MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

8. The Eigenfunction Method and its Applications to PDEs

8.1. **Linear partial differential equations.**

General description. Many mathematical physics problems lead to linear partial differential equations:

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial^2 u}{\partial t^2}, \tag{8.1}$$

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial u}{\partial t}, \tag{8.2}$$

where P,R and Q are functions of x, and $u = u(x,t)$.

Here is a list of partial differential equations researchers often encounter.

(I) *Heat Flow in a Rod:*

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho,\ (K = \text{thermal conductivity},\ c = \text{heat capacity},\ \rho = \text{density})$$

(II) *Vibration String*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = T/\rho \qquad (T \text{ is Tension, } \rho \text{ is mass per unit length})$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{F(u,t)}{\rho} \qquad \text{(Forced vibration)}$$

(II) *Vibration of Rectangular Membrane*

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \qquad c^2 = T/\rho \quad (T = \text{Tension},\ \rho = \text{surface density}).$$

(III) Vibration of Circular Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big) + \frac{F(r,\theta,t)}{\rho} \qquad \text{(Forced Vibration)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big) \qquad \text{(independent of direction, that is, } \theta)$$

Boundary and initial value conditions. However, the solutions of these partial differential equations are subjected to *boundary conditions* :

$$\begin{aligned} \alpha\, u(a,\, t) + \beta\, \frac{\partial u(a,\, t)}{\partial x} &= 0 \\ \gamma\, u(b,\, t) + \delta\, \frac{\partial u(b,\, t)}{\partial x} &= 0, \end{aligned} \tag{8.3}$$

for $t \geq 0$, $a \leq x \leq b$, where α, β, γ and δ are constants, and *initial condition*:

$$u(x, 0) = f(x), \tag{8.4}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \tag{8.5}$$

for $a \leq x \leq b$ where $f(x)$ and $g(x)$ are given continuous functions.

We note that the above boundary and initial conditions can be interpreted as:

$$\begin{aligned} \alpha \lim_{x\to a} u(x,t) + \beta \lim_{x\to a} \frac{\partial u}{\partial x}(x,t) &= 0, \\ \gamma \lim_{x\to b} u(x,t) + \delta \lim_{x\to b} \frac{\partial u}{\partial x}(x,t) &= 0, \end{aligned}$$

for $t \geq 0$, and

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \lim_{t\to 0} \frac{\partial u}{\partial t}(x,t) = g(x),$$

for $a \leq x \leq b$. We also assume that $(\alpha,\, \beta) \neq (0,\, 0)$ and $(\gamma,\, \delta) \neq (0,\, 0)$.

8.2. **Separation of variables method.** The idea is to write

$$u = u(x,t) = \Phi(x)\, T(t)$$

where $\Phi(x)$ and $T(t)$ are functions of x and t only respectively. We further assume this $u(x,\, t)$ satisfies the boundary conditions wrote down above. We substitute u into

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

and this gives

$$P\Phi'' T + R\Phi' T + Q\,\Phi\, T = \Phi T'',$$

or, after dividing both sides by $u = \Phi \cdot T$

$$\frac{P\Phi'' + R\Phi' + Q\Phi}{\Phi} = \frac{T''}{T}.$$

We observe that the left-side of the above equation is a function of x only, and the right-side is a function of t only. We deduce both sides must be equal to the same constant $-\lambda$, say. Thus we obtain

$$P\,\Phi'' + R\,\Phi' + Q\,\Phi + \lambda\Phi = 0, \tag{8.6}$$

and

$$T'' + \lambda T = 0. \tag{8.7}$$

It can easily be verified that the above boundary conditions for Φ becomes

$$\begin{aligned} \alpha\,\Phi(a) + \beta\,\Phi'(a) = 0 \\ \gamma\,\Phi(b) + \delta\,\Phi'(b) = 0. \end{aligned} \tag{8.8}$$

The second order ODE (8.6) and the boundary condition (8.8) is called a *Sturm-Liouville boundary value problem.*

For this first equation in Φ above, we will indicate that the Strum-Liouville problem has an infinite set of solutions Φ and their corresponding positive λ, that is,

$$\Phi = \Phi_n(x), \qquad \lambda = \lambda_n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda_n \to +\infty$ (see later). In the second equation in T, then

$$T = T_n(t) = A_n \cos(\sqrt{\lambda_n t}) + B_n \sin(\sqrt{\lambda_n})t,$$

$n = 0, 1, 2, \ldots$. Since the PDE is linear, The superposition principle gives

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.9}$$

if the series converges and we can differentiate term-by-term twice. We note that the $\{\Phi_n\}$ are in fact orthogonal called the *eigenfunctions* and $\{\lambda_n\}$ are called the *eigenvalues* corresponding to the eigenfunctions.

