k\big)^2} \exp\Big\{ \int^z \Big(\frac{2\alpha - 1 - s}{z} - p_1 - p_2 z - \cdots \Big)\, dz \Big\} \\
&\qquad = \frac{z^{-s-1}}{\big(\sum_{k=0}^\infty a_k z^k\big)^2} \exp\Big\{ - \int^z (p_1 + p_2 z + \cdots)\, dz \Big\} \\
&\qquad = z^{-s-1} g(z),
\end{aligned}$$

where $g(0) = 1/a_0^2$. Since $a_0 \neq 0$, so $\big(\sum_{k=0}^\infty a_k z^k\big)^{-2}$ is also analytic at $z = 0$. Thus, we may write $g(z) = \sum_{k=0}^\infty g_k z^k$. Substituting this series of $z^{-s-1}g(z)$ into $w(z)$ yields

$$\begin{aligned}
w(z) &= w_0(z) \int^z z^{-s-1} \sum_{k=0}^\infty g_k z^k \, dz \\
&= w_0(z) \Big(\sum_{k=0}^{s-1} \frac{g_k z^{k-s}}{k - s} + g_s \log z + \sum_{k=s+1}^\infty \frac{g_k z^{k-s}}{k - s} \Big).
\end{aligned}$$

Thus, we see that if $s = 0$ (double root α), then we have

$$w(z) = g_0 w_0(z) \log z + z^{\alpha+1} \sum_{k=0}^\infty b_k z^k,$$

where $g_0 \neq 0$. The solution has a logarithmic branch point. If, however, $s \neq 0$, then we have

$$\begin{aligned}
w(z) &= g_s w_0(z) \log z + z^{\alpha - s} \Big(\sum_{k=0}^{s-1} \frac{g_k z^k}{k - s} + \sum_{k=s+1}^\infty \frac{g_k z^k}{k - s} \Big) \\
&= g_s w_0(z) \log z + z^{\alpha'} \sum_{k=0}^\infty c_k z^k,
\end{aligned}$$

where g_s may be zero. If that is the case then the second solution has no logarithmic term. $\square$

Frobenius later simplified Fuchs's method by introducing his **Frobenius method** in 1873 [11] when he was at the age of 24, even before he presented his *Habilitationsschrift* (thesis). We shall return to its discussion when time permits later.

Exercise 4.9 (Whittaker and Watson)**.**

Show that the equation

$$w'' + \frac{1}{z}w' - m^2 w = 0$$

has power series solutions about $z = 0$ given by

$$w_1(z) = \sum_{k=0}^{\infty} \frac{m^{2k} z^{2k}}{2^{2k}(k!)^2},$$

and

$$w_2 = w_1(z)\log z - \sum_{k=1}^{\infty} \frac{m^{2k} z^{2k}}{2^{2k}(k!)^2}\Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\Big),$$

and that these series converge for all z.

Exercise 4.10. *Prove that the equation*

$$z^2 w'' + (z+1)w' + w = 0,$$

has an irregular singularity at $z = 0$, and show that one can find only one series solution about $z = 0$ with the method discussed int this chapter.

Exercise 4.11. *Show that the equation*

$$z^3 w'' + z^2 w' + w = 0,$$

has an irregular singularity at $z = 0$, and that it is impossible to find a series solution there.

Chapter 5

Hypergeometric Equations

Here comes next to consider second order differential equations with more than one regular singular points in the complex plane $\mathbb{C}$. When all singular points of the differential equation are regular singular, then we call the equation of **Fuchsian- type equation**. Those equations with three regular singular points are particularly interesting and are *very* classical. The canonical form of the equation is called **Gauss hypergeometric equation** which includes almost all the **special functions** that we usually encounter such as **Legendre equation**, **Laguerre equation** and **Hermite equation**, etc and their associated orthogonal polynomials. The canonical form of the equations with four regular singular points is called **Heun equation**, which includes important special cases such as **Lamé** equation, **Mathieu equation**, **Prolate/Oblate Spheroidal equation**, **Ellipsoidal equations**, etc, are still poorly understood.

5.1 Fuchsian-type Differential Equations

Theorem 5.1. *Let a_1, a_2, $\cdots a_n$ and ∞ be the regular singular points of*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0. \tag{5.1}$$

Then we have

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z-a_k}, \quad q(z) = \sum_{k=1}^{n} \Big(\frac{B_k}{(z-a_k)^2} + \frac{C_k}{z-a_k}\Big), \tag{5.2}$$

and the indicial equation at each a_k is given by

$$\alpha^2 + (A_k - 1)\alpha + B_k = 0.$$

Suppose $\{\alpha_k,\, \alpha_k'\}$ and $\{\alpha_\infty,\, \alpha_\infty'\}$ denote, respectively, the exponents of the corresponding indicial equations at a_1, a_2, $\cdots a_n$ and ∞. Then we have

$$(\alpha_\infty + \alpha_\infty') + \sum_{k=1}^{n} (\alpha_k + \alpha_k') = n - 1,$$

Proof. Since $p(z)$ has at most a first order pole at each of the points $a_1,\ a_2,\ \cdots a_n$, so the function $\phi(z)$ defind in

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k} + \varphi(z)$$

where $\varphi(z)$ is an entire function in $\mathbb{C}$, and where the A_k is the residue of $p(z)$ at a_k $(k = 1,\ 2,\ \cdots n)$. On the other hand, the ∞ is also a regular singular point of $p(z)$, it follows from Proposition 4.2.2 that $zp(z)$ is analytic at ∞. That is, $p(z) = O(1/z)$. So $\varphi(z) \to 0$ and hence must be the constant zero by applying Liouville's theorem from complex analysis. Hence

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}.$$

Similarly, we have

$$q(z) = \sum_{k=1}^{n} \Big(\frac{B_k}{(z - a_k)^2} + \frac{C_k}{z - a_k} \Big) + \psi(z),$$

where $\psi(z)$ is entire function. But $z^2 q(z)$ is analytic at ∞, so $q(z) = O(1/z^2)$ as $z \to \infty$. This shows that $\psi(z) \equiv 0$. The Proposition 4.2.2 again implies that

$$\sum_{k=1}^{n} C_k = 0.$$

Considering the Taylor expansions around the regular singular point a_k

$$(z - a_k)p(z) = \sum_{j=0}^{\infty} p_j \cdot (z - a_k)^j, \quad (z - a_k)^2 q(z) = \sum_{j=0}^{\infty} q_j \cdot (z - a_k)^j$$

and

$$w(z) = \sum_{j=0}^{\infty} a_j (z - a_k)^{\alpha + j}$$

yields

$$\alpha^2 + (A_k - 1)\alpha + B_k = 0.$$

for $k = 1,\ 2,\ \cdots,n$. We now deal with the indicial equation at $z = \infty$. Since the ∞ is also a regular singular point, so the Proposition 4.2.2 implies that

$$\begin{aligned}
\lim_{t\to 0} t\Big(\frac{2}{t} - \frac{1}{t^2}p\Big(\frac{1}{t}\Big)\Big) &= 2 - \lim_{z\to\infty} zp(z) \\
&= 2 - \lim_{z\to\infty} z\left\{\sum_{k=1}^{n} \frac{\prod_{\substack{j=1\\ j\neq k}}^{n} A_k(z-a_j)}{\prod_{j=1}^{n}(z-a_j)}\right\} \\
&= 2 - \lim_{z\to\infty} \frac{\big(\sum_{k=1}^{n} A_k\big)z^n + O(z^{n-1})}{z^n + O(z^{n-1})} \\
&= 2 - \sum_{k=1}^{n} A_k. \qquad (5.3)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\lim_{t\to 0} t^2 \cdot \frac{1}{t^4} q\Big(\frac{1}{t}\Big) &= \lim_{z\to\infty} z^2 q(z) \\
&= \lim_{z\to\infty} z^2 \sum_{k=1}^{n} \Big(\frac{B_k}{(z-a_k)^2} + \frac{C_k}{z-a_k}\Big) \\
&= \sum_{k=1}^{n} B_k + \lim_{z\to\infty} z \sum_{k=1}^{n} \frac{C_k}{1-a_k/z} \\
&= \sum_{k=1}^{n} B_k + \lim_{z\to\infty} z \sum_{k=1}^{n} C_k\big[1 + \frac{a_k}{z} + O\big(\frac{1}{z^2}\big)\big] \\
&= \sum_{k=1}^{n} B_k + 0 + \sum_{k=1}^{n} a_k C_k, \qquad (5.4)
\end{aligned}$$

because $\sum_{k=1}^{n} C_k = 0$. The indicial equation for the point $z = 0$ is of the form

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

where $p_0 = \lim_{z\to 0} zp(z)$ and $q_0 = \lim_{z\to 0} z^2 q(z)$. Thus, the point at infinity is given by

$$\alpha^2 + \Big(1 - \sum_{k=1}^{n} A_k\Big)\alpha + \sum_{k=1}^{n}(B_k + a_k C_k) = 0.$$

Let α_k, α'_k be the exponents of the indicial equation $\alpha^2 + (A_k - 1)\alpha + B_k = 0$ for $k = 1, 2, \cdots, n$, and α_∞, α'_∞ be the exponents of the indicial equation at ∞. Thus we deduce

$$\sum_{k=1}^{n}(\alpha_k + \alpha'_k) = -\sum_{k=1}^{n}(A_k - 1) = \sum_{k=1}^{n}(1 - A_k) = n - \sum_{k=1}^{n} A_k,$$

and $\alpha_\infty + \alpha'_\infty = 1 - \sum_{k=1}^{n} A_k$. Hence we deduce

$$\sum_{k=1}^{n}(\alpha_k + \alpha'_k) + (\alpha_\infty + \alpha'_\infty) = n - 1,$$

as required. $\square$

Exercise 5.2 (Kline). *We note that if ∞ is an ordinary point, then show that*

$$p(z) = \frac{2}{z} + \frac{p_2}{z^2} + \cdots ,$$

and

$$q(z) = \frac{q_4}{z^4} + \frac{q_5}{z^5} + \cdots .$$

Suppose the ∞ in Theorem 5.1 is an ordinary point. Then show that the following identities hold.

$$\sum_{k=1}^{n} A_k = 2,$$
$$\sum_{k=1}^{n} C_k = 0,$$
$$\sum_{k=1}^{n} (B_k + a_k C_k) = 0,$$
$$\sum_{k=1}^{n} (2a_k B_k + a_k^2 C_k) = 0.$$

5.2 Differential Equations with Three Regular Singular Points

We again consider differential equation in the form

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0. \tag{5.5}$$

The following statement was first written down by Papperitz in 1885.

Theorem 5.3. *Let the differential equation (5.5) has regular singular points at ξ, η and ζ in $\mathbb{C}$. Suppose the α, α'; β, β'; γ, γ' are, respectively, the exponents at the regular singular points ξ, η and ζ. Then the differential equation must assume the form*

$$\begin{aligned} \frac{d^2w}{dz^2} &+ \left\{\frac{1-\alpha-\alpha'}{z-\xi} + \frac{1-\beta-\beta'}{z-\eta} + \frac{1-\gamma-\gamma'}{z-\zeta}\right\}\frac{dw}{dz} - \\ &- \left\{\frac{\alpha\alpha'}{(z-\xi)(\eta-\zeta)} + \frac{\beta\beta'}{(z-\eta)(\zeta-\xi)} + \frac{\gamma\gamma'}{(z-\zeta)(\xi-\eta)}\right\}\times \\ &\times \frac{(\xi-\eta)(\eta-\zeta)(\zeta-\xi)}{(z-\xi)(z-\eta)(z-\zeta)}\, w = 0. \end{aligned} \tag{5.6}$$

Proof. The equation has regular singular points at ξ, η and ζ in $\mathbb{C}$. Thus the functions

$$P(z) = (z-\xi)(z-\eta)(z-\zeta)\,p(z),$$

and

$$Q(z) = (z-\xi)^2(z-\eta)^2(z-\zeta)^2\,q(z)$$

are both entire functions (complex differentiable everywhere). Besides, since the point at "∞" is an analytic point for both $p(z)$ and $q(z)$, so we deduce from the Exercise 5.2 that the functions that $P(z)$ and $Q(z)$ have degree two. That is,

$$p(z) = \frac{P(z)}{(z-\xi)(z-\eta)(z-\zeta)} = \frac{A}{z-\xi} + \frac{B}{z-\eta} + \frac{C}{z-\zeta},$$

where $A+B+C=2$ (by Exercise 5.2 again). Similarly,

$$\begin{aligned}(z-\xi)(z-\eta)(z-\zeta)q(z) &= \frac{Q(z)}{(z-\xi)(z-\eta)(z-\zeta)}\\ &= \frac{D}{z-\xi} + \frac{E}{z-\eta} + \frac{F}{z-\zeta}.\end{aligned}$$

Substitute $w(z) = \sum_{k=0}^{\infty} w_k (z-\xi)^{\alpha+k}$ into the differential equation and and note the coefficient q takes the form

$$q(z) = \frac{D}{(z-\xi)(z-\zeta)} + \frac{E(z-\xi)^2}{(z-\eta)^2(x-\zeta)} + \frac{F(z-\xi)^2}{(z-\eta)(z-\zeta)^2}.$$

So $q_0 = \frac{D}{(\xi-\eta)(\xi-\zeta)}$. Thus the indicial equation at $z=\xi$ is given by

$$\alpha(\alpha-1) + A\alpha + \frac{D}{(\xi-\eta)(\xi-\zeta)} = 0,$$

where $\alpha + \alpha' = -(A-1) = 1-A$ or $A = 1-\alpha-\alpha'$, and

$$\alpha\alpha' = \frac{D}{(\xi-\eta)(\xi-\zeta)},$$

or $D = (\xi-\eta)(\xi-\zeta)\alpha\alpha'$. Similarly, we have , at the regular singular points η, ζ that $B = 1-\beta-\beta'$ and $C = 1-\gamma-\gamma'$, and hence

$$E = (\eta-\xi)(\eta-\zeta)\beta\beta', \qquad F = (\zeta-\xi)(\zeta-\eta)\gamma\gamma'.$$

This yields the differential equation of the required form. $\square$

Remark 5.2.1. *We note that the constraint $A+B+C=2$ implies that $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.*

Example 5.2.1. *Suppose that a second order linear differential equation has only one regular singular point at $z = 0$ and infinity is an ordinary point. Show that the equation is of the form $zw'' + 2w' = 0$.*

