MATH4822E FOURIER ANALYSIS AND APPLICATIONS CHAPTER 1 INTRODUCTION

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1. Introduction

Joseph Fourier first published his theory in a book entitled "The Analytic Theory of Heat" in 1822. Since then the Fourier series and related topics have been standard tools in, but not limited to, physical and engineering sciences. As a results there are numerous monographs written on the subject. A quick search in Amazon.com under the title "Fourier" or in Wikepedia will return in hundreds of items. You may find numerous videos about the Fourier series even from UTube.

This course draws heavily on the following texts:

- G. P. Tolstov, "Fourier Series", Dover publication, 1976,
- G. B. Folland, "Fourier Analysis and its Applications", Brooks/Cole Publishing Company, 1992. Republished by American Mathematical Society,
- D. M. Bressoud, "A Radical Approach to Real Analysis", 2nd Ed., Mathematical Association of America, Washington, DC, 2007

We may quote directly from these texts from time to time. Full credits will be given to these authors when we do so. These notes may contain typos/errors and you are encouraged to let me know when you have spotted them. This is course will teach Fourier series as a technique of solving some important physical science problems, like what Fourier did almost two centuries ago, but we put equal weight in the vigorous reasoning behind. So it is also a mathematical analysis course. We will review all basics about mathematical analysis in the next chapter. This course is also about mathematics culture, both the past and the present. For we cannot really appreciate the value and place of the subject matter in science today if we do not know something of the past and the cause of investigation. Some knowledge of history will help us not to loss sight easily when scientific advancement has been accumulating at a very high speed at present. As a result, we may better position ourselves in exploring the unknown in the future. In fact, the story that set off by Fourier has just barely begun, despite its long history. We may have occasions to describe some new development in due course.

A short biography of Joseph Fourier

- Born on 21st March 1768, died on 16th May 1830.
- French mathematician, physicist. .
- Discovered the underlying equations for heat conduction
- Discovered new mathematical methods and techniques for solving these equations
- Applied his results to various situations and problems
- Used experimental evidence to test and check his results
- Discoverer of greenhouse effect

- He took a prominent part in his own district in promoting the French Revolution, serving on the local Revolutionary Committee. He was imprisoned briefly during the Terror but in 1795 was appointed to the école Normale Supérieure, and subsequently succeeded Joseph-Louis Lagrange at the école Polytechnique.(Wikipedia)
- Fourier accompanied Napoleon Bonaparte on his Egyptian expedition in 1798, as scientific adviser, and was appointed secretary of the Institut d'égypte.
- His name is one of the 72 names inscribed on the Eiffel Tower.

The Analytic Theory of Heat (1822)

Primary causes are unknown to US; but are subject to simple and constant laws, which may be discovered by observation, the study of them being the object of natural philosophy. Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form .one of the most important branches of general physics.

Crisis in Mathematics: Fourier Series

This section is taken directly from Bressoud.

The crisis struck four days before Christmas 1807. The edifice of calculus was shaken to its foundations. In retrospect, the difficulties had been building for decades. Yet while most scientists realized that something had happened, it would take fifty years before the full impact of the event was understood. the nineteenth century would see ever expanding investigations into the assumptions of calculus, an inspection and refitting of the structure from the footings to the pinnacle, so thorough a reconstruction that calculus was given a new name: *Analysis*. Few of those who witnessed the incident of 1807 would have recognized mathematics as it stood one hundred years later. the twentieth century was to open with a redefinition of the integral by Henri Lebesgue and an examination of the logical underpinnings of arithmetic by Bertrand Russell and Alfred North Whitehead, both direct consequences of the events set in motion in that critical year. The crisis was precipitated by the deposition at the Institut de France in Paris of a manuscript, *Theory of the Propagation of Heat in Solid Bodies*, by the 39-year old prefect of the department of Isère, Joseph Fourier.

Background to the Problem

Fourier began his investigations with the problem of describing the flow of heat in a very long and thin rectangular plate or lamina. He considered the situation where there is no heat loss from either face of the plate and the two long sides are held at a constant temperature which he set equal to 0. Heat is applied in some known manner to one of the short sides, and the remaining short side is treated as infinitely far away.