Substitute the infinite sum of u from (8.9) into the PDEs and after rearranging yields

$$\begin{aligned} &P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu - \frac{\partial^2 u}{\partial t^2} \\ &\qquad = P\sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial x^2} + R\sum_{k=0}^{\infty}\frac{\partial u_k}{\partial x} + Q\sum_{k=0}^{\infty} u_k - \sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial t^2} \\ &\qquad = \sum_{k=0}^{\infty}\Big(P\frac{\partial^2 u_n}{\partial x^2} + R\frac{\partial u_k}{\partial x} + Q\,u_k - \frac{\partial^2 u_k}{\partial t^2}\Big) = 0. \end{aligned}$$

The u in infinite sum (8.9) should satisfy the initial condition (8.4):

$$f(x) = u(x,\,0) = \sum_{k=0}^{\infty} T_k(0)\,\Phi_k(x),$$

and

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} T_k'(0)\,\Phi_k(x)$$

Thus the problem becomes to expand f and g by the orthogonal family of eigenfunctions $\{\Phi_n\}$:

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x), \qquad g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(t)$$

and requiring $T_k(0) = C_k$, $T_k'(0) = c_k$ for $k = 0, 1, 2, \ldots$ If $\lambda > 0$, then we could also work out the A_k and B_k for $T_k(t)$:

$$A_k = C_k, \qquad B_k = \frac{c_k}{\sqrt{\lambda_k}}, \qquad k = 0,\, 1,\, 2,\, \ldots$$

8.3. **An example of vibrating string.**

Example. Equation of Vibrating String

We consider a homogeneous string, stretched, and fastened at both ends ($x = 0$ and $x = l$). If the string is displaced by a small displacement and then released, then it will start to vibrate. Let $u(x,t)$ be the vertical displacement at the distance x and time t. We analyze the forces acting on a portion AB of the string: Then the difference of the tensions in the vertical direction is approximately

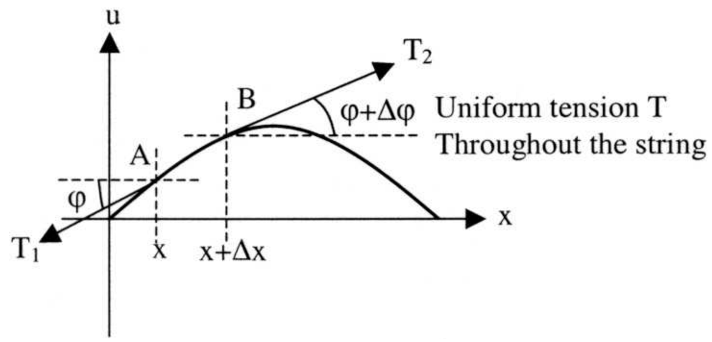

FIGURE 1.

measured by:

$$\begin{aligned}
&T \cdot \Big(\sin(\phi + \Delta\phi) - \sin\phi\Big) \approx T \cdot \Big(\frac{\sin(\phi+\Delta\phi)}{\cos(\phi+\Delta\phi)} - \frac{\sin\phi}{\cos\phi}\Big) \\
&= T \cdot \Big(\tan(\phi+\Delta\phi) - \tan\phi\Big) = T \cdot \Big(\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u}{\partial x}(x,t)\Big) \\
&= T \cdot \frac{\partial^2 u}{\partial x^2}\big(x + \theta\Delta x,\, t\big) \cdot \Delta x, \qquad 0 < \theta < 1.
\end{aligned}$$

Now the Newton's second law of motion ($F = ma$) gives

$$\underbrace{\rho\,\Delta x}_{\text{mass density}}\ \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = \underbrace{T\cdot\frac{\partial^u}{\partial x^2}\,\Delta x}_{\text{force}}, \qquad \rho \text{ is mass per unit length}$$