Proof. Let the equation be in the form $w''+p(z)w'+q(z)w = 0$. Since $z = 0$ is a regular singular point. So $P(z) = zp(z)$ is an entire function. But $z = \infty$ is an ordinary point, so $P(z) = zp(z) = z\big(\frac{A}{z} + \frac{p_2}{z^2} + \cdots\big) = A + \frac{p_2}{z} + \cdots$ as $z \to \infty$ by Exercise 5.2. We conclude that P is the constant A by Liouville's theorem. We also deduce that $A = 2$ since $p(z) = \frac{2}{z} + \frac{p_2}{z^2} + \cdots$. Similarly, we have $Q(z) = z^2 q(z)$ to be an entire function. But $z = \infty$ is an ordinary point, so $Q(z) = z^2 q(z) = z^2\big(\frac{q_4}{z^4} + \frac{q_5}{z^5} + \cdots\big) = \frac{q_4}{z^2} + \frac{q_4}{z^3} + \cdots \to 0$ as $z \to \infty$ by Exercise 5.2 again. Thus $Q(z) \equiv 0$ by Liouville's theorem again. Hence $q(z) \equiv 0$. We conclude that $w'' + \frac{2}{z}w' + 0 = 0$. □

Exercise 5.4. *Suppose a second order linear differential equation has both the origin and the infinity to be regular singular points, and that their respective exponents are $\{\alpha,\, \alpha'\}$ and $\{\alpha_\infty,\, \alpha'_\infty\}$. Derive the exact form of the differential equation and show that $\alpha+\alpha'+\alpha_\infty+\alpha'_\infty = 0$ and $\alpha\cdot\alpha' = \alpha_\infty\cdot\alpha'_\infty$ hold.*

If we now choose $\eta = \infty$ in the differential equation in Theorem 5.3, then we easily obtain

$$\frac{d^2w}{dz^2} + \Big\{\frac{1-\alpha-\alpha'}{z-\xi} + \frac{1-\gamma-\gamma'}{z-\zeta}\Big\}\frac{dw}{dz} - \Big\{\frac{\alpha\alpha'(\xi-\zeta)}{z-\xi} + \beta\beta' + \frac{\gamma\gamma'(\zeta-\xi)}{z-\zeta}\Big\}\times \times \frac{w}{(z-\xi)(z-\zeta)} = 0. \tag{5.7}$$

For convenience we could set $\xi = 0$ and $\gamma = 1$. So this differential equation will be further reduced to the form

$$\frac{d^2w}{dz^2} + \Big\{\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\Big\}\frac{dw}{dz} - \Big\{\frac{\alpha\alpha'(-1)}{z} + \beta\beta' + \frac{\gamma\gamma'(1-0)}{z-1}\Big\}\times \times \frac{w}{z(z-1)} = 0. \tag{5.8}$$

We note that the exponent pair at each of the singularities is exactly the same as before. Before we embark on more structural results, we need the following result.

Proposition 5.2.1. *Let*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{5.9}$$

to have a regular singular point at $z_0 = 0$ with exponent $\{\alpha,\, \alpha'\}$. Then the function $f(z)$ defined by $w(z) = z^\lambda f(z)$ satisfies the differential equation

$$\frac{d^2 f}{dz^2} + \Big(p(z) + \frac{2\lambda}{z}\Big)\frac{df}{dz} + \Big(q(z) + \frac{\lambda p(z)}{z} + \frac{\lambda(\lambda-1)}{z^2}\Big) f = 0, \tag{5.10}$$

with regular singular point also at $z = 0$ but the exponent pair is $\{\alpha - \lambda,\, \alpha' - \lambda\}$. If $z = \infty$, then the equation (5.10) has exponent pair $\{\alpha + \lambda,\, \alpha' + \lambda\}$ instead.

Proof. Let $w(z) = z^\lambda f(z)$. We leave the verification of the equation (5.10) as an exercise. Substitute the series

$$zp(z) = \sum_{k=0}^\infty p_k z^k, \qquad z^2 q(z) = \sum_{k=0}^\infty q_k z^k, \qquad f(z) = \sum_{k=0}^\infty a_k z^{\alpha+k}$$

into the differential equation (5.10) and consider the indicial equation:

$$\alpha(\alpha-1) + \big(p_0 + 2\lambda\big)\alpha + q_0 + \lambda p_0 + \lambda(\lambda-1) = 0,$$

or

$$\alpha^2 + \big(p_0 - 1 + 2\lambda\big)\alpha + q_0 + \lambda p_0 + \lambda(\lambda-1) = 0.$$

Let $\{\tilde\alpha,\, \tilde\alpha'\}$ be the exponent pair for the equation (5.10) at $z = 0$. But then $\alpha + \alpha' = 1 - p_0 - 2\lambda = \alpha + \alpha' - 2\lambda$, and

$$\begin{aligned}\tilde\alpha\tilde\alpha' &= q_0 + \lambda p_0 + \lambda(\lambda-1)\\ &= \alpha\alpha' - \lambda(\alpha+\alpha') + \lambda\\ &= (\alpha-\lambda)(\alpha'-\lambda).\end{aligned}$$

Solving the above two algebraic equations proves the first part of the proposition.

□

Exercise 5.5. *Complete the proof of the above Proposition of the transformation to ∞ and the change of the exponent pair.*

Remark 5.2.2. *If we apply the transformation $w(z) = (z-1)^\lambda f(z)$ to the differential equation (5.9), then the resulting equation (5.10) would still have $\{0,\, 1,\, \infty\}$ as regular singular points, but the exponent pair at $z = 1$ is now $\{\beta + \lambda,\, \beta' + \lambda\}$ instead. The idea could be apply to a more general situation. Suppose the differential equation (5.9) has $z = \xi$ as a regular singular point with exponent pair $\{\alpha,\, \alpha'\}$, then the resulting differential equation after the transformation $w(z) = (z-\xi)^\lambda f(z)$ would still has $z = \xi$ as a regular singular point but now the corresponding exponent pair is$\{\alpha + \lambda,\, \alpha' + \lambda\}$.*

5.3 Riemann's $P-$Scheme

The following fomulation of solution of Gauss hypergeometric equation appeared to be first put forward by B. Riemann (1826–1866) in 1857 [19].

Definition 5.3.1. *Following Riemann, we write the solution w of the equation (5.6) in Riemann's $P-$symbol:*

$$w(z) = P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}.$$

Thus a solution of equation (5.7) can be denoted by

$$w(z) = P\left\{\begin{array}{cccc} \xi & \infty & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\},$$

while that of (5.8) can be denoted by

$$w(z) = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}.$$

It follows from Proposition (5.2.1) that

Exercise 5.6.

$$z^p(z-1)^q P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha+p & \beta-p-q & \gamma+q\,; & z \\ \alpha'+p & \beta'-p-q & \gamma'+q & \end{array}\right\}.$$

Similarly,

Exercise 5.7.

$$z^{-p}(z-1)^{-q} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-p & \beta+p+q & \gamma-q\,; & z \\ \alpha'-p & \beta'+p+q & \gamma'-q & \end{array}\right\}.$$

Exercise 5.8. *Prove by transforming the differential equation that*

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 1 & 0 & \infty & \\ \alpha & \beta & \gamma\,; & \dfrac{1}{1-z} \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

In general, we have

Exercise 5.9.

$$\Big(\frac{z-\xi}{z-\eta}\Big)^p\Big(\frac{z-\zeta}{z-\eta}\Big)^q P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha+p & \beta-p-q & \gamma+q\,; & z \\ \alpha'+p & \beta'-p-q & \gamma'+q & \end{array}\right\}.$$

Exercise 5.10. *Let $M(z)=\frac{az+b}{cz+d}$, where $ad-bc\neq 0$, be a Möbius transformation. Let $M(\xi)=\xi_1$, $M(\eta)=\eta_1$, $M(\zeta)=\zeta_1$. Show that*

$$P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} \xi_1 & \eta_1 & \zeta_1 & \\ \alpha & \beta & \gamma\,; & z_1 \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

where $z_1=M(z)$.

5.4 Gauss Hypergeometric Equation

Let

$$t=M(z)=\frac{(z-\xi)(\eta-\zeta)}{(z-\eta)(\xi-\zeta)}$$

be a Möbius transformation. We deduce from Exercise 5.10 that

$$\begin{aligned} P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} &= P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & t \\ \alpha' & \beta' & \gamma' & \end{array}\right\} \\ &= t^\alpha(1-t)^\gamma P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-\alpha & \alpha+\beta+\gamma & \gamma-\gamma\,; & t \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\} \\ &= t^\alpha(1-t)^\gamma P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0\,; & t \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\}. \end{aligned}$$

It is customary to denote $\alpha+\beta+\gamma=a$, $\alpha+\beta'+\gamma=b$ and $1-c=\alpha'-\alpha$. Note that $c-a-b=1-\gamma-\gamma'-\beta-\beta'-2\gamma=\gamma+\gamma'-2\gamma=\gamma'-\gamma$. Then the last Riemann-Scheme becomes

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0\,; & t \\ 1-c & b & c-a-b & \end{array}\right\}$$

satisfies the differential equation

$$t(1-t)w''+\{c-(a+b+1)t\}w'-abw=0,$$

which is called the **Gauss hypergeometric equation**, where we have assumed that none of the exponent pair $\{0,\,1-c\}$, $\{0,\,c-a-b\}$ and $\{a,\,b\}$ is an integer.

The important **associated Legendre's differential equation**

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\Big\{\nu(\nu+1)-\frac{\mu^2}{1-z^2}\Big\}w=0$$

is a Gauss hypergeometric differential equation, where ν, μ are arbitrary complex constants, with singularities at ±1, ∞. In terms of Riemann's P notation, we have

$$w(z)=P\left\{\begin{array}{cccc} -1 & \infty & 1 & \\ \frac{\mu}{2} & \nu+1 & \frac{\mu}{2}\,; & t \\ -\frac{\mu}{2} & \nu & -\frac{\mu}{2} & \end{array}\right\}.$$

The equation is called **Legendre's differential equation** if $m=0$. Its solutions are called **spherical harmonics**. This equation is important because it is often encountered in solving boundary value problems of potential theory for spherical region, and hence the name of spherical harmonics.

5.5 Gauss Hypergeometric series

Let us consider power series solutions of the Gauss Hypergeometric equation

$$z(1-z)w''+\{c-(a+b+1)z\}w'-abw=0, \tag{5.11}$$

at the regulary singular point $z=0$. Substitute the power series $w=\sum_{k=0}^{\infty}a_kz^{k+\alpha}$ into the equation (5.11) yields the indicial equation

$$\alpha(\alpha+c-1)=0,$$

so that the coefficient a_k satisfies the recurrence relation

$$(\alpha+k)(\alpha+c+k-1)a_k=(\alpha+a+k-1)(\alpha+b+k-1)a_{k-1},$$

or

$$(\alpha+1+k-1)(\alpha+c+k-1)a_k=(\alpha+a+k-1)(\alpha+b+k-1)a_{k-1}.$$

This gives

$$
\begin{aligned}
a_k &= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}a_{k-1}\\
&= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}\frac{(\alpha+a+k-2)(\alpha+b+k-2)}{(\alpha+1+k-2)(\alpha+c+k-2)}a_{k-2}\\
&\cdots\cdots\\
&= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}\cdots\frac{(\alpha+a+0)(\alpha+b+0)}{(\alpha+1+0)(\alpha+c+0)}a_0\\
&= \frac{(\alpha+a)_k(\alpha+b)_k}{(\alpha+1)_k(\alpha+c)_k}a_0,
\end{aligned}
$$

where a_0 is an arbitrarily chosen constant and we recall that

$$(\alpha)_0 = 1, \qquad (\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1),$$

for each integer k. So we have obtained, assuming that $1-c$ is not an integer, two linearly independent power series solutions

$$w_1(z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k, \qquad w_2(z) = z^{1-c}\sum_{k=0}^{\infty}\frac{(a-c+1)_k(b-c+1)_k}{(2-c)_k k!}z^k.$$

Definition 5.5.1. *A **hypergeometric series** is a power series given in the form*

$$
\begin{aligned}
F(a,\,b;\,c;\,z) = {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right) &= \sum_{k=0}^{\infty}c_kz^k = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k\\
&= 1+\frac{ab}{c}\,z+\frac{a(a+1)b(b+1)}{c(c+1)\,1!}\,z^2+\\
&\quad+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,2!}\,z^3+\cdots \qquad (5.12)
\end{aligned}
$$

The series has its name because the ratio between two consecutive terms, that is $\frac{a_{k+1}}{a_k}$ is a *rational function* of k.

Theorem 5.11. *The series ${}_2F_1(a,\,b;\,c;\,z)$*

1. *converges absolutely in $|z|<1$;*
2. *converges absolutely if $\Re(c-a-b)>0$ if $|z|=1$;*
3. *converges conditionally if $0\geq\Re(c-a-b)>-1$ if $|z|=1$ and $z\neq1$;*
4. *diverges if $\Re(c-a-b)\leq-1$.*

Proof. 1. This first part is easy. For let $u_k = \frac{(a)_k(b)_k}{(c)_k k!} z^k$. Then

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{(a+k)(b+k)}{(c+k)(1+k)} z\right| \to |z|$$

as $k \to \infty$. Thus the hypergeometric series converges (uniformly) within the unit disc, and it diverges in $|z| > 1$ by the ratio test.

2. Recall that $\Gamma(z) = \lim_{k\to\infty} k! k^{z-1}/(z)_k$. But then the $k-$th coefficient of the power series behaves asymptotically like

$$\begin{aligned}\frac{(a)_k(b)_k}{(c)_k k!} &\approx \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{k!k^{a-1} \cdot k!k^{b-1}}{k!k^{c-1} \cdot k!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} k^{a+b-c-1} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{k^{c+1-a-b}},\end{aligned}$$

for all $n \geq N$ say. But then

$$\left|\sum_N^\infty \frac{(a)_k(b)_k}{(c)_k k!} z^k\right| \leq \left|\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right| \sum_N^\infty \frac{1}{|k^{c+1-a-b}|} = \left|\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right| \sum_N^\infty \frac{1}{k^{\Re(c+1-a-b)}}$$

which is known to be convergent when $\Re(c-a-b) > 0$ since $\sum \frac{1}{k^{1+\delta}}$ converges when $\Re\delta > 0$.

3. This follows easily from (2).

4. This follows easily from (2).

□

In any case, the series will never converge on $|z| > 1$. We will see how one can "continue" the solution outside the unit disc.

Exercise 5.12. *Give an alternative proof of the second part of the Theorem 5.11 based directly on the Theorem 3.18 or the Theorem 3.1.*

Definition 5.5.2. *Let $a_1, a_2, \cdots a_p$ and $b_1, b_2, \cdots b_q$ be given complex constants. We define* ***generalized Hypergeometric function*** *by*

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle|\, z\right) = \sum_{-\infty}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k k!} z^k. \tag{5.13}$$

Exercise 5.13. *Prove that the generalized hypergeometric series (5.13)*

1. *conveges absolutely for all z if $p \leq q$ and for $|z| < 1$ if $p = q+1$;*
2. *diverges for all $x \neq 0$ if $p > q+1$.*

Exercise 5.14. *Prove that the generalized hypergeometric series (5.13)* ${}_{q+1}F_q$ *with* $|z| = 1$,

1. *converges absolutely if* $\Re(\sum b_j - \sum a_j) > 0$;
2. *converges conditionally if* $z \neq 1$ *and* $0 \geq \Re(\sum b_j - \sum a_j) > -1$;
3. *the diverges for all* $z \neq 1$ *if* $\Re(\sum b_j - \sum a_j) \leq -1$.