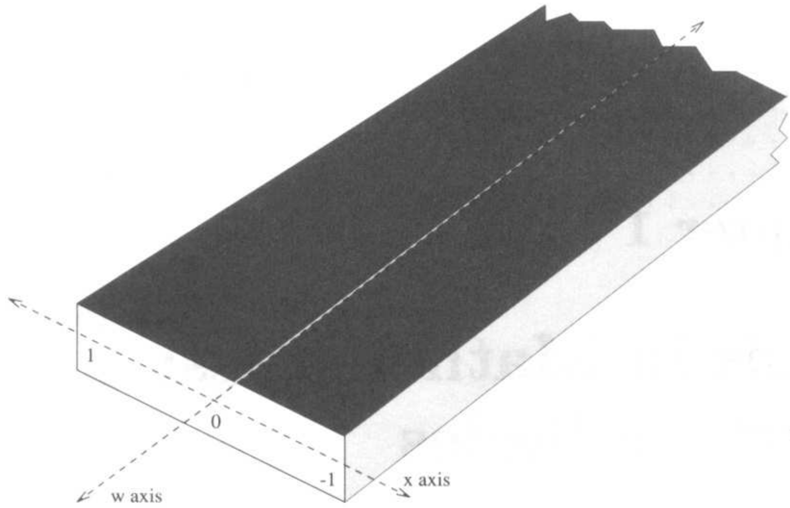

FIGURE 1. Fourier's thin plate.

This sheet can be represented in the $wx-$plane by a region bounded below by $x = -1$, above by $x = 1$, and on the left by $w = 0$. It was a constant temperature of 0 along the top and bottom edges so that if $z(w, x)$ represents the temperature at the point (w, x), then

$$z(w, -1) = z(w, 1) = 0, \qquad w > 0. \tag{1.1}$$

The known temperature distribution along the left-hand edge is described as a function of x:

$$z(0, x) = f(x). \tag{1.2}$$

Fourier restricted himself to the case where f is an *even*function of x: $f(-x) = f(x)$. The first and most important example he considered was that of a constant temperature normalized to

$$z(0, x) = f(x) = 1. \tag{1.3}$$

The task was to find a stable solution under these constraints.

Fourier began by demonstrating that a stationary solution satisfies the differential equation now known as Laplace's equation:

$$\frac{\partial^2 z}{\partial w^2} + \frac{\partial^2 z}{\partial x^2} = 0 \tag{1.4}$$

Pierre Simon Laplace (1749-1827) and others had come across the equation in various contexts. In modern terminology, it is simply the observation that when the flow of heat (∇z) has reached a state of equilibrium, it is incompressible $(\nabla \cdot \nabla z = 0)$. The equation is different from what we call *heat equation* with the time evolution is taken into consideration.

To solve his partial differential equation (1.4), Fourier introduced a technique that is standard today. He researched for special solutions of the form

$$z = \phi(w)\psi(x). \tag{1.5}$$

When z is of this form, equation (1.4) reduces to

$$\phi''(w)\psi(x) + \phi(w)\psi''(x) = 0, \tag{1.6}$$

or, assuming the second derivatives are not zero. That is,

$$\frac{\phi(w)}{\phi''(w)} + \frac{\psi(x)}{\psi''(x)}$$

or

$$\frac{\phi(w)}{\phi''(w)} = -\frac{\psi(x)}{\psi''(x)}. \tag{1.7}$$

The left side of equation (1.7) is independent of x while the right side is independent of w. This implies that both sides are independent of both w and x, and so each of these ratios is constant. It follows that the sign of $\psi(x)$ is either always the same as the sign of $\psi''(x)$ or is always the opposite. Equation (1.1) tells us that

$$\psi(-1) = \psi(1) = 0,$$

and so $\dfrac{\psi(x)}{\psi''(x)}$ must be negative:

$$\frac{\phi(w)}{\phi''(w)} = A > 0, \qquad \frac{\psi(x)}{\psi''(x)} = -A < 0,$$

for some positive constant A. Fourier set $A = 1/n^2$ and solved for $\phi(w) = c_1e^{-nw} + c_3e^{nw}$ and $\psi(x) = c_2 \cos nx + c_4 \sin nx$. The coefficient of $\sin nx$ must be zero because ψ is an even function of x. He then argued that c_3 must be zero because the temperature will approach 0 as we move away from the source of heat at $w = 0$ at the $x-$axis. He had found a solution:

$$z(w,\, x) = ae^{-nw} \cos nx,$$

where a and n are unknown constants. The general solution is a sum of such functions:

$$z = a_1e^{-n_1w} \cos n_1x + a_2e^{-n_2w} \cos n_2x + a_3e^{-n_3w} \cos n_3x + \dots. \tag{1.8}$$

Equation (1.1) holds if and only if each n_i is an odd multiple of $\dfrac{\pi}{2}$:

$$n_1 = \frac{\pi}{2}, \quad n_2 = \frac{3\pi}{2}, \quad n_3 = \frac{5\pi}{2}, \cdots$$

The temperature distribution along the left-hand edge, $z(0,\, x) = f(x)$, implies that

$$\begin{aligned} f(x) &= a_1e^{-n_1\cdot 0} \cos n_1x + a_2e^{-n_2\cdot 0} \cos n_2x + a_3e^{-n_3\cdot 0} \cos n_3x + \dots \\ &= a_1 \cos\frac{\pi x}{2} + a_2 \cos\frac{3\pi x}{2} + a_3 \cos\frac{5\pi x}{2} + \cdots, \end{aligned} \tag{1.9}$$

Fourier had reduced his problem to that of taking an even function and expressing it as a possibly infinite sum of cosines, what we today call a *Fourier series*. His next step was to demonstrate how to accomplish this.

