MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

8. The Eigenfunction Method and its Applications to PDEs

8.1. **Linear partial differential equations.**

General description. Many mathematical physics problems lead to linear partial differential equations:

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial^2 u}{\partial t^2}, \tag{8.1}$$

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial u}{\partial t}, \tag{8.2}$$

where P,R and Q are functions of x, and $u = u(x,t)$.

Here is a list of partial differential equations researchers often encounter.

(I) *Heat Flow in a Rod:*

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho, \ (K = \text{thermal conductivity}, \ c = \text{heat capacity}, \ \rho = \text{density})$$

(II) *Vibration String*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = T/\rho \qquad (T \text{ is Tension, } \rho \text{ is mass per unit length})$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{F(u,t)}{\rho} \qquad \text{(Forced vibration)}$$

(II) *Vibration of Rectangular Membrane*

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \qquad c^2 = T/\rho \quad (T = \text{Tension}, \ \rho = \text{surface density}).$$

(III) Vibration of Circular Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big) + \frac{F(r,\theta,t)}{\rho} \qquad \text{(Forced Vibration)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big) \qquad \text{(independent of direction, that is, } \theta)$$

Boundary and initial value conditions. However, the solutions of these partial differential equations are subjected to *boundary conditions* :

$$\begin{aligned} \alpha\, u(a,\,t) + \beta\, \frac{\partial u(a,\,t)}{\partial x} &= 0 \\ \gamma\, u(b,\,t) + \delta\, \frac{\partial u(b,\,t)}{\partial x} &= 0, \end{aligned} \tag{8.3}$$

for $t \geq 0$, $a \leq x \leq b$, where α, β, γ and δ are constants, and *initial condition*:

$$u(x, 0) = f(x), \tag{8.4}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \tag{8.5}$$

for $a \leq x \leq b$ where $f(x)$ and $g(x)$ are given continuous functions.

We note that the above boundary and initial conditions can be interpreted as:

$$\begin{aligned} \alpha \lim_{x\to a} u(x,t) + \beta \lim_{x\to a} \frac{\partial u}{\partial x}(x,t) &= 0, \\ \gamma \lim_{x\to b} u(x,t) + \delta \lim_{x\to b} \frac{\partial u}{\partial x}(x,t) &= 0, \end{aligned}$$

for $t \geq 0$, and

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \lim_{t\to 0} \frac{\partial u}{\partial t}(x,t) = g(x),$$

for $a \leq x \leq b$. We also assume that $(\alpha,\,\beta) \neq (0,\,0)$ and $(\gamma,\,\delta) \neq (0,\,0)$.

8.2. **Separation of variables method.** The idea is to write

$$u = u(x,t) = \Phi(x)\, T(t)$$

where $\Phi(x)$ and $T(t)$ are functions of x and t only respectively. We further assume this $u(x,\,t)$ satisfies the boundary conditions wrote down above. We substitute u into

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

and this gives

$$P\Phi'' T + R\Phi' T + Q\,\Phi\, T = \Phi T'',$$

or, after dividing both sides by $u = \Phi \cdot T$

$$\frac{P\Phi'' + R\Phi' + Q\Phi}{\Phi} = \frac{T''}{T}.$$

We observe that the left-side of the above equation is a function of x only, and the right-side is a function of t only. We deduce both sides must be equal to the same constant $-\lambda$, say. Thus we obtain

$$P\,\Phi'' + R\,\Phi' + Q\,\Phi + \lambda\Phi = 0, \tag{8.6}$$

and

$$T'' + \lambda T = 0. \tag{8.7}$$

It can easily be verified that the above boundary conditions for Φ becomes

$$\begin{aligned} \alpha\,\Phi(a) + \beta\,\Phi'(a) &= 0 \\ \gamma\,\Phi(b) + \delta\,\Phi'(b) &= 0. \end{aligned} \tag{8.8}$$

The second order ODE (8.6) and the boundary condition (8.8) is called a *Sturm-Liouville boundary value problem.*

For this first equation in Φ above, we will indicate that the Strum-Liouville problem has an infinite set of solutions Φ and their corresponding positive λ, that is,

$$\Phi = \Phi_n(x), \qquad \lambda = \lambda_n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda_n \to +\infty$ (see later). In the second equation in T, then

$$T = T_n(t) = A_n \cos(\sqrt{\lambda_n t}) + B_n \sin(\sqrt{\lambda_n})t,$$

$n = 0, 1, 2, \ldots$. Since the PDE is linear, The superposition principle gives

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.9}$$

if the series converges and we can differentiate term-by-term twice. We note that the $\{\Phi_n\}$ are in fact orthogonal called the *eigenfunctions* and $\{\lambda_n\}$ are called the *eigenvalues* corresponding to the eigenfunctions.