We note that the hypergeometric function includes many important functions that we have encountered. For examples, we have

$$\begin{aligned}
(1-z)^{-a} &= {}_1F_0\Big(\begin{matrix} a \\ - \end{matrix} \Big| -z\Big);\\
e^z &= z\,{}_0F_0\Big(\begin{matrix} - \\ - \end{matrix} \Big| z\Big);\\
\log(1+z) &= z\,{}_2F_1\Big(\begin{matrix} 1,1 \\ 2 \end{matrix} \Big| -z\Big);\\
\sin z &= z\,{}_0F_1\Big(\begin{matrix} - \\ \frac{3}{2} \end{matrix} \Big| -\frac{z^2}{4}\Big);\\
\cos z &= z\,{}_0F_1\Big(\begin{matrix} - \\ \frac{1}{2} \end{matrix} \Big| -\frac{z^2}{4}\Big);\\
\sin^{-1} z &= z\,{}_2F_1\Big(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \Big| z^2\Big)
\end{aligned}$$

5.6 Uniform Convergence II

We also need to extend the idea of uniform convergence for infinite integrals, discussed in Section 3.6, to include integrals of finite intervals. Recall that $f(z) = u(z) + iv(z)$ where $u(z)$, $v(z)$ are real-valued functions. So it suffices to consider real-valued functions.

Definition 5.6.1. *Let* $a < b$ *be extended numbers, that is* $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$, *and* $I \in \mathbb{R}$. *Let* $f(x, y)$ *be a real-valued integrable function with respect to* x *over the interval* $a < x < b$, *and for* $y \in I$. *Suppose that the integral*

$$\phi(y) = \int_a^b f(x, y)\, dx \tag{5.14}$$

converges for all values of y *in* I. *Then the integral is said to be* ***uniformly convergent on*** I *if, for any given* $\varepsilon > 0$, *we can find real numbers* $\tilde{a}$, $\tilde{b} \in (a, b)$, *depending on* ε, *but not on* y, *such that*

$$\left|\phi(y) - \int_{\hat{a}}^{\hat{b}} f(x, y)\, dx\right| < \varepsilon,$$

whenever $a < \hat{a} < \tilde{a},\ \tilde{b} < \hat{b} < b.$

Then we obviously have an analogue of Lemma 3.6.1, a $M-$test, for this new definition.

Lemma 5.6.1. *Suppose that* $f(a,\, b) \times I \to \mathbb{R}$*, where* f*,* $(a,\, b)$*,* I *are defined as in the above definition, such that* f *is integrable in any subinterval of* $(a,\, b)$ *for each* $y \in I$*. If there is a real function* g *that is absolutely integable on* $(a,\, b)$ *such that* $|f(x,\, y)| \le g(x)$ *for* $(x,\, y) \in (a,\, b) \times I$*, then*

$$\int_a^b f(x,\, y)\, dx$$

converges uniformly on I*.*

Proof. Given $\varepsilon > 0$, there exist $\tilde{a},\, \tilde{b} \in (a,\, b)$ such that

$$\left(\int_a^{\tilde{a}} + \int_{\tilde{b}}^b\right) g(x)\, dx < \varepsilon.$$

Thus for each $a < \hat{a} < \tilde{a}$ and $\tilde{b} < \hat{b} < b$, and $y \in I$

$$\begin{aligned}\left|\int_a^b f(x,\, y)\, dx - \int_{\hat{a}}^{\hat{b}} f(x,\, y)\, dx\right| &\le \int_a^{\hat{a}} |f(x,\, y)|\, dx + \int_{\hat{b}}^b |f(x,\, y)|\, dx \\ &\le \int_a^{\hat{a}} g(x)\, dx + \int_{\hat{b}}^b g(x)\, dx < \varepsilon.\end{aligned}$$

□

It can be shown that if the integral $\int_a^b f(x,\, y)\, dx$ converges uniformly on I, then $\lim_{y\to y_0} \int_a^b f(x,\, y)\, dx = \int_a^b \lim_{y\to y_0} f(x,\, y)\, dx$. One has "complex analogues" of the above definition and $M - test$. If, in addition, we assume that $f(t,\, z)$ is analytic function of z in certain compact subset D of $\mathbb{C}$, and that the integral $\phi(z) = \int_a^b f(t,\, z)\, dt$ converges uniformly for all $z \in D$ (extending the "real" definition in an obvious sense), then $\lim_{z\to z_0} \int_a^b f(t,\, z)\, dt = \int_a^b \lim_{z\to z_0} f(t,\, z)\, dt$ and $\phi(z)$ is analytic in D. We shall omit the detailed description of their proofs.

5.7 Euler's Integral Representation

Next we introduce an integral representation of the hypergeomertric function due to Euler [10, Vol. 12, pp. 221-230] in this section, which is very important for the understanding of many classical special functions which are special cases of it.

Theorem 5.15. *Let $\Re c > \Re b > 0$. Then*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt, \qquad (5.15) $$

where z lies in the $\mathbb{C}$ cut along the infinite ray $[1, +\infty)$. We assume that $(1-zt)^{-a}$ in its principal branch and $\arg t = 0 = \arg(1-t)$.

Proof. We recall from Theorem 3.11 since $\Re(c-b) > 0$, so that

$$ \begin{aligned} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} & \int_0^1 t^{b+k-1}(1-t)^{c-b-1}\, dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+k,\, c-b) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b+k)\Gamma(c-b)}{\Gamma(c+k)} \\ &= \frac{(b)_k}{(c)_k}. \end{aligned} $$

Substitute this equation into (5.12) gives

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \int_0^1 t^{b+k-1}(1-t)^{c-b-1}\, dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, dt \sum_{k=0}^{\infty} \frac{(a)_k}{k!} (zt)^k, \end{aligned} $$

where we have interchanged the summation and integration signs because of uniform convergence of the summation. In fact, we have

$$ \begin{aligned} \left| \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \int_0^1 t^{b+k-1}(1-t)^{c-b-1}\, dt \right| &\le \sum_{k=0}^{\infty} \frac{|(a)_k|}{k!} |z|^k \int_0^1 |t^{b+k-1}(1-t)^{c-b-1}|\, dt \\ &\le \sum_{k=0}^{\infty} \frac{|(a)_k|}{k!} |z|^k \int_0^1 t^{\Re(b)+1-k}(1-t)^{\Re(c-b)-1}\, dt \\ &= \sum_{k=0}^{\infty} \frac{|(a)_k|}{k!} B(\Re b + k,\, \Re(c-b))|z|^k \\ &= \frac{\Gamma(\Re b)\Gamma(\Re(c-b))}{\Gamma(\Re c)} \sum_{k=0}^{\infty} \frac{|(a)_k|(\Re b)_k}{(\Re c)_k k!} |z|^k. \end{aligned} $$

Besides, we have

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}(zt)^k = (1-zt)^{-a},$$

for $0 \le t \le 1$ and $|z| < 1$. Hence we have proved

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt$$

under the assumption that $\Re c > \Re b > 0$ and $|z| < 1$. The next step is to extend the above integral to $\mathbb{C}\backslash[1 + \infty)$ and is still analytic there.

Let us consider the bounded region

$$\rho \le |1-z| \le R, \qquad |\arg(1-z)| \le \pi - \delta, \tag{5.16}$$

where $\rho > 0$, $\delta > 0$ are arbitrarily small and $R > 0$ is arbitrarily large. Then for t in $0 \le t \le 1$, the integrand

$$t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}$$

is continuous in t and z. Besides, the integrand is analytic in z. In order to show that (5.15) is analytic, we let $M = \max_{z\in(5.16)} |(1-zt)^{-a}|$, then

$$\left| \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \right| \le M \int_0^1 t^{\Re b-1}(1-t)^{\Re(c-b)-1}\, dt$$

for all z lies in (5.16). But the last integral is a beta integral which is convergent since $\Re c > \Re b > 0$. This shows that the integral (5.15) is uniform convergence in the region (5.16). Hence the integral is analytic in (5.16) by the last section. But both ρ, R and δ are arbitrary. So the integral represents an analytic function in $\mathbb{C}\backslash[1, \infty)$. It follows that the earlier assumption that $|z| < 1$ can be dropped by the principal of analytic continuation. $\square$

Lemma 5.7.1 (Abel's lemma)**.** *Let $m \le M$ be two positive numbers. Suppose that*

$$b_1 \ge b_2 \ge \cdots \ge b_n \ge 0,$$

and that

$$m \le a_1 + a_2 + \cdots + a_n \le M,$$

for all n. Then

$$b_1 m \le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le b_1 M$$

holds for all values of n.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then

$$\begin{aligned}
&a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n \\
&= b_1s_1 + b_2(s_2 - s_1) + b_3(s_3 - s_2) + \cdots + b_n(s_n - s_{n-1}) \\
&= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + b_ns_n \\
&\le M(b_1 - b_2) + M(b_2 - b_3) + \cdots + M(b_{n-1} - b_n) + b_nM \\
&= Mb_1.
\end{aligned}$$

This yields the required upper bound. The lower bound can be established similarly. $\square$

Exercise 5.16. *Complete the proof of the Abel lemma for the lower bound.*

We recall that if a power series $\sum_{k=0}^{\infty} a_k z_k$ has a radius of convergence $R > 0$, then it converges uniformly on any $|z| \le r < R$. Let $r < \rho < R$. On the other hand, we have $|a_n z^n| = |a_n|\rho^n < K$ to hold for all n. But then for $|z| \le r$

$$|a_n z^n| = \left|a_n \rho^n \left(\frac{z}{\rho}\right)^n\right| < K\left(\frac{z}{\rho}\right)^n.$$

The series $\sum_{k=0}^{\infty} a_k z_k$ therefore converges uniformly in $|z| \le r$. The question is if we can extend the uniform convergence to the boundary $|z| = R$.

Theorem 5.17 (Abel's theorem)**.** *Suppose the series $\sum_{k=0}^{\infty} a_k = s$ exists and that the power series $\sum_{k=0}^{\infty} a_k z_k$ has a radius of convergence equal to unity. Then the series converges uniformly on $0 \le x \le 1$, and*

$$\lim_{x\to 1} \sum_{k=0}^{\infty} a_k x^k = s.$$

Example 5.7.1. *A simple example illustrating Able's theorem is the power series $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ in $|x| < 1$. Abel's theorem yields*

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots.$$

Proof. Let

$$s_{n,\,p} = a_n + a_{n+1} + \cdots + a_p.$$

Since $\sum a_k$ converges, so given $\varepsilon > 0$, we choose an integer N such that $|s_{n,\,p}| < \varepsilon$ whenever $p > n > N$. But the sequence $\{x^n\}$ is non-increasing for $0 \le x \le 1$, so Abel's lemma implies that

$$|a_n x^n + a_{n+1}x^{n+1} + \cdots + a_p x^p| < \varepsilon x^n \le \varepsilon$$

which implies uniform convergence on $[0,\,1]$. $\square$

We remark that one can "extend" the region of uniform convergence from [0, 1] to a compact set near $z = 1$. But the analysis is more complicated. See Copson [8, pp. 100-101]. Conversely, one can ask if $f(x) = \sum_{k=0}^{\infty} a_k x^k = s$ as $x \to 1$, then does $\sum_{k=0}^{\infty} a_k$ also converge (to s?). But this is *not* true in general. If one imposes the extra assumption that $a_k = o(1/k)$, then $\sum_{k=0}^{\infty} a_k$ does converge. This is called a *Tauberian-type theorem.* See [21, p. 10].

We are ready for

Theorem 5.18 (Gauss 1812)**.** *Suppose* $\Re(c-a-b) > 0$. *Then*

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

holds.

Proof. Suppose that $\Re(c-a-b) > 0$. Then Theorem 5.11 implies that $F(a,\, b;\, c;\, 1)$ converges. Abel's theorem asserts that

$$F(a,\, b;\, c;\, 1) = \lim_{x \to 1-} F(a,\, b;\, c;\, x).$$

Let us assume in addition that $\Re c > \Re b > 0$. Then we can make use of the Euler integral representation. Then Euler's representation gives

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} &= {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1 \right) = \lim_{x \to 1-} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x \right) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \lim_{x \to 1-} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1-a}\, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b,\, c-a-b) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\end{aligned}$$

where we have taken the limit sign into the integral. This can be justified if we can show that the Euler integral converges uniformly for $z = x \in [0, 1]$. Notice that

$$1 - t \le |1 - xt| \le 1,$$

for $0 \le t,\, x \le 1$. In particular, we have

$$|t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}| \le t^{\Re b-1}(1-t)^{\lambda-1},$$

where

$$\lambda = \begin{cases} \Re(c-a-b), & \text{if } \Re a > 0; \\ \Re(c-b), & \text{if } \Re a < 0. \end{cases} \tag{5.17}$$

But the integral

$$\int_0^1 t^{\Re b-1}(1-t)^{\lambda-1}\, dt$$

is convergent. This shows that the Euler integral converges uniformly for $0 \le x \le 1$, and hence justifying the interchange of limit and integral signs by the last section. Hence this Gauss's formula has been verified when both inequalities

$$\Re(c-b-a) > 0, \qquad \text{and} \qquad \Re c > \Re b > 0$$

hold. Although one can apply the principal of analytic continuation to remove the second inequality above, we prefer to give a direct proof below. Let us write,

$$F(a,\, b;\, c;\, z) = \sum_{k=0}^{\infty} A_k z^k, \quad F(a,\, b;\, c+1;\, z) = \sum_{k=0}^{\infty} B_k z^k.$$

Then it is routine to check that

$$c(c-a-b)A_k - (c-a)(c-b)B_k = \frac{(a)_k(b)_k}{k!(c+1)_{k-1}}\Big[c-a-b-\frac{(c-a)(c-b)}{c+k}\Big]$$

and

$$c(kA_k - (k+1)A_{k+1}) = \frac{(a)_k(b)_k}{k!(c+1)_{k-1}}\Big[k - \frac{(a+k)(b+k)}{c+k}\Big]$$

hold and that the right-hand sides of the above identities are equal. Thus, we deduce

$$c(c-a-b)A_k = (c-a)(c-b)B_k + ckA_k - c(k+1)A_{k+1}.$$

Hence

$$c(c-a-b)\sum_{k=0}^{n} A_k = (c-a)(c-b)\sum_{k=0}^{n} B_k - c(n+1)A_{n+1}.$$

Now let $n \to \infty$ to establish

$$F(a,\, b;\, c;\, 1) = \frac{(c-a)(c-b)}{c(c-a-b)} F(a,\, b;\, c+1;\, 1)$$

since

$$(n+1)A_{n+1} \approx \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{(n+1)^{a-b-c}} \to 0$$

and $\Re(c-b-a) > 0$. Thus

$$\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)} F(a,\, b;\, c;\, 1) = \frac{\Gamma(c+k-a)\Gamma(c+k-b)}{\Gamma(c+k)\Gamma(c+k-a-b)} F(a,\, b;\, c{+}k;\, 1) \to 1 \tag{5.18}$$

as $k \to \infty$, as can be verified (see later). This completes the proof under the assumption that $\Re(c-b-a) > 0$. $\square$