Here was the crux of the crisis. Infinite sums of trigonometric functions had appeared before. Daniel Bernoulli (1700-1782) proposed such sums in 1753 as solutions to the problem of modelling the vibrating string. They had been summarily dismissed by the greatest mathematician of the time, Leonhard Euler (1707-1783). Perhaps Euler scented the danger they presented to his understanding of calculus. the committee that reviewed Fourier's manuscript: Laplace, Joseph Louis Lagrange (1736-1813), Sylvestre Francois Lacroix (1765-1843), and Gaspard Monge (1746-1818), echoed Euler's dismissal in an unenthusiastic summary written by Simeon Denis Poisson (1781-1840). Lagrange was later to make his objections explicit. Well into the 1820s, Fourier series would remain suspect because they contradicted the established wisdom about the nature of functions.

Fourier did more than suggest that the solution to the heat equation lay in his trigonometric series. He gave a simple and practical means of finding those coefficients, the a_i. In so doing, he produced a vast array of verifiable solutions to specific problems. Bernoulli's proposition could be debated endlessly with little effect for it was only theoretical. Fourier's method could actually be implemented. It could not be rejected without forcing the question of why it seemed to work.

There are problems with Fourier series, but they are subtler than anyone realized in that winter of 1807-08. It was not until the 1850s that Berhard Riemann (1826-1866) and Karl Weierstrass (1815-1897) would sort out the confusion that had greeted Fourier and clearly delineate the real questions.

Solution and Objections

While Fourier described the cosine expansion of many different even functions, all of the relevant techniques and difficulties can be found in his first example: the expansion of $f(x) = 1$. Several different approaches to finding the coefficients, the a_n, are proposed. The one that has become standard is to use the observation that

$$\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi x}{2}\right) \cos\left(\frac{(2n-1)\pi x}{2}\right) dx = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases} \tag{1.10}$$

We follow Fourier and assume that our even function f can be expressed as a cosine series:

$$\begin{aligned} f(x) &= a_1 \cos\left(\frac{\pi x}{2}\right) + a_2 \cos\left(\frac{3\pi x}{2}\right) + a_3 \cos\left(\frac{5\pi x}{2}\right) + \cdots \\ &= \sum_{m=1}^{\infty} a_m \cos\left(\frac{(2m-1)\pi x}{2}\right). \end{aligned} \tag{1.11}$$

Fourier now argues that a_n can be calculated by evaluating the following integral:

$$
\begin{aligned}
&\int_{-1}^{1} \cos\Big(\frac{(2n-1)\pi x}{2}\Big)dx \\
&= \int_{-1}^{1} \Big[\sum_{m=1}^{\infty} a_m \cos\Big(\frac{(2m-1)\pi x}{2}\Big)\Big] \cos\Big(\frac{(2n-1)\pi x}{2}\Big)dx \\
&= \sum_{m=1}^{\infty} a_m \int_{-1}^{1} \cos\Big(\frac{(2m-1)\pi x}{2}\Big) \cos\Big(\frac{(2n-1)\pi x}{2}\Big)dx \\
&= a_1\cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + \cdots + a_n \cdot 1 + a_{n+1}\cdot 0 + \dots \\
&= a_n.
\end{aligned}
\tag{1.12}
$$

For our particular case, $f(x) = 1$, the coefficients are

$$
\begin{aligned}
a_n &= \int_{-1}^{1} 1\cdot \cos\Big(\frac{(2n-1)\pi x}{2}\Big)dx \\
&= \frac{2}{(2n-1)\pi}\Big[\sin\Big(\frac{(2n-1)\pi x}{2}\Big)\Big]_{-1}^{1} \\
&= \frac{4}{(2n-1)\pi}(-1)^{n-1}.
\end{aligned}
$$

It follows that

$$
f(x) = 1 = \frac{4}{\pi}\Big[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots\Big]. \tag{1.13}
$$