Dividing both sides by Δx gives

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = \frac{T}{\rho}$$

which is the equation for *free vibration* of the string. Since both ends of the string are fixed, so the boundary and initial conditions are given, respectively, by

$$\begin{aligned} u(0,t) = 0 = u(l,t), &\qquad t \geq 0 \\ \text{and} \qquad u(x,0) = f(x), &\qquad \frac{\partial u}{\partial t}(x,0) = g(x) \end{aligned}$$

where f and g are continuous functions and vanish for $x = 0$, l. We now apply the method of Sturm-Liouville to

$$u = u(x,t) = \Phi(x)T(x)$$

to get

$$\Phi(x)\,T''(t) = a^2\Phi''\,(x)T(t).$$

That is,

$$\frac{\Phi''}{\Phi} = \frac{T''}{a^2T} = -\lambda.$$

Thus,

$$\begin{aligned} \Phi'' + \lambda\Phi = 0, \\ T'' + a^2\lambda T = 0, \end{aligned}$$

subject to $u(0,t) = \Phi(0)T(t) = 0 = \Phi(l)T(t) = u(l,t)$ for all $t \geq 0$, that is, subject to $\Phi(0) = 0 = \Phi(l)$. We will assume λ is positive, so we write λ^2 instead:

$$\Phi'' + \lambda^2\Phi = 0, \qquad T'' + a^2\lambda^2 T = 0.$$

The general solution of the first equation is

$$\Phi(x) = c_1\cos\lambda x + c_2\sin\lambda x.$$

But the boundary condition gives

$$\Phi(0) = 0 = c_1\cos 0 + c_2\sin 0 = c_1,$$

that is, $c_1 = 0$, and so

$$0 = \Phi(l) = c_2 \sin \lambda l.$$

But $c_2 \neq 0$, so $\lambda = \dfrac{\pi k}{l}$. But the above analysis works for all λ_k, $k = 0, 1, 2, \ldots$, we obtain $\lambda_k = \dfrac{\pi k}{l}$, $k = 0, 1, 2, \ldots$ and

$$\Phi_k(x) = \sin \frac{\pi k x}{l}, \qquad k = 0, 1, 2, \ldots$$

Thus, the second differential equation gives

$$T_k(t) = A_k \cos(a\lambda_k t) + B_k \sin(a\lambda_k t), \qquad k = 0, 1, 2, \ldots$$

Hence

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t) \\ &= \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{a\pi k t}{l}\right) + B_k \sin\left(\frac{a\pi k t}{l}\right) \right] \sin\left(\frac{\pi k x}{l}\right). \end{aligned}$$

We now apply the initial condition to $u(x,\, t)$. So we require

$$\begin{aligned} f(x) = u(x,0) &= \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi k x}{l}\right) \\ \text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x,0) &= \sum_{k=0}^{\infty} B_k \left(\frac{a\pi k}{l}\right) \sin\left(\frac{\pi k x}{l}\right) \end{aligned}$$

are just the Fourier series of f and g with respect to $\{\sin \dfrac{\pi k x}{l}\}$. Thus

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l f(x) \, \sin \frac{\pi k x}{l} \, dx, \\ B_k &= \frac{2}{a\pi k} \int_0^l g(x) \, \sin \frac{\pi k x}{l} \, dx. \end{aligned}$$

8.4. **Remarks on separation of variables.** Here are some reasons that one wants to study linear second order ODEs

- Solve PDEs in $\mathbb{R}^3$ such as Laplace Eqn. ($\Delta^2\Phi = 0$), Helmholtz Eqn. [$(\Delta^2 + k^2)\Phi = 0$], etc.;
- *Separation of variables* of the PDEs under different curvilinear orthogonal coordinate systems giving various ODEs;
- *Boundary value* (Sturm-Liouville type)problems;
- Some of these linear ODEs are better understood (*Bessel* Eqn.) than the others (*Spheroidal Wave* Eqn.)
- Almost all of these Eqns are *ancient*.