Exercise 5.19. *Complete the proof for (5.18).*

Corollary 5.1 (Chu (1303)-Vandermonde (1770))**.** *Let n be a positive integer. Then*

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} \,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n}. \tag{5.19}$$

Remark 5.7.1. *According to Askey's account [3], the Chinese mathematician S.-S. Chu [7] had already discovered the essential form of the identity. Askey described Chu's result was "absolutely incredible" as even before adequate notation had been developed [3, p. 60].*

5.8 Kummer's 24 Solutions

We now consider an *internal symmetry* of the hypergeometric equation/function. Recall that the standard hypergeometric equation (5.11) has two power series solutions whose characteristic exponents 0 and $1-c$ are recorded in the Riemann-Scheme

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

We have the following observation

$$\begin{aligned} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ \underline{1-c} & b & c-a-b & \end{matrix}\right\} &= z^{1-c} P\left\{\begin{matrix} 0 & \infty & 1 & \\ c-1 & a-c+1 & 0\,; & z \\ \underline{0} & b-c+1 & c-a-b & \end{matrix}\right\} \\ &= z^{1-c} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 1-c' & a' & 0\,; & z \\ \underline{0} & b' & c-a-b & \end{matrix}\right\}, \end{aligned}$$

where $a' = a-c+1,\ b' = b-c+1$ and $c' = 2-c$. The above transformation indicates that that one of the original power series solution of equation (5.11) with characteristic exponent $1-c$ can be written in terms of the characteristic exponent 0 power series solution $F(a',\, b';\, c';\, z)$ represented by the above last Riemann-Scheme. Thus we have two solutions

$$w_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ 1-c \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k,$$

$$w_2(z) = z^{1-c}{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} \,\middle|\, z\right) = z^{1-c}\sum_{k=0}^{\infty} \frac{(a-c+1)_k(b-c+1)_k}{(2-c)_k k!} z^k,$$

which are precisely two linearly independent solutions found in Section 5.5. We recall that we have assumed that $1-c$ is not an integer. Otherwise, there will be a logarithm term in the solution (see later). This gives two series solutions around the regular singular point $z=0$.

We now consider the regular singular point $z=1$. Just as seen from question 2 of Worksheet 07 that $1/(1-z)$ preserves the hypergeometric equation but permuted the singularities, we now consider the transformation $z = 1-t$. This will transform the hypergeometric equation to the form

$$t(1-t)\frac{d^2w(t)}{dt^2} + [(1+a+b-c)-(a+b+1)t]\frac{dw(t)}{dt} - abw(t) = 0,$$

where $w(t) = y(z)$ where $t = 1-z$. That is, we have shown that the relation

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & 1-z \\ c-a-b & b & 1-c & \end{matrix}\right\} \tag{5.20}$$

holds. Observe that

$$\begin{aligned} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & \underline{0} & ;\ z \\ 1-c & b & c-a-b & \end{matrix}\right\} &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0 & ;\ 1-z \\ c-a-b & b & 1-c & \end{matrix}\right\} \\ &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a' & 0 & ;\ 1-z \\ 1-c' & b' & c'-a'-b' & \end{matrix}\right\} \end{aligned}$$

where $a' = a,\ b' = b$ and $c' = 1+a+b-c$. We obtain

$$w_3(z) = {}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a+b-c \end{matrix} \,\middle|\, 1-z\right).$$

Applying the transformation from w_1 to w_2 above to w_3 and identifying $t = 1-z$ and the parameters a', b' and c' yields the fourth solution

$$w_4(z) = z^{1-c'}{}_2F_1\left(\begin{matrix} a'-c'+1,\ b'-c'+1 \\ 2-c' \end{matrix} \,\middle|\, t\right) = z^{c-a-b}{}_2F_1\left(\begin{matrix} c-b,\ c-a \\ 1+c-a-b \end{matrix} \,\middle|\, 1-z\right).$$

Similarly, one can obtain two solutions around the regular singular point ∞:

Exercise 5.20.

$$w_5(z) = (-z)^{-a}{}_2F_1\Big(\begin{matrix} a,\ a-c+1 \\ a-b+1 \end{matrix}\Big|\ \frac{1}{z}\Big),$$

where $c-a-b$ is not an integer, and

$$w_6(z) = (-z)^{-b}{}_2F_1\Big(\begin{matrix} b,\ b-c+1 \\ b-a+1 \end{matrix}\Big|\ \frac{1}{z}\Big),$$

where $a-b$ is not an integer. The negative signs are introduced for convinience sake (see later).

Hence we have obtained six solutions $w_1(z), \cdots, w_6(z)$. Let us now take $w_1(z)$, that is the standard solution, as an example on how to generate more solutions (where we have used underline on the appropriate exponent to indicate on which solution that we are considering):

$$\begin{aligned} w_1(z) &= {}_2F_1\Big(\begin{matrix} a,b \\ c \end{matrix}\Big|\ z\Big) \\ &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\} \\ &= (1-z)^{c-a-b}P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & c-b & a+b-c\,; & z \\ 1-c & c-a & 0 & \end{matrix}\right\}. \end{aligned}$$

Thus

$${}_2F_1\Big(\begin{matrix} a,b \\ c \end{matrix}\Big|\ z\Big) = C(1-z)^{c-a-b}{}_2F_1\Big(\begin{matrix} c-a,c-b \\ c \end{matrix}\Big|\ z\Big),$$

for some constant C. If we choose the branch of $(1-z)^{c-a-b}$ such that it equals 1 when $z=0$, then $C=1$ since both sides of the equation are equal to 1. Thus, we have

$$w_1(z) = {}_2F_1\Big(\begin{matrix} a,b \\ c \end{matrix}\Big|\ z\Big) = (1-z)^{c-a-b}{}_2F_1\Big(\begin{matrix} c-a,c-b \\ c \end{matrix}\Big|\ z\Big),$$

where $|\arg(1-z)| < \pi$. On the other hand, we have

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & 0 & a\,; & \dfrac{z}{z-1} \\ 1-c & c-a-b & b & \end{matrix}\right\}$$

$$= \left(1-\frac{z}{z-1}\right)^a P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{matrix}\right\}$$

$$= (1-z)^{-a} P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{matrix}\right\}.$$

Thus, we have

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, z\right) = (1-z)^{-a}{}_2F_1\left(\begin{matrix} a, & c-b \\ & c \end{matrix}\,\middle|\, \frac{z}{z-1}\right),$$

provided that $|\arg(1-z)| < \pi$. Interchanging the a and b in the above analysis yields another format:

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, z\right) = (1-z)^{-b}{}_2F_1\left(\begin{matrix} b, & c-a \\ & c \end{matrix}\,\middle|\, \frac{z}{z-1}\right),$$

provided that $|\arg(1-z)| < \pi$. Putting these cases together, we have

$$\begin{aligned} w_1(z) &= {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, z\right) \\ &= (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}\,\middle|\, z\right) = (1-z)^{-b}{}_2F_1\left(\begin{matrix} b, & c-a \\ & c \end{matrix}\,\middle|\, \frac{z}{z-1}\right) \\ &= (1-z)^{-a}{}_2F_1\left(\begin{matrix} a, & c-b \\ & c \end{matrix}\,\middle|\, \frac{z}{z-1}\right) \\ &= (1-z)^{-b}{}_2F_1\left(\begin{matrix} b, & c-a \\ & c \end{matrix}\,\middle|\, \frac{z}{z-1}\right). \end{aligned}$$

Similar formulae exist for the remaining $w_2(z), \cdots, w_6(z)$, where each solution has three additional variartions, thus forming a total of twenty four solutions, which are called **Kummer's 24** solutions. We list two sets of these formulae below.

Exercise 5.21.

$$\begin{aligned} w_2(z) &= z^{1-c}{}_2F_1(a+1-c,\, b+1-c;\, 2-c;\, z) \\ &= z^{1-c}(1-z)^{c-a-b}{}_2F_1(1-a,\, 1-b;\, 2-c;\, z) \\ &= z^{1-c}(1-z)^{c-a-1}{}_2F_1(a+1-c,\, 1-b;\, 2-c;\, z/(z-1)) \\ &= z^{1-c}(1-z)^{c-b-1}{}_2F_1(b+1-c,\, 1-a;\, 2-c;\, z/(z-1)) \end{aligned}$$

and

Exercise 5.22.

$$\begin{aligned}
w_5(z) &= (-z)^{-a}{}_2F_1(a+1-c,\, a+1-b;\, 1-b;\, 1/z) \\
&= (-z)^{-a}(1-z)^{c-a-b}{}_2F_1(1-b,\, c-b;\, a+1-b;\, 1/z) \\
&= (1-z)^{-a}{}_2F_1(a,\, c-b;\, a+1-b;\, 1/(1-z)) \\
&= (-z)^{1-c}(1-z)^{c-a-1}{}_2F_1(a+c-1,\, 1-b;\, a+1-b;\, 1/(1-z))
\end{aligned}$$

One can also interpret the above twenty four solutions by considering that there are six Möbius transformations that permute the three regular singular points $\{0,\, 1,\, \infty\}$:

$$z, \qquad 1-z, \qquad \frac{z}{z-1}, \qquad \frac{1}{z}, \qquad \frac{1}{z-1}, \qquad 1-\frac{1}{z}.$$

This list gives all the possible permutations of the regular singularities. Notice the function

$$z^\rho(1-z)^\sigma{}_2F_1(a^*,\, b^*;\, c^*;\, z^*),$$

where ρ and σ can be chosen so that exactly one of the exponents at $z=0$ and $z=1$ becomes zero. But we have two choices of exponents at either of the points (0 or 1) so that there are a total of $6\times 2\times 2 = 24$ combinations. We refer to Bateman's project Vol. I [9] for a complete list of these twenty four solutions.

5.9 Analytic Continuation

The Kummer 24 solutions, when viewed as power series solutions, will have different regions of convergence. For example, the formula

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right) = (1-z)^{-b}{}_2F_1\left(\begin{matrix} b,\ c-a \\ c \end{matrix}\,\middle|\, \frac{z}{z-1}\right),$$

indicates that although the left-side will converge for $|z|<1$, the right-side, requires

$$\left|\frac{z}{z-1}\right| < 1$$

instead, which is equivalent to having $\Re z < \frac{1}{2}$. That is, the right-side converges for all z in $\Re z < \frac{1}{2}$. One can also view the power series ${}_2F_1(a,\, b;\, c;\, z)$ being analytically continued into the $\Re z < \frac{1}{2}$. Of course, this is not quite the same as the analytical continuation provided by the Euler integral representation. However, one can use this as a trick to obtain use identities, such as one shown here.

We note that

$$w_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ 1-c \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k,$$

$$w_2(z) = z^{1-c}{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} \,\middle|\, z\right) = z^{1-c}\sum_{k=0}^{\infty} \frac{(a-c+1)_k(b-c+1)_k}{(2-c)_k k!} z^k$$

are two linearly independent solutions (why?) of the hypergeometric equation. A third solution given by

$$w_3(z) = {}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a+b-c \end{matrix} \,\middle|\, 1-z\right),$$

when considered as a power series, has its radius of convergence $|1-z| < 1$. The regions of convergence of w_1, w_2 and w_3 has an overlap region. The $\{w_1, w_2\}$ being a fundamdental set, so that the w_3 can be written as a linear combination of w_1, w_2 in this overlap region. We have

Theorem 5.23.

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a+b-c \end{matrix} \,\middle|\, 1-z\right) \\ &= A\,{}_2F_1\left(\begin{matrix} a, b \\ 1-c \end{matrix} \,\middle|\, z\right) + B\,z^{1-c}{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} \,\middle|\, z\right), \end{aligned} \tag{5.21}$$

where

$$A = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}, \quad \text{and} \quad B = \frac{\Gamma(c-1)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}. \tag{5.22}$$

Proof. It remains to evaluate the constants A, B in (5.21). To do so, we suppose $z = 0$ and $\Re c < 1$. Then the (5.21) becomes

$${}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a+b-c \end{matrix} \,\middle|\, 1\right) = A\,{}_2F_1\left(\begin{matrix} a, b \\ 1-c \end{matrix} \,\middle|\, 0\right) = A.$$

Now Gauss's summation formula (Theorem 5.18) yields

$$A = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}.$$

We further assume that $\Re(c-a-b) > 0$ and set $z = 1$ in (5.21). This gives

$$1 = A\,\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(b-a)} + B\,\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}.$$

Applying Gauss's reflection on Gamma function to the last equation yields

$$\begin{aligned}
&B\,\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\\
&= 1-\frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\cdot\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(b-a)}\\
&= 1-\frac{\pi/\sin\pi c\cdot\pi/\sin\pi(c-a-b)}{\pi/\sin\pi(c-a)\cdot\pi/\sin\pi(c-b)}\\
&= \frac{\sin\pi c\cdot\sin\pi(c-a-b)-\sin\pi(c-a)\cdot\sin\pi(c-b)}{\sin\pi c\cdot\sin\pi(c-a-b)}\\
&= -\frac{\sin\pi a\cdot\sin\pi b}{\sin\pi c\cdot\sin\pi(c-a-b).}
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= -\frac{\sin\pi a\cdot\sin\pi b}{\sin\pi c\cdot\sin\pi(c-a-b)}\cdot\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-c)\Gamma(c-a-b)}\\
&= -\frac{\Gamma(c)\Gamma(1-c)\Gamma(1+a+b-c)}{(1-c)\Gamma(1-c)\Gamma(a)\Gamma(b)}\\
&= \frac{\Gamma(c-1)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)},
\end{aligned}$$

as required. □

Exercise 5.24. *Show that*

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, z\right)\\
&= C\,(-z)^{-a}{}_2F_1\left(\begin{matrix}a,\ a-c+1\\ a-b+1\end{matrix}\,\middle|\,\frac{1}{z}\right)+D(-z)^{-b}{}_2F_1\left(\begin{matrix}b,\ b-c+1\\ b-a+1\end{matrix}\,\middle|\,\frac{1}{z}\right),
\end{aligned}$$

where

$$C=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}\quad\textit{and}\quad D=\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}.$$

We again refer to Bateman's project Vol. I for a complete list of these twenty four solutions.

