MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

8. The Eigenfunction Method and its Applications to PDEs

8.1. **Linear partial differential equations.**

General description. Many mathematical physics problems lead to linear partial differential equations:

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial^2 u}{\partial t^2}, \tag{8.1}$$

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial u}{\partial t}, \tag{8.2}$$

where P,R and Q are functions of x, and $u = u(x,t)$.

Here is a list of partial differential equations researchers often encounter.

(I) *Heat Flow in a Rod:*

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho, \ (K = \text{thermal conductivity}, \ c = \text{heat capacity}, \ \rho = \text{density})$$

(II) *Vibration String*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = T/\rho \qquad (T \text{ is Tension, } \rho \text{ is mass per unit length})$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{F(u,t)}{\rho} \qquad \text{(Forced vibration)}$$

(II) *Vibration of Rectangular Membrane*

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \qquad c^2 = T/\rho \quad (T = \text{Tension}, \ \rho = \text{surface density}).$$

(III) Vibration of Circular Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big) + \frac{F(r,\theta,t)}{\rho} \qquad \text{(Forced Vibration)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big) \qquad \text{(independent of direction, that is, } \theta)$$

Boundary and initial value conditions. However, the solutions of these partial differential equations are subjected to *boundary conditions* :

$$\begin{aligned} \alpha\, u(a,\, t) + \beta\, \frac{\partial u(a,\, t)}{\partial x} &= 0 \\ \gamma\, u(b,\, t) + \delta\, \frac{\partial u(b,\, t)}{\partial x} &= 0, \end{aligned} \tag{8.3}$$

for $t \geq 0$, $a \leq x \leq b$, where α, β, γ and δ are constants, and *initial condition*:

$$u(x, 0) = f(x), \tag{8.4}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \tag{8.5}$$

for $a \leq x \leq b$ where $f(x)$ and $g(x)$ are given continuous functions.

We note that the above boundary and initial conditions can be interpreted as:

$$\alpha \lim_{x\to a} u(x,t) + \beta \lim_{x\to a} \frac{\partial u}{\partial x}(x,t) = 0,$$
$$\gamma \lim_{x\to b} u(x,t) + \delta \lim_{x\to b} \frac{\partial u}{\partial x}(x,t) = 0,$$

for $t \geq 0$, and

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \lim_{t\to 0} \frac{\partial u}{\partial t}(x,t) = g(x),$$

for $a \leq x \leq b$. We also assume that $(\alpha,\, \beta) \neq (0,\, 0)$ and $(\gamma,\, \delta) \neq (0,\, 0)$.

8.2. **Separation of variables method.** The idea is to write

$$u = u(x,t) = \Phi(x)\, T(t)$$

where $\Phi(x)$ and $T(t)$ are functions of x and t only respectively. We further assume this $u(x,\, t)$ satisfies the boundary conditions wrote down above. We substitute u into

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

and this gives

$$P\Phi'' T + R\Phi' T + Q\,\Phi\, T = \Phi T'',$$

or, after dividing both sides by $u = \Phi \cdot T$

$$\frac{P\Phi'' + R\Phi' + Q\Phi}{\Phi} = \frac{T''}{T}.$$

We observe that the left-side of the above equation is a function of x only, and the right-side is a function of t only. We deduce both sides must be equal to the same constant $-\lambda$, say. Thus we obtain

$$P\,\Phi'' + R\,\Phi' + Q\,\Phi + \lambda\Phi = 0, \tag{8.6}$$

and

$$T'' + \lambda T = 0. \tag{8.7}$$

It can easily be verified that the above boundary conditions for Φ becomes

$$\begin{aligned} \alpha\,\Phi(a) + \beta\,\Phi'(a) &= 0 \\ \gamma\,\Phi(b) + \delta\,\Phi'(b) &= 0. \end{aligned} \tag{8.8}$$

The second order ODE (8.6) and the boundary condition (8.8) is called a *Sturm-Liouville boundary value problem.*

For this first equation in Φ above, we will indicate that the Strum-Liouville problem has an infinite set of solutions Φ and their corresponding positive λ, that is,

$$\Phi = \Phi_n(x), \qquad \lambda = \lambda_n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda_n \to +\infty$ (see later). In the second equation in T, then

$$T = T_n(t) = A_n \cos(\sqrt{\lambda_n t}) + B_n \sin(\sqrt{\lambda_n})t,$$

$n = 0, 1, 2, \ldots$. Since the PDE is linear, The superposition principle gives

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.9}$$

if the series converges and we can differentiate term-by-term twice. We note that the $\{\Phi_n\}$ are in fact orthogonal called the *eigenfunctions* and $\{\lambda_n\}$ are called the *eigenvalues* corresponding to the eigenfunctions.

Substitute the infinite sum of u from (8.9) into the PDEs and after rearranging yields

$$\begin{aligned} P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu - \frac{\partial^2 u}{\partial t^2} & \\ &= P\sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial x^2} + R\sum_{k=0}^{\infty}\frac{\partial u_k}{\partial x} + Q\sum_{k=0}^{\infty} u_k - \sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial t^2} \\ &= \sum_{k=0}^{\infty}\Big(P\,\frac{\partial^2 u_n}{\partial x^2} + R\frac{\partial u_k}{\partial x} + Q\,u_k - \frac{\partial^2 u_k}{\partial t^2}\Big) = 0. \end{aligned}$$

The u in infinite sum (8.9) should satisfy the initial condition (8.4):

$$f(x) = u(x,\,0) = \sum_{k=0}^{\infty} T_k(0)\,\Phi_k(x),$$

and

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} T_k'(0)\,\Phi_k(x)$$

Thus the problem becomes to expand f and g by the orthogonal family of eigenfunctions $\{\Phi_n\}$:

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x), \qquad g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(t)$$

and requiring $T_k(0) = C_k$, $T_k'(0) = c_k$ for $k = 0, 1, 2, \ldots$ If $\lambda > 0$, then we could also work out the A_k and B_k for $T_k(t)$:

$$A_k = C_k, \qquad B_k = \frac{c_k}{\sqrt{\lambda_k}}, \qquad k = 0,\, 1,\, 2,\, \ldots$$

8.3. **An example of vibrating string.**

Example. Equation of Vibrating String

We consider a homogeneous string, stretched, and fastened at both ends ($x = 0$ and $x = l$). If the string is displaced by a small displacement and then released, then it will start to vibrate. Let $u(x,t)$ be the vertical displacement at the distance x and time t. We analyze the forces acting on a portion AB of the string: Then the difference of the tensions in the vertical direction is approximately

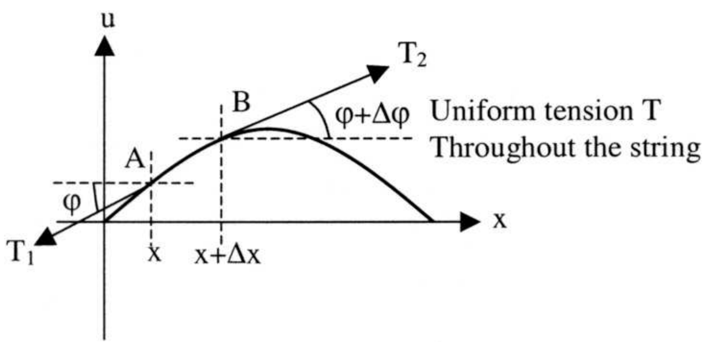

Figure 1

measured by:

$$\begin{aligned} T \cdot \Big(\sin(\phi + \Delta\phi) - \sin\phi\Big) &\approx T \cdot \Big(\frac{\sin(\phi+\Delta\phi)}{\cos(\phi+\Delta\phi)} - \frac{\sin\phi}{\cos\phi}\Big) \\ = T \cdot \Big(\tan(\phi+\Delta\phi) - \tan\phi\Big) &= T \cdot \Big(\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u}{\partial x}(x,t)\Big) \\ = T \cdot \frac{\partial^2 u}{\partial x^2}\big(x + \theta\Delta x,\, t\big) \cdot \Delta x, &\qquad 0 < \theta < 1. \end{aligned}$$

Now the Newton's second law of motion ($F = ma$) gives

$$\underbrace{\rho\,\Delta x}_{\text{mass density}} \quad \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = \underbrace{T\cdot\frac{\partial^u}{\partial x^2}\,\Delta x}_{\text{force}}, \qquad \rho \text{ is mass per unit length}$$

Dividing both sides by Δx gives

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = \frac{T}{\rho}$$

which is the equation for *free vibration* of the string. Since both ends of the string are fixed, so the boundary and initial conditions are given, respectively, by

$$u(0,t) = 0 = u(l,t), \qquad t \geq 0$$

$$\text{and} \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where f and g are continuous functions and vanish for $x = 0$, l. We now apply the method of Sturm-Liouville to

$$u = u(x,t) = \Phi(x)T(x)$$

to get

$$\Phi(x)\,T''(t) = a^2\Phi''\,(x)T(t).$$

That is,

$$\frac{\Phi''}{\Phi} = \frac{T''}{a^2T} = -\lambda.$$

Thus,

$$\Phi'' + \lambda\Phi = 0,$$
$$T'' + a^2\lambda T = 0,$$

subject to $u(0,t) = \Phi(0)T(t) = 0 = \Phi(l)T(t) = u(l,t)$ for all $t \geq 0$, that is, subject to $\Phi(0) = 0 = \Phi(l)$. We will assume λ is positive, so we write λ^2 instead:

$$\Phi'' + \lambda^2\Phi = 0, \qquad T'' + a^2\lambda^2 T = 0.$$

The general solution of the first equation is

$$\Phi(x) = c_1\cos\lambda x + c_2\sin\lambda x.$$

But the boundary condition gives

$$\Phi(0) = 0 = c_1\cos 0 + c_2\sin 0 = c_1,$$

that is, $c_1 = 0$, and so

$$0 = \Phi(l) = c_2 \sin \lambda l.$$

But $c_2 \neq 0$, so $\lambda = \frac{\pi k}{l}$. But the above analysis works for all λ_k, $k = 0, 1, 2, \ldots$, we obtain $\lambda_k = \frac{\pi k}{l}$, $k = 0, 1, 2, \ldots$ and

$$\Phi_k(x) = \sin \frac{\pi k x}{l}, \qquad k = 0, 1, 2, \ldots$$

Thus, the second differential equation gives

$$T_k(t) = A_k \cos(a\lambda_k t) + B_k \sin(a\lambda_k t), \qquad k = 0, 1, 2, \ldots$$

Hence

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t) \\ &= \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{a\pi k t}{l}\right) + B_k \sin\left(\frac{a\pi k t}{l}\right) \right] \sin\left(\frac{\pi k x}{l}\right). \end{aligned}$$

We now apply the initial condition to $u(x, t)$. So we require

$$f(x) = u(x,0) = \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi k x}{l}\right)$$

$$\text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} B_k \left(\frac{a\pi k}{l}\right) \sin\left(\frac{\pi k x}{l}\right)$$

are just the Fourier series of f and g with respect to $\{\sin \frac{\pi k x}{l}\}$. Thus

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l f(x) \sin \frac{\pi k x}{l}\, dx, \\ B_k &= \frac{2}{a\pi k} \int_0^l g(x) \sin \frac{\pi k x}{l}\, dx. \end{aligned}$$

To be continued ...