Question: Under what $3D$ curvilinear orthogonal coordinate systems (u_1, u_2, u_3) do we have a solution of the (elliptic PDE) Helmholtz Eqn

$$(\Delta^2 + k^2)\Phi = 0,$$

to be solved by *separation of variables* of the form

$$\Phi(\mathbf{r}) = \Phi_1(u_1) \cdot \Phi_2(u_2) \cdot \Phi_3(u_3) \quad ?$$

Theorem 8.1 (Eisenhart (1934))**.** *There are precisely eleven curvilinear orthogonal coordinate systems in each of which the Helmholtz equation separates.*

("Separable Systems of Stäckel." Ann. Math. **35** (1934), 284–305.) (Morse & Feshbach, "Methods of Theoretical Physics I". NY: McGraw-Hill, pp. 125–

The Eleven Coordinate Systems:

(1) Cartesian	$x = x,\quad y = y,\quad z = z$	
(2) Cylindrical	$x = \rho\cos\phi,\quad y = \rho\sin\phi,\quad z = z$ $\rho \geqslant 0,\quad -\pi < \phi \leqslant \pi$	
(3) Spherical polar	$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi$ $z = r\cos\theta$ $r \geqslant 0,\quad 0 \leqslant \theta \leqslant \pi,\quad -\pi < \phi \leqslant \pi$	
(4) Parabolic cylinder	$x = u^2 - v^2,\quad y = 2uv,\quad z = z$ $u \geqslant 0,\quad -\infty < v < \infty$	Half plane
(5) Elliptic cylinder	$x = f\cosh\xi\cos\eta,\quad y = f\sinh\xi\sin\eta$ $z = z$ $\xi \geqslant 0,\quad -\pi < \eta \leqslant \pi$	Infinite strip Plane with straight aperture
(6) Rotation-paraboloidal	$x = 2uv\cos\phi,\quad y = 2uv\sin\phi$ $z = u^2 - v^2$ $u \geqslant 0,\quad v \geqslant 0,\quad -\pi < \phi \leqslant \pi$	Half-line
(7) Prolate spheroidal	$x = \ell\sinh u\sin v\cos\phi$ $y = \ell\sinh u\sin v\sin\phi$ $z = \ell\cosh u\cos v$ $u \geqslant 0,\quad 0 \leqslant v \leqslant \pi,\quad -\pi < \phi \leqslant \pi$	Finite line segment Two half-lines
(8) Oblate spheroidal	$x = \ell\cosh u\sin v\cos\phi$ $y = \ell\cosh u\sin v\sin\phi$ $z = \ell\sinh u\cos v$ $u \geqslant 0,\quad 0 \leqslant v \leqslant \pi,\quad -\pi < \phi \leqslant \pi$	Circular plate (disc) Plane with circular aperture
(9) Paraboloidal	$x = \frac{1}{2}\ell(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)$ $y = 2\ell\cosh\alpha\cos\beta\sinh\gamma$ $z = 2\ell\sinh\alpha\sin\beta\cosh\gamma$ $\alpha \geqslant 0,\quad -\pi < \beta \leqslant \pi,\quad \gamma \geqslant 0$	Parabolic plate Plane with parabolic aperture
(10) Elliptic conal	$x = kr\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta,\quad y = (ik/k')r\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta$ $z = (1/k')r\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta$ $r \geqslant 0,\quad -2K < \alpha \leqslant 2K$ $\beta = K + iu,\quad 0 \leqslant u \leqslant 2K'$	Plane sector (including quarter-plane)
(11) Ellipsoidal	$x = k^2\ell\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma$ $y = (-k^2\ell/k')\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma$ $z = (i\ell/k')\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma$ α, β as in (10), $\gamma = iK' + w,\quad 0 < w \leqslant K$	Elliptic plate Plane with elliptic aperture

FIGURE 2. (Arscott & Darai 1981 IMA Appl. Math.)

11 Curvilinear Orthogonal Coordinate systems

(1) Cartesian;
(2) Cylindrical;
(3) Spherical polar;
(4) Parabolic cylinderical;
(5) Elliptic cylinderical;
(6) Rotation paraboloidal;
(7) Prolate paraboloidal;
(8) Oblate paraboloidal;
(9) Paraboloidal;
(10) Elliptic conal;
(11) Ellipsoidal.