Chapter 6

Barnes's Integral

6.1 Introduction

Our discussion below is in fact a special case of an integral transform called **Mellin transform**[1]. Roughly speaking, if

$$F(s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

provided the integral exists of course, then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds$$

holds for a certain class of functions. This second transform serves as a kind of inversion of the original transform. Since we shall only deal with special cases of it, so we will not further discuss it here. Interested reader should consult [15].

Barnes (1904–1910)[2] published a series of papers that further developed Mellin's method. It is said that integral of Barnes-type have been little used before Barnes's time. He showed in [5] that one can represent ${}_2F_1(a,\, b;\, c;\, z)$ in terms of an inverse Mellin's integral. This allows an alternative development of the theory of many classial special functions which are special cases of ${}_2F_1$ in terms of Barnes's integrals. This and other related integrals are called **Mellin-Barnes integrals** or simply **Barnes integrals**.

Example 6.1.1. *Recall that the integral representation of the Gamma function is defined by*

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx, \qquad \Re s > 0.$$

[1] R. H. Mellin (1854–1933) was a Finnish mathematician. He served on the editorial board of *Acta Mathematica* for the period 1908–1933.

[2] E. W. Barnes (1874–1953) was an English mathematician and Bishopric of Birmingham (1924–1952).

Then
$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)\,ds, \qquad c>0,\ \Re z>0.$$

Proof. We consider a rectangular contour L_N formed by the vertices
$$c \pm iR, \qquad c - \big(N + \frac{1}{2}\big) \pm iR, \qquad N \in \mathbb{N}.$$
It is easy to see that the Gamma function has poles $\{0,\,-1,\,-2,\,\cdots,-N\}$ lie inside the contour L. Recall form Theorem 3.5 that
$$\Gamma(z) = \Psi(z) + \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+z)k!},$$
where Ψ is an entire function. So the residue at $z=-k$ is $(-1)^k/k!$. Hence the corresponding residues of the integrand $x^{-s}\Gamma(s)$ are (see Theorem 3.5))
$$\frac{(-1)^j x^j}{j!}, \qquad j = 0,\,1,\,2,\,\cdots,N.$$
Cauchy's residue theorem yields
$$\begin{aligned}\frac{1}{2\pi i}\int_{L_R} x^{-s}\Gamma(s)\,ds &- \frac{1}{2\pi i}\int_{c-iR}^{c+iR} x^{-s}\Gamma(s)\,ds\\ &= \frac{1}{2\pi i}\int_{L_R\setminus[c-iR,\,c+iR]} x^{-s}\Gamma(s)\,ds\\ &\to 0\end{aligned}$$
as $R\to\infty$, after letting t in $s = c-(N+1/2)+iR$ in
$$|\Gamma(c-(N+1/2)+iR)| = \sqrt{2\pi}|R|^{c-\frac{1}{2}}e^{-\pi|R|/2}\big[1+O(1/|R|)\big]$$
as $|R|\to\infty$, where $c-(N_1+1/2)\le c-(N+1/2)\le c-(N_2+1/2)$ (see [1, Cor. 1.4.4]) and letting $N\to\infty$ with Corollary 3.1. □

Exercise 6.1. *Complete the proof of the above example.*

6.2 Barnes's Integral

It can be shown (exercise) that
$$\int_0^\infty x^{s-1}{}_2F_1\Big(\begin{matrix}a,\ b\\ c\end{matrix}\Big|\,-x\Big)\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)},$$
provided that $\min(\Re a,\,\Re b) > \Re s > 0$ (the additional condition $\Re c > \Re b$ can be removed). This the Mellin transform at the formal level. This gives a possible form for a contour integral form of ${}_2F_1$.

Exercise 6.2. *Justify the above Mellin transform of ${}_2F_1$.*

Theorem 6.3 (Barnes (1908))**.** *Let $|z| < 1$ and $|\arg(-z)| < \pi$. Then*

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s\,ds, \quad (6.1)$$

provided none of the a and b is zero or a negative integer and none of the $c-1$, $a-b$, $a+b-c$ is an integer or zero, and where the path of integration is along the imaginary axis, modified, if necessary, by loops to make the poles of $\Gamma(-s)$ lie to the right of the path, and those of $\Gamma(a+s)\Gamma(b+s)$ to its left. Moreover, the integral represents an analytic function in $|\arg(-z)| < \pi$, thus providing an analytic continuation of the ${}_2F_1$ outside $|z| < 1$.

Proof. Let R_n denote the closed contour formed by the four sides of the rectangle with vertices

$$\pm iN, \qquad N + \frac{1}{2} \pm iN,$$

where N is a postive integer greater than $\max(|\Im a|, |\Im b|)$, so that no pole of the integrand lie on R_N. We then have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{R_N} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s\,ds \\
&= \sum \Big\{ \texttt{residues of } \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s \texttt{ at the poles within } R_N \Big\} \\
&= -\sum_{k=0}^{N} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(k+1)}\,z^k,
\end{aligned}$$

since

$$\operatorname{Res}_{z=k}((-z)^s\Gamma(-s)) = \lim_{z\to k}(z-k)((-z)^s\Gamma(-s)) = -(-z)^k\frac{(-1)^k}{k!} = -\frac{z^k}{\Gamma(k+1)}.$$

Hence

$$\frac{1}{2\pi i}\int_{-iN}^{iN} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s\,ds = \sum_{k=0}^{N} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(k+1)}\,z^k + \frac{1}{2\pi i}L_R,$$

where

$$L_R = \int_{-iN}^{-iN+N+\frac{1}{2}} + \int_{N+\frac{1}{2}-iN}^{N+\frac{1}{2}+iN} - \int_{iN}^{N+\frac{1}{2}+iN} := L_1 + L_2 - L_3,$$

say. Let us assume that $|s|$ is large and $|\arg z| < \pi$. Then

$$\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)} \approx s^{\alpha+i\beta}\big(1+o(1)\big),$$

where $\alpha + i\beta = a+b-c-1$, and so

$$\left|\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)}\right| \approx |s|^{\alpha}|s^{i\beta}|\big(1+o(1)\big) = |s|^{\alpha}e^{-\beta\arg s}\big(1+o(1)\big),$$

by applying the estimate (3.1) ($\Gamma(x) \approx \sqrt{2\pi}x^{x-1/2}e^{-x}, \qquad |x| \to \infty$ in $|\arg x| \le \pi - \delta$). It follows from this estimate that the estimate on $L_1 + L_2 + L_3$ is given by

$$\left|\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)}\right| < AN^{\alpha}, \tag{6.2}$$

where the constant A is independent of N. We deduce from this estimate that for s lies on L_1, that is $s = T - iN,\ 0 \le T \le N + \frac{1}{2}$. Applying Euler's reflection formula for large N yield

$$|\Gamma(-s)\Gamma(1+s)| = \left|\frac{\pi}{\sin(-\pi s)}\right| = \frac{2\pi}{|e^{\pi(N+iT)} - e^{-\pi(N+iT)}|} < 4\pi e^{-\pi N},$$

and

$$|(-z)^s| = |z|^T e^{N\arg(-z)} < |z|^T e^{(\pi-\varepsilon)N}.$$

Substituting the above estimates into L_1 yields

$$\begin{aligned}
|L_1| &= \left|\int_{-iN}^{-iN+N+\frac{1}{2}} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s\,ds\right| \\
&< \left|\int_{-iN}^{-iN+N+\frac{1}{2}} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)}\Gamma(-s)\Gamma(s+1)\,(-z)^s\,ds\right| \\
&< 4\pi AN^{\alpha}e^{-\varepsilon N}\int_0^{N+\frac{1}{2}} |z|^T\,dT \\
&< 2(2N+1)\pi AN^{\alpha}e^{-\varepsilon N}
\end{aligned}$$

since $|z| < 1$ by assumption, thus proving $L_1 \to 0$ as $N \to \infty$. Similar argument can be applied to L_3 to show $L_3 \to 0$ as $N \to \infty$. We omit the details.

It remains to consider L_2. Let s lie on L_2. That is, $s = N + \frac{1}{2} + it$, where $-N \le t \le N$. We have

$$|\Gamma(-s)\Gamma(1+s)| = \frac{\pi}{|\sin\pi(N+\frac{1}{2}+it)|} = \frac{\pi}{|\cosh\pi t|} < 2\pi e^{-\pi|t|},$$

and

$$|(-z)^s| = |z|^{N+\frac{1}{2}} e^{-t\arg(-z)}.$$

It follows that

$$\begin{aligned}
|L_2| &= \left| \int_{N+\frac{1}{2}-iN}^{N+\frac{1}{2}+iN} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds \right| \\
&= \left| \int_{N+\frac{1}{2}-iN}^{N+\frac{1}{2}+iN} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)} \Gamma(-s)\Gamma(s+1) \, (-z)^s \, ds \right| \\
&\le 2\pi(AN^\alpha)|z|^{N+\frac{1}{2}} \int_{-N}^{N} e^{-t\arg(-z)} e^{-\pi|t|} \, dt \\
&\le 2\pi(AN^\alpha)|z|^{N+\frac{1}{2}} \int_{-N}^{N} e^{-\varepsilon|t|} \, dt \\
&< 4\pi AN^\alpha |z|^{N+\frac{1}{2}}/\varepsilon,
\end{aligned}$$

which tends to zero for each $\varepsilon > 0$ as $N \to \infty$ since $|z| < 1$.

We have thus shown that

$$\frac{1}{2\pi i} \int_{-iN}^{iN} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds - \sum_{k=0}^{N} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(k+1)} z^k = \frac{1}{2\pi i} L_R \to 0,$$

as $R(N) \to \infty$ when $|z| < 1$ and $|\arg(-z)| < \pi$. Thus the infinite contour integral formula (6.1) holds.

Let us now consider D be any closed region of the cut plane $|\arg(-z)| < \pi$. Thus, D must lie in $|\arg(-z)| \le \pi - \varepsilon$ for some $\varepsilon > 0$. Let us re-examine the integrand

$$\Psi(z,\, s) = \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s$$

where $s = it$, and t is real and $|t| \ge t_0$. Then one can follow the previous technique to show that

$$|\Psi(z,\, s)| < K|t|^\alpha e^{-\varepsilon|t|},$$

where K is independent of z. holds as $|t| \to \infty$. This shows that the integral $\int_{-i\infty}^{i\infty} \psi(z,\, s),\, ds$ converges uniformly on any closed region D that lies in $|\arg(-z)| \le \pi - \varepsilon$, and hence is analytic there. This completes the proof. $\square$

Chapter 7

Frobenius's Method

We continue our invertigation stated in Chapter 4 on how to construct solutions of differential equations of the type

$$L[w] := \frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{7.1}$$

with a regular singular point at the origin. We recall that Fuchs [12] developed a definitive theory of solutions on such equations[1] with finite number of regular singular points in the $\hat{\mathbb{C}}$. There is, however, a shortcoming that Fuchs's method is "long and more difficult than they need be", as quoted from Gray's historical account on Fuchs's theory [13, p. 56]. A young student George Frobenius (1849–1917) from University of Berlin proposed an effective method in 1873 to simplify Fuchs's method. This new method has since been called **Frobenius's method** taught in most elementary differential equation courses. The method is particularly effective for dealing with the situation when the indicial equation of the second order ordinary differential equation possesses a double root or two distinct roots but differs by an integer. The method also works for higer order differential equations. Interested reader should consult Poole [16] for more background information about the method.

7.1 Double roots case

Let us recall that we write

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k, \qquad z^2q(z) = \sum_{k=0}^{\infty} q_k z^k,$$

[1]Fuchs's original theory allows him to deal with higher order linear differential equation of regular singular type.

where the coefficients p_0, q_0 and q_1 are not all zero. Let

$$w(z) = \sum_{k=0}^{\infty} a_k z^{\alpha+k},$$

where α is a constant to be determined. Multiply z^2 on both sides of the differential equation (7.1) and substitute this expansion $w(z)$ into (7.1) yields, after simplification,

$$L[w] = z^{\alpha} \sum_{k=0}^{\infty} \Big[F(\alpha+k)a_k + \sum_{j=0}^{k-1} a_j \big((\alpha+j)p_{k-j} + q_{k-j} \big) \Big] z^k, \tag{7.2}$$

where we have demanded that

$$F(\alpha+k)a_k + \sum_{j=0}^{k-1} a_j \big((\alpha+j)p_{k-j} + q_{k-j} \big) = 0, \qquad k \geq 1 \tag{7.3}$$

holds, and where we have set

$$F(\alpha+k) = (\alpha+k)(\alpha+k-1) + p_0(\alpha+k) + q_0 \tag{7.4}$$

and

$$F(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0 = (\alpha-\alpha_1)(\alpha-\alpha_2) \tag{7.5}$$

when $k = 0$. The equation $F(\alpha) = 0$ gives the two (characteristic) exponents α_1, α_2 of the indicial equation. Thus, the expression (7.2) becomes

$$L[w] = a_0 z^{\alpha}(\alpha-\alpha_1)(\alpha-\alpha_2),$$

so that $L[w] = 0$ when $\alpha = \alpha_1$ or $\alpha = \alpha_2$. Hence we have obtained two linearly independent solutions. Suppose that $\alpha = \alpha_1$ is a double root. Then we have

$$L[w] = a_0 z^{\alpha}(\alpha-\alpha_1)^2, \tag{7.6}$$

and one could only find one solution of the form $w(z) = z^{\alpha} \sum_{k=0}^{\infty} a_k z^k$. In order to find a second solution, let us consider

$$L\Big[\frac{\partial w}{\partial \alpha}\Big] = a_0 \log z (\alpha-\alpha_1)^2 + a_0 z^{\alpha} \cdot 2(\alpha-\alpha_1), \tag{7.7}$$

which equals zero when $\alpha = \alpha_1$. Then both $w(z) = z^{\alpha} \sum_{k=0}^{\infty} a_k z^k$ and

$$w_2(z) = \Big(\frac{\partial w}{\partial \alpha}\Big)_{\alpha=\alpha_1} = w(z)\log z + z^{\alpha_1} \sum_{k=1}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_1} z^k, \tag{7.8}$$

are solutions to $L[\cdot] = 0$ when $\alpha = \alpha_1$. This is precisely the first case described in Theorem 4.8.