We recall that our original problem was to find the distribution of heat, $z(w, x)$, when we hold the side at $x = 0$ at the constant temperature 1 and the sides at $x = -1$ and $x = 1$ at the constant temperature 0. The solution (see Figure 2) is given by

$$
\begin{aligned}
z(w,x) = \frac{4}{\pi}\Big[& e^{-\frac{\pi w}{2}}\cos\frac{\pi x}{2} - \frac{1}{3}e^{-\frac{3\pi w}{2}}\cos\frac{3\pi x}{2} \\
&+ \frac{1}{5}e^{-\frac{5\pi w}{2}}\cos\frac{5\pi x}{2} - \frac{1}{7}e^{-\frac{7\pi w}{2}}\cos\frac{7\pi x}{2} + \cdots\Big].
\end{aligned}
\tag{1.14}
$$

Equation (1.13) has an interesting corollary. If we set $x = 0$ and multiply both sides by π, then we see that

$$
\pi = 4\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big). \tag{1.15}
$$

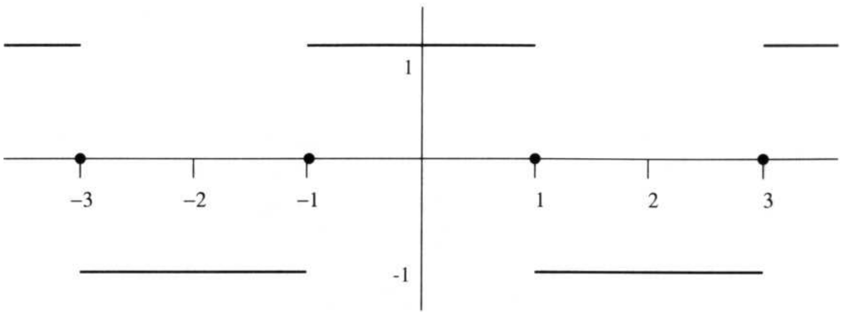

FIGURE 2. $f(x) = \frac{4}{\pi}\Big[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots\Big].$

THE OBJECTIONS

Fourier was quick to realize that equation (1.13) is only valid for $-1 < x < 1$. If we replace x by $x+2$ in the $n-$th summand, then it changes sign:

$$\cos\Big(\frac{(2n-1)\pi(x+2)}{2}\Big) = \cos\Big(\frac{(2n-1)\pi x}{2} + (2n-1)\pi\Big) = -\cos\Big(\frac{(2n-1)\pi x}{2}\Big).$$

It follows that for x between 1 and 3, equation (1.13) becomes

$$f(x) = -1 = \frac{4}{\pi}\Big[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots\Big] \tag{1.16}$$

In general, $f(x+2) = -f(x)$. The function represented by this cosine series has a graph which alternates between -1 and $+1$ as shown in Figure 2.

This is a very strange behaviour. Equation (1.13) seems to be saying that our cosine series is the constant function 1. Equation (1.16) says that our series is not constant. Moreover, to the mathematicians of 1807, Figure 2 did not look like the graph of a function. Functions were polynomials; roots, powers, and logarithms; trigonometric functions and their inverses; and whatever could be built up by addition, subtraction, multiplication, division, or composition of these functions. Functions had graphs with unbroken curves. Functions had derivatives and Taylor series. Fourier's cosine series flew in the face of everything that was *known* about the behaviour of functions. Something must be dreadfully wrong.

In retrospect, there is another flaw in his reasoning. That is his assumption in equation (1.12) that he could interchange his summation and his integral:

$$\int_{-1}^{1}\sum_m a_m \to \sum_m a_m \int_{-1}^{1}.$$

It would be some years before anyone realized that this exchange, which is perfectly legal when the summation is finite, can lead to errors when the summation is infinite.

The process of finding this cosine series was not where his paper was attacked. It was the cosine series itself that presented problems. These infinite summations cast doubts on what scientists thought they knew about the nature of functions, about continuity, about differentiability and integrability. If Fourier's disturbing series were to be accepted, then all of calculus needed to be rethought.

Lagrange thought he found the flaw in Fourier's work in the question of convergence: whether the summation approaches a single value as more terms are taken. He asserted that the cosine series,

$$\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots ,$$

does not have a well-defined value for all x. His reason for believing this was that the series consisting of the absolute values of the coefficients,

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots ,$$

grows without limit. In fact, Fourier's cosine expansion of $f(x) = 1$ *does* converges for any x, as Fourier demonstrated a few years later. The complete justification of the use of these infinite trigonometric series would have to wait twenty-two years for the work of Peter Gustav Lejeune Dirichlet (1805-1859), a young German, who, in 1807, when Fourier deposited his manuscript, was two years old.