The Classification Table

Separation of Laplace and Helmholtz equations in various co-ordinate systems

	Ordinary differential equations arising from separation of	
Co-ordinate system	Laplace equation	Helmholtz equation
(1) Cartesian	(Trivial)	(Trivial)
(2) Cylindrical	(Trivial)	Bessel's
(3) Spherical polar	Associated Legendre	Associated Legendre and spherical Bessel
(4) Parabolic cylinder	(Trivial)	Weber's
(5) Elliptic cylinder	(Trivial)	Mathieu's
(6) Rotation-paraboloidal	Bessel's	Confluent hypergeom (Tricomi's)
(7) Prolate spheroidal	Associated Legendre	Spheroidal wave
(8) Oblate spheroidal	Associated Legendre	Spheroidal wave
(9) Paraboloidal	Mathieu's	Whittaker–Hill
(10) Elliptic conal	Lamé's	Spherical Bessel and Lamé's
(11) Ellipsoidal	Lamé's	Ellipsoidal wave

FIGURE 3. (Arscott & Darai 1981 IMA Appl. Math.)

The Equation Table

Associated Legendre:	$(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left\{n(n+1) - \dfrac{m^2}{(1-x^2)}\right\} y = 0$
Bessel:	$x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-n^2)y = 0$
Spherical Bessel:	$x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + [x^2-n(n+1)]y = 0$
Weber's:	$\dfrac{d^2y}{dx^2} + (\lambda-\tfrac{1}{4}x^2)y = 0$
Confluent hypergeometric:	$x\dfrac{d^2y}{dx^2} + (\gamma-x)\dfrac{dy}{dx} - \alpha y = 0$
Mathieu's:	$\dfrac{d^2y}{dx^2} + (\lambda-2q\cos 2x)y = 0$
Spheroidal wave:	$(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left\{\lambda - \dfrac{\mu^2}{(1-x^2)} + \gamma^2(1-x^2)\right\} y = 0$
Lamé's:	$\dfrac{d^2y}{dx^2} + [h-n(n+1)k^2 \operatorname{sn}^2 x]y = 0$
Whittaker–Hill:	$\dfrac{d^2y}{dx^2} + (a+b\cos 2x+c\cos 4x)y = 0$
Ellipsoidal wave:	$\dfrac{d^2y}{dx^2} - (a+bk^2\operatorname{sn}^2 x+qk^4\operatorname{sn}^4 x)y = 0$

FIGURE 4. (Arscott & Darai 1981 IMA Appl. Math.)

8.5. **Sturm-Liouville Boundary Value Problems.** We assume that the function P in (8.1) does not vanish. As a result, we can rewrite the equation (8.1) into a *self-adjoint form*:

Lemma 8.2. *The equation*

$$P\Phi'' + R\Phi' + Q\Phi = -\lambda\Phi \tag{8.10}$$

can be written in the form

$$(p\Phi')' + q\Phi = -\lambda r\Phi,$$

where p, q and r are continuous functions of x on $[a, b]$, p is positive and has a continuous derivative, and r is positive.

Proof. Multiply the equation (8.10) on both sides by r. Then we require:

$$\begin{aligned} rP\Phi'' + rR\Phi' + rQ\Phi &= -\lambda r\Phi, \\ p\Phi'' + p'\Phi' + q\Phi &= -\lambda r\Phi \end{aligned}$$

to be the same. So it is sufficient that we require

$$\begin{aligned} & p' = rR \qquad \text{and} \qquad p = rP, \\ \text{that is, if} \qquad & \frac{p'}{p} = \frac{R}{P} \qquad \text{and} \qquad r = \frac{p}{P}, \\ \text{or equivalently} \qquad & p = e^{\int \frac{R}{P}} \qquad \text{and} \qquad r = \frac{1}{P}e^{\int \frac{R}{P}} \end{aligned}$$

which is well-defined since P is positive, and the conclusion for p, q and r thus follow. □

Lemma 8.3. *Let*

$$L(\Phi) = \frac{d}{dx}\Big(P\frac{d\Phi}{dx}\Big) + q\Phi. \tag{8.11}$$

Then for any two twice differentiable functions Φ and Ψ, we have

$$\Phi L(\Psi) - \Psi L(\Phi) = \frac{d}{dx}\big(p(\Phi\Psi' - \Phi'\Psi)\big). \tag{8.12}$$

Proof. Direct verification. □

Lemma 8.4. *If Φ and Ψ satisfy the boundary condition*

$$\begin{aligned} \alpha\Phi(a) + \beta\Phi'(a) &= 0, \\ \gamma\Phi(b) + \delta\Phi'(b) &= 0 \end{aligned} \tag{8.13}$$

then

$$\Phi\Psi' - \Phi'\Psi\Big|_{x=a} = 0 = \Phi\Psi' - \Phi'\Psi\Big|_{x=b}. \tag{8.14}$$