7.2 Roots differ by an integer

Suppose that $\alpha_1 = \alpha_2 + s$ for some positive integer s. Then the previous method no longer holds. Note that the coefficient a_0 is a free parameter. We define

$$a_0 = a_0'(\alpha - \alpha_2),$$

where a_0' is now arbitrary. Then we deduce, as before, that

$$L[w] = L[w] = a_0'\, z^\alpha (\alpha - \alpha_1)(\alpha - \alpha_2)^2$$

holds instead. Similarly, we have

$$\begin{aligned} L\Big[\frac{\partial w}{\partial \alpha}\Big] &= a_0'\, z^\alpha (\alpha - \alpha_1)(\alpha - \alpha_2)^2 \log z + a_0' z^\alpha (\alpha - \alpha_2)^2 \\ &\quad + a_0' z^\alpha (\alpha - \alpha_1) \cdot 2(\alpha - \alpha_2). \end{aligned} \tag{7.9}$$

Then it follows that in addition to $(w)_{\alpha=\alpha_1}$ and $(w)_{\alpha=\alpha_2}$, the $\big(\frac{\partial w}{\partial \alpha}\big)_{\alpha=\alpha_2}$ is also a solution to the original differential equation. We first observe that the two power series solutions obtained for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ differs only by a constant multiple. To see this, we note that $a_0 = a_0'(\alpha - \alpha_2)$ by the above construction, so the recurrence relation (7.3) implies that each of $a_1, a_2, \cdots, a_{s-1}$ has a common factor $(\alpha - \alpha_2)$, and thus vanish when $\alpha = \alpha_2$. That is,

$$(a_1)_{\alpha=\alpha_2},\ (a_2)_{\alpha=\alpha_2},\ \cdots,\ (a_{s-1})_{\alpha=\alpha_2} = 0.$$

When $k = s$ in (7.3), then $F(\alpha_2 + s) = F(\alpha_1) = 0$. So a_s is arbitrary (as discussed in Chapter 4). For the a_k when $k > s$, we write $k = k' + s$, $a_j' = a_{j+s}$ are determined by

$$\begin{aligned} &F(\alpha_2 + k)a_k + \sum_{j=0}^{k-1} a_j\big[(\alpha_2 + j)p_{k-j} + q_{k-j}\big] \\ &= F(\alpha_2 + s + k')a_{s+k'} + \Big(\sum_{j=0}^{s-1} + \sum_{j=s}^{s+k'-1}\Big) a_j\big[(\alpha_2 + j)p_{k-j} + q_{k-j}\big] \\ &= F(\alpha_1 + k')a_{s+k'} + \sum_{j=0}^{k'-1} a_{s+j}\big[(\alpha_1 + j)p_{k-(j+s)} + q_{k-(j+s)}\big] \\ &= F(\alpha_1 + k')a_{s+k'} + \sum_{j=0}^{k'-1} a_{s+j}\big[(\alpha_1 + j)p_{k'-j} + q_{k'-j}\big] \\ &= F(\alpha_1 + k')a_{k'}' + \sum_{j=0}^{k'-1} a_j'\big[(\alpha_1 + j)p_{k'-j} + q_{k'-j}\big], \end{aligned}$$

for $k' \geq 1$, which is identical to the recurence relation (7.3). So we have essentially shown that

$$(w)_{\alpha=\alpha_2} = z^{\alpha_2} z^s \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k = z^{\alpha_1} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k$$

holds, where $(a_0)_{\alpha=\alpha_1}$, thus proving that $(w)_{\alpha=\alpha_2}$ is a constant multiple of $(w)_{\alpha=\alpha_1}$.

Thus a second linearly independent solution is given by

$$\begin{aligned}
w_2(z) = \Big(\frac{\partial w}{\partial \alpha}\Big)_{\alpha=\alpha_2} &= \log z \cdot z^{\alpha_2} \sum_{k=s}^{\infty} (a_k)_{\alpha=\alpha_2} z^k + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= w_1(z) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= \Big(z^{\alpha_1} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^k\Big) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k \\
&= \Big(z^{\alpha_2} \sum_{k=0}^{\infty} (a_k)_{\alpha=\alpha_1} z^{k+s}\Big) \log z + z^{\alpha_2} \sum_{k=0}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=\alpha_2} z^k,
\end{aligned}$$

which is precisely the second case described in Theorem 4.8.

7.3 Hypergeometric Functions: degenerate cases

It suffices to consider the case when c is an integer and around $z = 0$, for the other cases can be covered by applying Riemann's $P-$scheme. We first consider the $c = 1$. Then the two exponents at $z = 0$ are double root $\alpha_1 = \alpha_2 = 0$. The first solution is $w_1(z) = {}_2F_1(a,\, b;\, c;\, z)$. The second solution follows from (7.8) that

$$w_2(z) = w_1(z) \log z + \sum_{k=1}^{\infty} \Big(\frac{\partial a_k}{\partial \alpha}\Big)_{\alpha=0} z^k.$$

Recall that the coefficients satisfies the recurrence relation:

$$\begin{aligned}
a_k &= \frac{(\alpha + a + k - 1)(\alpha + b + k - 1)}{(\alpha + 1 + k - 1)(\alpha + c + k - 1)} a_{k-1} = \frac{(\alpha + a)_k (\alpha + b)_k}{(\alpha + 1)_k (\alpha + c)_k} a_0 \\
&= \frac{\Gamma(\alpha + 1)\Gamma(\alpha + c)}{\Gamma(\alpha + a)\Gamma(\alpha + b)} \cdot \frac{\Gamma(\alpha + a + k)\Gamma(\alpha + b + k)}{\Gamma(\alpha + 1 + k)\Gamma(\alpha + c + k)},
\end{aligned}$$

where we have chosen $a_0 = 1$. Thus,

$$\begin{aligned}
w_2(z) = {}_2F_1(a,\, b;\, c;\, z) \log z + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k \big\{\psi(a + k) + \psi(b + k) - \psi(c + k) \\
- \psi(1 + k) - \psi(a) - \psi(b) + \psi(c) + \psi(1)\big\},
\end{aligned} \tag{7.10}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Putting $c = 1$ and consider the case when **both a and b are not negative integers**. Then one can further write the above solution in the form

$$w_2(z) = {}_2F_1(a,\, b;\, 1;\, z) \log z \qquad (7.11)$$
$$+ \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} z^k \big\{\psi(a+k) + \psi(b+k) - \psi(c+k) - \psi(1+k)\big\},$$

where we have subtracted a multiple $-\psi(a) - \psi(b) + 2\psi(1)$ of ${}_2F_1(a,\, b;\, c;\, z)$ from the above solution.

If, however, **one of a and b is an negative integer**, $a = -n$ say, then one of the terms from $\psi(a+k)$ will have a pole for $k = n$, so that the above solution is undefined. But it is still possible to rewrite the (7.10) by making use of the identity

$$\psi(1-z) = \psi(z) + \pi \cot \pi z$$

into a form that is manageable. For we can write

$$\lim_{a \to -n} \{(\psi(a+k) - \psi(a)\} = \lim_{a \to -n} \{(\psi(1-a-k) - \psi(1-a)\} = \psi(1+n-k) - \psi(1+n),$$

which is finite, provided $k \le n$. Substituting this expression back into (7.11) and take away a constant multiple of the first solution yield

$$w_2(z) = {}_2F_1(-n,\, b;\, 1;\, z) \log z$$
$$+ \sum_{k=1}^{n} \frac{(-n)_k (b)_k}{(k!)^2} z^k \big\{\psi(1+n-k) - \psi(1+n) + \psi(b+k) - 2\psi(1+k)\big\}. \qquad (7.12)$$

The case when **both a, b are negative** can be handled in a similar fashion.

If $c \ge 2$, then we have $\alpha_2 = 1 - c$ is a negative integer. That is, we have $s = c - 1$. The situtation becomes more tedious to describe. We merely quote the results here.

If **both a and b are not negative integers**, then

$$w_2(z) = {}_2F_1(a,\, b;\, s+1;\, z) \log z$$
$$+ \frac{s!}{\Gamma(a)\Gamma(b)} \sum_{k=1}^{s} (-1)^{k-1} (k-1)! \frac{\Gamma(a-k)\Gamma(b-k)}{(s-k)!} z^{-k}$$
$$+ \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(s+1)_k k!} z^k \big\{\psi(a+k) + \psi(b+k) - \psi(s+1+k) - \psi(1+k)\big\}.$$

If, however, **one of a or b is a negative integer** (a say), the

$$\begin{aligned} w_2(z) = {}_2F_1(-n,\, b;\, s+1;\, z)\log z \\ & - \frac{s!}{\Gamma(b)} \sum_{k=1}^{s} \frac{(k-1)!\Gamma(b-k)}{(s-k)!(n+1)_k} z^{-k} \\ & + \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(s+1)_k k!} z^k \big\{\psi(1+n-k) + \psi(b+k) - \psi(s+1+k) - \psi(1+k)\big\}. \end{aligned}$$

Chapter 8

Confluent Hypergeometric Functions

We shall discuss the confluent hypergeometric equations or the Kummer equations and their solutions in this section. Each of these equations has two free parameters, which can be transformed to a Whittaker equation, or a Coulomb-Wave equation. Each confluent hypergeometric equation can also be transformed to a Bessel equation after a suitable choice its two free parameters. These equations have extensive applications.

8.1 The derivation of the equation

Let us recall that the Gauss hypergeometric equation is given by

$$z(1-z)\frac{d^2y}{dz^2}+[c-(a+b+1)z]\frac{dy}{dz}-ab\,y=0 \tag{8.1}$$

which has regular singular points at 0, 1, ∞ in the extended complex plane $\hat{\mathbf{C}}$.

This equation has the **hypergeometric series**

$$y(z)={}_2F_1(a,\,b;\,c;\,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{8.2}$$

which has a radius of convergence equals to unity as one of its solutions provided $b\neq 0,\,-1,\,-2,\cdots$. A transformation of $Y(x)=y(x/b)$ in the (8.2) yields

$$z(1-z/b)\frac{d^2Y}{dz^2}+[c-(a+b+1)z/b]\frac{dY}{dx}-a\,Y=0 \tag{8.3}$$

which clearly has regular singular points at 0, b, ∞. Letting $b\to\infty$ in the equation (8.3) results in the

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-a\,y=0, \tag{8.4}$$

which is called the **confluent hypergeometric equation**. Clearly the same transformation z/b applied to (8.1) and letting $b \to \infty$ results in a series

$$y_1(z) = \Phi(a,\, c; z) := {}_1F_1\,[a;\, c; z] = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}. \tag{8.5}$$

This solution is called **Kummer function** or **Kummer series**. We note that the hypergeometric series has radius of convergence 1, and after the above substitution of z/b results in a series that has radius of convergence equal to $|b|$. So the Kummer series has an infinite radius of convergence. Thus, the Kummer function (8.5) is an entire function. It is readily checked that the equation (8.4) has a regular singular point at 0 and an irregular singular point at ∞. The original Bateman project adopted [9, p. 248] the notation $\Phi(a,\, c;\, x)$ the authors attributed the credit to Humbert. Another popular notation for the $\Phi(a,\, c;\, x)$ is $M(a,\, c;\, x)$. We note that the Φ is undefined when c is a negative integer. We will not discuss a modification of the function. See [9].

8.2 Tricomi Functions

A second linearly independent solution of (8.4) can be obtained from a second solution of (8.1), namely

$$x^{1-c}{}_2F_1(1+a-c,\, 1+b-c;\, 2-c;\, x) \tag{8.6}$$

by making the change of variable $z = x/b$ in this ${}_2F_1$ and let $b \to \infty$ as in the earlier case. This yields

$$y_2 = x^{1-c}{}_1F_1(1+a-c;\, 2-c;\; x), \tag{8.7}$$

provided that $c \neq -1,\, 0,\, 1,\, \cdots$. However, a more important second linearly independent solution to $\Phi(a;\, c;\, x)$ is defined by

$$\Psi(a;\, c;\, x) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a;\, c;\, x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1;\, 2-c;\, x). \tag{8.8}$$

According to [9, Vol. 1, section 6.6, (7)], the function Ψ is defined by Tricomi (1927) with the notation G. The notation $U(a;\, c;\, x)$ [2, section 13.3], [20, section 1.3] is also commonly used. We refer the readers to Slater [20, section 1.3] for the list of notations for Ψ that were used in the older literature.

8.3 Relation with Bessel functions

We define the **Bessel function of first kind of order** ν to be the complex function represented by the power series

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{\nu+2k}}{\Gamma(\nu+k+1)\,k!} = z^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2})^{\nu+2k}}{\Gamma(\nu+k+1)\,k!}\, z^{2k}. \tag{8.9}$$

Here ν is an arbitrary complex constant.

Let us set $a = \nu + \frac{1}{2}$, $c = 2\nu + 1$ and replace z by $2iz$ in the Kummer series.

That is,

$$\begin{aligned}
\Phi(\nu+1/2,\, 2\nu+1; 2iz) &= \sum_{k=0}^{\infty} \frac{(\nu+\frac{1}{2})_k}{(2\nu+1)_k\,k!}(2iz)^k \\
&= {}_1F_2\left(\begin{matrix}\nu+\frac{1}{2}\\ 2\nu+1\end{matrix}\,\middle|\, 2iz\right) \\
&= e^{iz}{}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix}\,\middle|\, -\big(\frac{z}{2}\big)^2\right) \\
&= e^{iz}\sum_{k=0}^{\infty} \frac{(-1)^k}{(\nu+1)_k\,k!}\frac{z^{2k}}{2^{2k}} \\
&= e^{iz}\sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(\nu+1)}{\Gamma(\nu+k+1)\,k!}\big(\frac{z}{2}\big)^{2k} \\
&= e^{iz}\Gamma(\nu+1)\big(\frac{z}{2}\big)^{-\nu}\cdot\big(\frac{z}{2}\big)^{\nu}\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\big(\frac{z}{2}\big)^{2k} \\
&= e^{iz}\Gamma(\nu+1)\big(\frac{z}{2}\big)^{-\nu}\cdot J_\nu(z),
\end{aligned}$$

where we have applied Kummer's second transformation

$${}_1F_1\left(\begin{matrix}a\\ 2a\end{matrix}\,\middle|\, 4x\right) = e^{2x}\,{}_0F_1\left(\begin{matrix}-\\ a+\frac{1}{2}\end{matrix}\,\middle|\, x^2\right) \tag{8.10}$$

in the third step above, and the identity $(\nu)_k = \Gamma(\nu+k)/\Gamma(\nu)$ in the fourth step.

Since the confluent hypergeometric function $\Phi(\nu+1/2,\, 2\nu+1; 2iz)$ satisfies the hypergeometric equation

$$z\frac{d^2y}{dz^2} + (c-z)\frac{dy}{dz} - a\,y = 0 \tag{8.11}$$

with $a = \nu + 1/2$ and $c = 2\nu + 1$. Substituting $e^{iz}\Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu} \cdot J_\nu(z)$ into the (8.11) and simplfying lead to the equation

$$z^2\frac{d^2y}{dz^2} + x\frac{dy}{dz} + (z^2 - \nu^2)y = 0. \tag{8.12}$$

This is called the **Bessel equation with parameter** ν. It is clearly seen that $z = 0$ is a regular singular point and that $Z = \frac{1}{z}$ has an irregular singular point at $Z = \frac{1}{0} = \infty$ for the transformed equation with $Y(z) = y(1/z)$.