Proof. The equations

$$\alpha\Phi(a) + \beta\Phi'(a) = 0,$$
$$\alpha\Psi(a) + \beta\Psi'(a) = 0$$

have non-trivial solutions for α and β if and only if

$$\begin{vmatrix} \Phi(a) & \Phi'(a) \\ \Psi(a) & \Psi'(a) \end{vmatrix} = 0.$$

Similar argument gives the condition at $x = b$. □

Lemma 8.5. *Let L be the Sturm-Loiuville operator given in (8.11) and*

$$L(\Phi) = -\lambda r\Phi,$$
$$L(\Psi) = -\lambda r\Psi$$

and both Φ and Ψ satisfy the same boundary condition in (8.13) at $x = a$ and b. Then Φ and Ψ are orthogonal on $[a, b]$ with respect to the function r (called the orthogonal weight function).

Proof. Since

$$\Psi\, L(\Phi) - \Phi\, L(\Psi) = (\mu - \lambda)\, r\, \Phi\, \Psi.$$

But (8.12) gives

$$\begin{aligned} \frac{d}{dx}\, p\, (\Phi\Psi' - \Phi'\Psi) &= \Psi\, L(\Phi) - \Phi\, L(\Psi) \\ &= (\mu - \lambda)\, r\, \Phi\, \Psi. \end{aligned}$$

The (8.14) now gives

$$0 = p(\Phi\Psi' - \Phi'\Psi)\Big|_{x=a}^{x=b} = (\mu - \lambda) \int_a^b r\, \Phi\, \Psi\; dx.$$

This proves that Φ and Ψ are orthogonal with respect to the weight function r over $[a, b]$. □

Lemma 8.6. *Let L be the Sturm-Loiuville operator given in (8.11) and*

$$L(\Phi) = -\lambda\, r\, \Phi.$$

Then λ must be real.

Proof. For suppose λ is complex with $\lambda = \mu + i\nu$, $\nu \neq 0$ and $\Phi = \phi + i\psi$. Then

$$\big(p(\phi' + i\psi')\big)' + q(\phi + i\nu) = -(\mu + i\nu)\, r\, (\phi + i\psi).$$

Taking complex conjugate of this equation gives

$$\big(p(\phi' - i\psi')\big)' + q(\phi - i\nu) = -(\mu - i\nu)\, r\, (\phi - i\psi).$$

which implies that $\phi - i\nu = \overline{\Phi}$ is an eigenvector and $\overline{\lambda} = \mu - i\nu$ is the corresponding eigenvalue. If we now follow the argument used in Lemma 8.5, then we obtain

$$\int_a^b r\, \Phi\, \overline{\Phi}\; dx = \int_a^b r\, (\phi^2 + \psi^2)\; dx > 0$$

contradicting that Φ and $\overline{\Phi}$ are orthogonal. $\square$

Theorem 8.7. *If $r > 0$, $q \leq 0$ and if the boundary conditions imply that*

$$p\, \Phi\, \Phi' \Big|_a^b \leq 0, \tag{8.15}$$

then all the eigenvalues of the boundary value problem for

$$(p\Phi')' + q\, \Phi = -\lambda\, r\, \Phi$$

are non-negative.

Proof. We multiply both sides of

$$(p\Phi')' + q\Phi = -\lambda r \Phi.$$

by Φ and integrate both sides of the resulted equation over $(a,\, b)$ to get

$$P\, \Phi'\, \Phi \Big|_{x=a}^{x=b} - \int_a^b P\, \Phi'^2\, dx + \int_a^b q\Phi^2\; dx = -\lambda \int_a^b r\Phi^2\; dx.$$

It follows from the hypothesis that $\lambda \geq 0$. In addition, $\lambda = 0$ only if $q \equiv 0$, $\Phi' \equiv 0$ over $[a,\, b]$. That is, if Φ is a constant eigenvector. $\square$

Remark. The assumption (8.15) includes

(1) $\Phi(a) = 0 = \Phi(b)$,
(2) $\Phi'(a) = 0 = \Phi'(b)$,
(3) $\Phi'(a) - h\Phi(a) = 0$, $\Phi'(b) + H\Phi(b) = 0$, where h and H are non-negative constants.

To be continued ...