The above derivation shows that the Bessel function of the first kind with order ν is a special case of the confluent hypergeometric functions with specifically chosen parameters.

8.4 Relation with Whittaker equation

The transformation $y = x^{-c/2}e^{x/2}W$, $a = \frac{1}{2} - \kappa + \mu$ and $c = 1 + 2\mu$ will transform the confluent hypergeometic equation to the so-called **Whittaker equation**

$$\frac{d^2W}{dx^2} + \Big(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\Big)W = 0.$$

8.5 Relation with Coulomb Wave equation

The transformation $y = e^{ix}x^{-L-1}F_L(\eta,\, x)/C_L(\eta)$, and with $a = L + 1 - i\eta$, $b = 2L + 2$ and $z = 2ix$ transforms the $M(a,\, b;\; z)$ to the **Coulomb Wave equation**

$$\frac{d^2w}{dx^2} + \Big(1 - \frac{2\eta}{x} - \frac{L(L+1)}{x^2}\Big)w = 0,$$

where $w = c_1F_L(\eta,\, x) + c_2G_L(\eta,\, x)$ are the **regular and irregular Coulomb wave functions**.

8.6 Differential Contiguous Relations of Kummer Functions

The relation

$$\Phi\,(a;\, c;\, x) = e^x\,\Phi\,(c - a;\, c;\, -x) \tag{8.13}$$

thus obtained is called **Kummer's first transformation**.

$$x^{1-c}\Phi\,(1 + a - c;\, 2 - c;\, x) = x^{1-c}e^x\,\Phi\,(1 - a;\, 2 - c;\, -x). \tag{8.14}$$

Two Φ functions are said to be **contiguous** if any one of their corresponding parameters differs by one. If we adopt the Gauss notation for hypergeometric functions that $\Phi(a\pm)$ denotes $\Phi(a\pm 1;\ c;\ x)$, and $\Phi(c\pm)$ denotes $\Phi(a;\ c\pm 1;\ x)$, then the $\Phi(a\pm)$ or the $\Phi(c\pm)$ are contiguous to $\Phi(a;\ c;\ x)$. We assume that c is not an integer because $\Phi(a;\ c;\ x)$ will be undefined then. The following relations can be deduced from Gauss's differential contiguous relations. But they can be verified directly. We differentiate the series expansion of Φ (since it is absolutely convergent)

$$\frac{d}{dx}\Phi(a;\ c;\ x) = \frac{a}{c}\,\Phi(a+1;\ c+1;\ x) = \frac{a}{c}\,\Phi(a+;\ c+) \tag{8.15}$$

and similarly,

$$\frac{d^k}{dx^k}\Phi(a;\ c;\ x) = \frac{(a)_k}{(c)_k}\,\Phi(a+k;\ c+k;\ x), \tag{8.16}$$

for any positive integer k. In addition, the following differential relations are readily verified. Thus,

$$\frac{d}{dx}\,x^a\Phi(a;\ c;\ x) = ax^{a-1}\,\Phi(a+1;\ c;\ x), \tag{8.17}$$

$$\frac{d}{dx}\,x^{c-1}\Phi(a;\ c;\ x) = (c-1)x^{c-2}\,\Phi(a;\ c-1;\ x) \tag{8.18}$$

and more generally,

$$\frac{d^k}{dx^k}\,x^{a+k-1}\Phi(a;\ c;\ x) = (a)_k x^{a-1}\,\Phi(a+k;\ c;\ x), \tag{8.19}$$

$$\frac{d^k}{dx^k}\,x^{c-1}\Phi(a;\ c;\ x) = (-1)^k(1-c)_k x^{c-1-k}\,\Phi(a;\ c-k;\ x), \tag{8.20}$$

for every positive integer k. Applying Kummer's first theorem (8.13) to (8.16) yields

$$\begin{aligned}\frac{d^k}{dx^k}\,e^{-x}\Phi(a;\ c;\ x) &= \frac{d^k}{dx^k}\,\Phi(c-a;\ c;\ -x)\\ &= \frac{(-1)^k(c-a)_k}{(c)_k}\,\Phi(c-a+k;\ c+k;\ -x)\\ &= \frac{(-1)^k(c-a)_k}{(c)_k}\,e^{-x}\Phi(a;\ c+k;\ x).\end{aligned} \tag{8.21}$$

Similarly, the application of Kummer's first theorem (8.13) to (8.19) and (8.20) yield, respectively,

$$\begin{aligned}\frac{d^k}{dx^k}\,e^{-x}x^{c-a+k-1}\Phi(a;\ c;\ x) &= \frac{d^k}{dx^k}\,x^{c-a+k-1}\Phi(c-a;\ c;\ -x)\\ &= (c-a)_k\,x^{c-a-1}\Phi(c-a+k;\ c;\ -x)\\ &= (c-a)_k\,x^{c-a-1}e^{-x}\Phi(a-k;\ c;\ x),\end{aligned} \tag{8.22}$$

and

$$\frac{d^k}{dx^k}\, e^{-x} x^{c-1} \Phi(a;\ c;\ x) = \frac{d^k}{dx^k}\, x^{c-1} \Phi(a-c;\ c;\ -x)$$
$$= (-1)^k (1-c)_k\, x^{c-k-1} \Phi(a-c;\ c-k;\ -x)$$
$$= (-1)^k e^{-x} (1-c)_k\, x^{c-k-1} \Phi(a-k;\ c-k;\ x). \tag{8.23}$$

The (8.17) can be written in the form

$$\Phi(a+1;\ c;\ x) = \Phi(a;\ c;\ x) + \frac{x}{a}\, \Phi'(a;\ c;\ x), \tag{8.24}$$

and that of (8.18) is

$$\Phi(a;\ c-1;\ x) = \Phi(a;\ c;\ x) + \frac{x}{c-1}\, \Phi'(a;\ c;\ x). \tag{8.25}$$

The special cases of (8.21), (8.22) and (8.23) when $k = 1$ are, respectively,

$$\Phi(a;\ c+1;\ x) = \frac{c}{c-a}\, \Phi(a;\ c;\ x) - \frac{c}{c-a}\, \Phi'(a;\ c;\ x), \tag{8.26}$$

$$\Phi(a-1;\ c;\ x) = \Big(1 - \frac{x}{c-a}\Big)\, \Phi(a;\ c;\ x) + \frac{x}{c-a}\, \Phi'(a;\ c;\ x), \tag{8.27}$$

and

$$\Phi(a-1;\ c-1;\ x) = \Big(1 - \frac{x}{c-1}\Big)\, \Phi(a;\ c;\ x) + \frac{x}{c-1}\, \Phi'(a;\ c;\ x), \tag{8.28}$$

8.7 Contiguous Relations of Kummer Functions Φ

One can deduce from the fifteen contiguous relations exist for the hypergeometric functions that there are six contiguous relatoins for the Kummer functions. One can also use power series expansions to derive the relations. Alternatively they can also be derived from the differential contiguous relations (8.15), (8.25)–(8.28) by eliminating the Φ' below. They are

$$(c-a)\, \Phi(a-1;\ c;\ x) + (2a-c+x)\, \Phi(a;\ c;\ x) - a\, \Phi(a+1;\ c;\ x) = 0, \tag{8.29}$$

$$c(c-1)\, \Phi(a;\ c-1;\ x) - c(c-1+x)\, \Phi(a;\ c;\ x) + (c-a)x\, \Phi(a;\ c+1;\ x) = 0, \tag{8.30}$$

$$(1+a-c)\, \Phi(a;\ c;\ x) - a\, \Phi(a+1;\ c;\ x) + (c-1)\, \Phi(a;\ c-1;\ x) = 0, \tag{8.31}$$

$$c\, \Phi(a;\ c;\ x) - c\, \Phi(a-1;\ c;\ x) - x\, \Phi(a;\ c+1;\ x) = 0, \tag{8.32}$$

$$c(a+x)\, \Phi(a;\ c;\ x) - (c-a)x\, \Phi(a;\ c+1;\ x) - ac\, \Phi(a+1;\ c;\ x) = 0, \tag{8.33}$$

$$(a-1+x)\, \Phi(a;\ c;\ x) + (c-a)\, \Phi(a-1;\ c;\ x) - (c-1)\, \Phi(a;\ c-1;\ x) = 0. \tag{8.34}$$

Each function $\Phi(a \pm n;\, c \pm m)$, $n, m = 0, 1, 2, \cdots$, is said to be **associated** with $\Phi(a;\, c;\, x)$. Because of the (8.15), thus the derivative Φ' is always contiguous with Φ. In fact, any three associated Kummer functions satisfies a homogeneous functional equation with polynomial coefficients in x by repeated applications of the contiguous relations above. For example, the continguous relations (8.31) and (8.32) yields

$$c(1-c-x)\,\Phi(a;\, c;\, x)+c(c-1)\,\Phi(a-1;\, c-1;\, x)-ax\,\Phi(a+1;\, c+1;\, x)=0. \tag{8.35}$$

Chapter 9

Bessel Equations

It can be shown that the Wronskian of J_ν and $J_{-\nu}$ is given by (G. N. Watson "A Treatise On The Theory Of Bessel Functions", pp. 42–43):

$$W(J_\nu,\, J_{-\nu}) = -\frac{2\sin\nu\pi}{\pi z}. \tag{9.1}$$

This shows that the J_ν and $J_{-\nu}$ forms a **fundamental set of solutions** when ν is not equal to an integer. In fact, when $\nu = n$ is an integer, then we can easily check that

$$J_{-n}(z) = (-1)^n J_n(z). \tag{9.2}$$

Thus it requires an extra effort to find another linearly independent solution. It turns out a second linearly independent solution is given by

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \tag{9.3}$$

when ν is not an integer. The case when ν is an integer n is defined by

$$Y_n(z) = \lim_{\nu\to n}\frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}. \tag{9.4}$$

The Y_ν so defined is linearly independent with J_ν for *all* values of ν.

In particular, we obtain

$$Y_n(z) = \frac{-1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{z}{2}\Big)^{2k-n} \tag{9.5}$$

$$+\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{n+2k}}{k!(n+k)!}\Big[2\log\frac{z}{2} - \psi(k+1) - \psi(k+n+1)\Big] \tag{9.6}$$

for $|\arg z| < \pi$ and $n = 0,\, 1,\, 2,\, \cdots$ with the understanding that we set the sum to be 0 when $n = 0$. Here the $\psi(z) = \Gamma'(z)/\Gamma(z)$. We note that the function is unbounded when $z = 0$.

9.1 Fourier-Bessel Series

Theorem 9.1. *Suppose the real function $f(r)$ is piecewise continuous in $(0,\, a)$ and of bounded variation in every subset $[r_1,\, r_2] \subset (0,\, a)$. If*

$$\int_0^a \sqrt{r}|f(r)|\, dr < \infty, \tag{9.7}$$

*then the **Fouier-Bessel series***

$$\sum_{k=1}^{\infty} c_k J_\nu\big(x_{\nu k}\frac{r}{a}\big) \tag{9.8}$$

converges to $f(r)$ at every continuity point of $f(r)$ and to

$$\frac{1}{2}[f(r+0)+f(r-0)] \tag{9.9}$$

at every discontinuity point of $f(r)$.

The $x_{\nu k}$ are the zero of J_ν $k = 0,\, 1,\, \cdots$. Orthogonality of the bessel functions omitted.

9.2 Physical Applications of Bessel Functions

We consider **radial vibration of circular membrane**. We assume that an elastic circular membrane (radius ℓ) can vibrate and that the material has a uniform density. Let $u(x,\, y,\, t)$ denote the displacement of the membrane at time t from its equilibrium position. We use polar coordinate in the $xy-$plane by the change of variables:

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{9.10}$$

Then the corresponding equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \tag{9.11}$$

can be transformed to the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big). \tag{9.12}$$

Since the membrane has uniform density, so $u = u(r,\, t)$ that is, it is independent of the θ. Thus we have

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r}\Big). \tag{9.13}$$

The boundary condition is $u(\ell, t) = 0$, and the initial conditions take the form

$$u(r, t) = f(r), \qquad \frac{\partial u(r, 0)}{\partial t} = g(r). \tag{9.14}$$

Separation of variables method yields

$$u(r, t) = R(r)T(t), \tag{9.15}$$

which satisfies the boundary condition $u(\ell, t) = 0$. Thus

$$RT'' = c^2\Big(R''T + \frac{1}{r}R'T\Big). \tag{9.16}$$

Hence

$$\frac{R'' + (1/r)R'}{R} + \frac{T''}{c^2T}; = -\lambda^2, \tag{9.17}$$

where the λ is a constant. Thus

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0 \tag{9.18}$$

$$T'' + c^2\lambda^2 T = 0. \tag{9.19}$$

We notice that the first equation is Bessel equation with $\nu = 0$. Thus its general solution is given by

$$R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r). \tag{9.20}$$

. Since Y_0 is unbounded when $r = 0$, so $C_2 = 0$. Thus the boundary condition implies that

$$J_0(\lambda\ell) = 0, \tag{9.21}$$

implying that $\mu = \lambda\ell$ is a zero of $J_0(\mu)$. Setting $C_1 = 1$, we obtain for each integer $n = 1, 2, 3, \cdots$,

$$R_n(r) = J_0(\lambda_n r) = J_0(\frac{\mu_n}{\ell}r), \tag{9.22}$$

where $\mu_n = \lambda_n\ell$ is the $n-$th zero of $J_0(\mu)$. Thus we have

$$u_n(r, t) = (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \cdot J_0(\lambda_n r), \tag{9.23}$$

for $n = 1, 2, 3, \cdots$. Thus the general solution is given by

$$\sum_{n=1}^{\infty} u_n(r, t) = \sum_{n=1}^{\infty}(A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \cdot J_0(\lambda_n r), \tag{9.24}$$

and by

$$f(r) = u(r, 0) = \sum_{n=1}^{\infty} A_n \cdot J_0(\lambda_n r), \tag{9.25}$$

and

$$g(r) = \frac{\partial u(r,0)}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} B_n c\lambda_n \cdot J_0(\lambda_n r). \tag{9.26}$$

Fourier-Bessel Series theory implies that

$$A_n = \frac{2}{\ell^2 J_1^2(\mu_n)} \int_0^{\ell} r f(r) J_0(\lambda_n r)\, dr, \tag{9.27}$$

$$B_n = \frac{2}{c\lambda_n \ell^2 J_1^2(\mu_n)} \int_0^{\ell} r g(r) J_0(\lambda_n r)\, dr. \tag{9.28}$$

9.3 Basic Properties of Bessel Functions

The general reference for Bessel functions is G. N. Watson's classic: "A Treatise on the Theory of Bessel Functions", published by Cambridge University Press in 1922 [22].

9.3.1 Zeros of Bessel Functions

See A. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions with formulas, graphs and mathematical tables", 10th Edt., National Bureau of Standards, 1964.

9.3.2 Recurrence Formulae for J_ν

We consider arbitrary complex ν.

$$\begin{aligned}
\frac{d}{dz} z^\nu J_\nu(z) &= \frac{d}{dz} \frac{(-1)^k z^{2\nu+2k}}{2^{\nu+2k}\, k!\Gamma(\nu+k+1)} \\
&= \frac{d}{dz} \frac{(-1)^k z^{2\nu-1+2k}}{2^{\nu-1+2k}\, k!\Gamma(\nu+k)} \\
&= z^\nu J_{\nu-1}(z).
\end{aligned}$$

But the left side can be expanded and this yields

$$z J_\nu'(z) + \nu J_\nu(z) = z J_{\nu-1}(z). \tag{9.29}$$

Similarly,

$$\frac{d}{dz} z^{-nu} J_\nu(z) = -z^{-\nu} J_{\nu+1}(z). \tag{9.30}$$

and this yields

$$z J_\nu'(z) - \nu J_\nu(z) = -z J_{\nu+1}(z). \tag{9.31}$$

Substracting and adding the above recurrence formulae yield

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{9.32}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z). \tag{9.33}$$

9.3.3 Generating Function for J_n

Jacobi in 1836 gave

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} t^k J_k(z). \tag{9.34}$$

Many of the forumulae derived above can be obtained from this expression.

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} c_k(z) t^k \tag{9.35}$$

for $0 < |t| < \infty$. We multiply the power series

$$e^{\frac{zt}{2}} = 1 + \frac{(z/2)}{1!}t + \frac{(z/2)^2}{1!}t^2 + \cdots \tag{9.36}$$

and

$$e^{-\frac{zt}{2}} = 1 - \frac{(z/2)}{1!}t + \frac{(z/2)^2}{1!}t^2 - \cdots \tag{9.37}$$

Multiplying the two series and comparing the coefficients of t^k yield

$$c_n(z) = J_n(z), \quad n = 0,\, 1,\, \cdots \tag{9.38}$$

$$c_n(z) = (-1)^n J_{-n}(z), \quad n = -1,\, -2,\, \cdots\,. \tag{9.39}$$

Thus

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = J_0(z) + \sum_{k=1}^{+\infty} J_k[t^k + (-1)^k t^{-k}]. \tag{9.40}$$

9.3.4 Lommel's Polynomials

Iterating the recurrence formula

$$J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1} \tag{9.41}$$

with respect to ν a number of times give

$$J_{\nu+k}(z) = P(1/z)J_\nu(z) - Q(1/z)J_{\nu-1}. \tag{9.42}$$

Lommel (1871) [See Watson, pp. 294–295] found that

$$J_{\nu+k}(z) = R_{k,\nu}(z)J_\nu(z) - R_{k-1,\nu+1}J_{\nu-1}. \tag{9.43}$$

9.3.5 Bessel Functions of half-integer Orders

One can check that

$$J_{-\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \cos z, \qquad J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z. \tag{9.44}$$

Moreover,

$$J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \Big(\frac{d}{z\,dz}\Big)^n \frac{\sin z}{z}, \qquad n = 0,\, 1,\, 2,\, \cdots.$$

Thus applying a recurrence formula and using the Lommel polynomials yield

$$J_{n+\frac{1}{2}}(z) = R_{n,\nu}(z) J_{\frac{1}{2}}(z) - R_{n-1,\nu+1} J_{-\frac{1}{2}}(z) \tag{9.45}$$

That is, we have

$$J_{n+\frac{1}{2}}(z) = R_{n,\nu}(z) \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z - R_{n-1,\nu+1} \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z. \tag{9.46}$$

9.3.6 Formulae for Lommel's polynomials

For each fixed ν, the Lommel polynomials are given by

$$R_{n\,\nu}(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!(\nu)_{n-k}}{k!(n-2k)!(\nu)_k} \left(\frac{2}{z}\right)^{n-2k} \tag{9.47}$$

where the $[x]$ means the largest integer not exceeding x. These Lommel polynomials have remarkable properties. Since

$$J_{-\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \cos z, \qquad J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z \tag{9.48}$$

and $\sin^2 x + \cos^2 x = 1$; we now have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = 2(-1)^n \frac{R_{2n,\,\frac{1}{2}-n}(z)}{\pi z}. \tag{9.49}$$

That is, we have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = \frac{2}{\pi z} \sum_{k=0}^{n} \frac{(2z)^{2n-2k}(2n-k)!(2n-2k)!}{[(n-k)!]^2 k!}. \tag{9.50}$$

A few special cases are

1. $J^2_{\frac{1}{2}}(z) + J^2_{-\frac{1}{2}}(z) = \dfrac{2}{\pi z}$;

2. $J^2_{\frac{3}{2}}(z) + J^2_{-\frac{3}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{1}{z^2}\Big)$;

3. $J^2_{\frac{5}{2}}(z) + J^2_{-\frac{5}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{3}{z^2} + \dfrac{9}{z^4}\Big)$;

4. $J^2_{\frac{7}{2}}(z) + J^2_{-\frac{7}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{6}{z^2} + \dfrac{45}{z^4} + \dfrac{225}{z^6}\Big)$

9.3.7 Orthogonality of the Lommel Polynomials

Let us set

$$h_{n,\,\nu}(z) = R_{n,\,\nu}(1/z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!(\nu)_{n-k}}{k!(n-2k)!(\nu)_k} \Big(\frac{z}{2}\Big)^{n-2k}, \tag{9.51}$$

then the set $\{h_{n\,\nu}(z)$ is called the *modified Lommel polynomials.* Since the Bessel functions $J_\nu(z)$ satisfies a three-term recurrence relation, so the Lommel polynomials inherit this property:

$$2z(n+\nu)h_{n\,\nu}(z) = h_{n+1,\,\nu}(z) + h_{n-1,\,\nu}(z) \tag{9.52}$$

with *initial conditions*

$$h_{0,\,\nu}(z) = 1, \qquad h_{1,\,\nu}(z) = 2\nu z. \tag{9.53}$$

If one start with a different set of *initial conditions*

$$h^*_{0,\,\nu}(z) = 0, \qquad h^*_{1,\,\nu}(z) = 2\nu, \tag{9.54}$$

then the sequence $\{h^*_{n,\,\nu}(z)\}$ generated by the (9.52) is called the *associated Lommel polynomials.* It is known that a three-term recurrence relation for polynomials with the coefficients of as in (9.52) will generate a set of orthogonal polynomials on $(-\infty,\,+\infty)$. That is, there is a probability measure α onb $(-\infty,\,+\infty)$ with $\int_{-\infty}^{+\infty} d\alpha = 1$

Theorem 9.2 (A. A. Markov, 1895)**.** *Suppose the set of* $\{p_n(z)\}$*of orthogonal polynomials with its measure* α *supported on a bounded internal* $[a,\,b]$, *then*

$$\lim_{n\to\infty} \frac{p^*_{n\,\nu}(z)}{p_{n\,\nu}(z)} = \int_a^b \frac{d\alpha(t)}{z-t} \tag{9.55}$$

holds uniformly for $z \notin [a,\,b]$.

Since we know

Theorem 9.3 (Hurwitz)**.** *The limit*

$$\lim_{n\to\infty}\frac{(z/2)^{\nu+n}R_{n,\,\nu+1}(z)}{\Gamma(n+\nu+1)}=J_\nu(z), \tag{9.56}$$

holds uniformly on compact subsets of $\mathbf{C}$*.*

So we have

Theorem 9.4. *For* $\nu>0$*, the polynomials* $\{h_{n\,\nu}(z)$ *are orthogonal with respect to a discrete measure* α_ν *normalized to have* $\int_{-\infty}^{+\infty}d\alpha(t)=1$*, and*

$$\int_{\mathbf{R}}\frac{d\alpha(t)}{z-t}=2\nu\frac{J_\nu(1/z)}{J_{\nu-1}(1/z)}. \tag{9.57}$$

One can use this formula to recover the measure $\alpha(t)$. Thus the Lommel "polynomials" was found to have very distinct properties to be complete in $L^2(-1,+1)$.

9.4 Integral formaulae

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\cos zt\,dt,\quad \Re(\nu)>-1/2,\quad |\arg z|<\pi. \tag{9.58}$$

Or equivalently,

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_{0}^{\pi}\cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta,\quad \Re(\nu)>-1/2,\quad |\arg z|<\pi, \tag{9.59}$$

where $t=\cos\theta$.

Writing $H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)$. Then

Theorem 9.5 (Hankel 1869)**.**

$$H_\nu^{(1)}(z)=\Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}}\frac{e^{i[z-\nu\pi/2-\pi/4]}}{\Gamma(\nu+1/2)}\int_0^{\infty\cdot\exp(i\beta)}e^{-u}u^{\nu-\frac{1}{2}}\Big(1+\frac{iu}{2z}\Big)^{\nu-\frac{1}{2}}du, \tag{9.60}$$

where $|\beta|<\pi/2$*.*

9.5 Asymptotic Behaviours of Bessel Functions

Expanding the integrand of Henkel's contour integral by binomial expansion and after some careful analysis, we have

Theorem 9.6. *For $-\pi < \arg z < 2\pi$,*

$$H_\nu^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2}-\nu)_m(\frac{1}{2}+\nu)_m}{(2ix)^m m!} + R_p^{(1)}(x)\right], \quad (9.61)$$

and for $-2\pi < \arg x < \pi$,

$$H_\nu^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2}-\nu)_m(\frac{1}{2}+\nu)_m}{(2ix)^m m!} + R_p^{(2)}(x)\right], \quad (9.62)$$

where

$$R_p^{(1)}(x) = O(x^{-p}) \qquad \text{and} \qquad R_p^{(2)}(x) = O(x^{-p}), \quad (9.63)$$

as $x \to +\infty$, uniformly in $-\pi+\delta < \arg x < 2\pi-\delta$ and $-2\pi+\delta < \arg x < \pi-\delta$ respectively.

The two expansions are valid simultaneously in $-\pi < \arg x < \pi$.

We thus have $J_\nu(z) = \frac{1}{2}(H_\nu^{(1)}(z) + H_\nu^{(2)}(z))$. So

Theorem 9.7. *For $|\arg z| < \pi$, we have*

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \Big[\cos\big(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\big) \sum_{k=0}^{\infty} \frac{(-1)^k(\nu,\, 2k)}{(2z)^{2k}} - \sin\big(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\big) \sum_{k=0}^{\infty} \frac{(-1)^k(\nu,\, 2k+1)}{(2z)^{2k+1}}\Big]$$

9.5.1 Addition formulae

Schläfli (1871) derived

$$J_\nu(z+t) = \sum_{k=-\infty}^{\infty} J_{\nu-k}(t) J_k(z). \quad (9.64)$$

Theorem 9.8 (Neumann (1867)). *Let z, Z, R forms a triangle and let ϕ be the angle opposite to the side R, then*

$$J_0\big\{\sqrt{(Z^2+z^2-2Zz\cos\phi)}\big\} = \sum_{k=0}^{\infty} \epsilon_k J_k(Z) J_k(z) \cos k\phi, \quad (9.65)$$

where $\epsilon_0 = 1$, $\epsilon_k = 2$ for $k \geq 1$.

We note that Z, z can assume complex values. There are generalizations to $\nu \neq 0$.

Bibliography

[1] Andrews, A., Askey, R. and Roy, R. "Special Functions", Encyclopedia Math. Appl. No. 71, Camb. Univ. Press, Cambridge 1999.

[2] M. Abramowitz and I. A. Stegun (ed.), "Handbook of Mathematical Functions", National Bureau of Standards, Applied Mathematics Series 55, Washington, 1964.

[3] R. Askey, "Orthogonal Polynomials and Special Functions", SIAM, Philadelphia, 1975.

[4] Bak, J. and Newman, D. J., "Complex Analysis", 2nd Edt., Springer-Verlag 1999.

[5] Barnes, E. W., "A new development of the theory of the hypergeometic functions", Proc. Lond. Math. Soc. (2), **6** (1908), 141-177.

[6] Brown, J. W. and Churchill, R. V., "Complex Variables and Applications", 8th Edition., McGraw Hill, 2009.

[7] Chu, Shih-Chieh, "Ssu Yuan Yü Chien (Precious Mirror of the Four Elements)", 1303 (Chinese).

[8] Copson, E. T., "An Introduction to the Theory of Complex Functions of A Complex Variable", Clarendon Press, Oxford, 1935.

[9] Erdélyi, A. "Higher Transendental Functions", Vol. I, II, III, A. Erdélyi, ed., McGraw-Hill. Reprinted by Krieger Publishing Co., Malabar, FL 1981.

[10] Euler, L., "Institutiones Calculi Integralis, II", Opera Omnia, Ser. I, Vols. 11-13, 1769.

[11] Frobenius, F. G., "Über die Integration der linearen Differentialgleichungen", *J. Reine Angew. Math.* **76** (1873), 214-235.

[12] Fuchs, L. I., "Zur Theorie der linearen Differentialgleichungen mit veräanderlichen Coefficienten", *J. Reine Angew. Math.* **66** (1866), 121-160.

[13] Gray, J. J., "Linear Differential Equations and Group Theory from Riemann to Poincaré", 2nd Edition, Birkhäuser, Boston, Basel, Berlin, 2000.

[14] Hardy, G. H., "Divergent Series", Clarendon Press, Oxford, 1949 (Reprinted by Amer. Math. Soc. 2000).

[15] Paris, R. B. and Kaminski, D., "Asymptotics and Mellin-Barnes Integrals", Camb. Univ. Press, 2001.

[16] Poole, E. G. C., "Introduction to the Theory of Linear Differential Equations", Clarendon Press, Oxford, 1936.

[17] Lebedev, N. N., "Special Functions and Their Applications", translated by R. A. Silverman from Russian, Prentice-Hall, 1965.

[18] Rainville, E. D., "Special Functions", MacMillan Company, 1960, New York.

[19] Riemann, B., "Beiträge zur Theorie der durch Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen", K. Ges. Wiss. Göttingen, **7** (1857) 3-32 (in *Werke* 67-83) You can find the paper from D. R. Wilkins's edited "Gesammelte Mathematische Werke".

[20] Slater, L. J., "Confluent Hypergeometric Functions", Camb. Univ. Press, 1960.

[21] Titchmarsh, E. C., "The Theory of Functions", 2nd Edt., Oxford Univ. Press, Oxford, 1939.

[22] Watson, G. N. "A Treatise on the Theory of Bessel Functions", Camb. Univ. Press, 1922

[23] Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis* (4th ed.), Cambridge University Press, 1927, (reprinted in 1992).

Prepared in $\LaTeX\,2_\varepsilon$ by RCJ