Substitute the infinite sum of u from (8.9) into the PDEs and after rearranging yields

$$\begin{aligned} &P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu - \frac{\partial^2 u}{\partial t^2} \\ &\qquad = P\sum_{k=0}^{\infty} \frac{\partial^2 u_k}{\partial x^2} + R\sum_{k=0}^{\infty} \frac{\partial u_k}{\partial x} + Q\sum_{k=0}^{\infty} u_k - \sum_{k=0}^{\infty} \frac{\partial^2 u_k}{\partial t^2} \\ &\qquad = \sum_{k=0}^{\infty} \Big(P\frac{\partial^2 u_n}{\partial x^2} + R\frac{\partial u_k}{\partial x} + Q\,u_k - \frac{\partial^2 u_k}{\partial t^2} \Big) = 0. \end{aligned}$$

The u in infinite sum (8.9) should satisfy the initial condition (8.4):

$$f(x) = u(x,\,0) = \sum_{k=0}^{\infty} T_k(0)\,\Phi_k(x),$$

and

$$g(x) = \frac{\partial u}{\partial t}(x, 0) = \sum_{k=0}^{\infty} T_k'(0)\, \Phi_k(x)$$

Thus the problem becomes to expand f and g by the orthogonal family of eigenfunctions $\{\Phi_n\}$:

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x), \qquad g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(t)$$

and requiring $T_k(0) = C_k$, $T_k'(0) = c_k$ for $k = 0, 1, 2, \ldots$ If $\lambda > 0$, then we could also work out the A_k and B_k for $T_k(t)$:

$$A_k = C_k, \qquad B_k = \frac{c_k}{\sqrt{\lambda_k}}, \qquad k = 0,\, 1,\, 2,\, \ldots$$

8.3. **An example of vibrating string.**

Example. Equation of Vibrating String

We consider a homogeneous string, stretched, and fastened at both ends ($x = 0$ and $x = l$). If the string is displaced by a small displacement and then released, then it will start to vibrate. Let $u(x, t)$ be the vertical displacement at the distance x and time t. We analyze the forces acting on a portion AB of the string: Then the difference of the tensions in the vertical direction is approximately

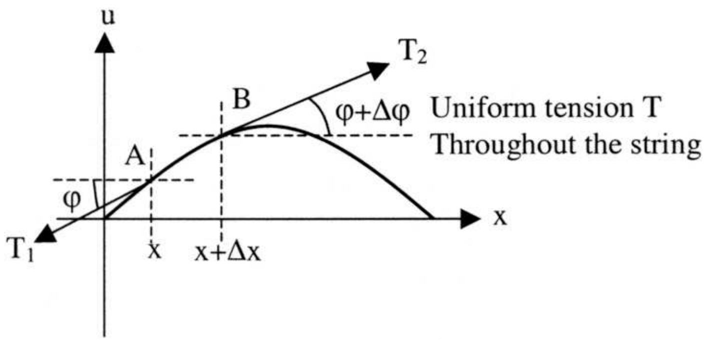

Figure 1.

measured by:

$$\begin{aligned}
&T \cdot \Big(\sin(\phi + \Delta\phi) - \sin\phi\Big) \approx T \cdot \Big(\frac{\sin(\phi + \Delta\phi)}{\cos(\phi + \Delta\phi)} - \frac{\sin\phi}{\cos\phi}\Big) \\
&= T \cdot \Big(\tan(\phi + \Delta\phi) - \tan\phi\Big) = T \cdot \Big(\frac{\partial u(x + \Delta x, t)}{\partial x} - \frac{\partial u}{\partial x}(x, t)\Big) \\
&= T \cdot \frac{\partial^2 u}{\partial x^2}\big(x + \theta\Delta x,\, t\big) \cdot \Delta x, \qquad 0 < \theta < 1.
\end{aligned}$$

Now the Newton's second law of motion ($F = ma$) gives

$$\underbrace{\rho\,\Delta x}_{\text{mass density}}\ \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = \underbrace{T\cdot\frac{\partial^u}{\partial x^2}\,\Delta x}_{\text{force}}, \qquad \rho \text{ is mass per unit length}$$

Dividing both sides by Δx gives

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = \frac{T}{\rho}$$

which is the equation for *free vibration* of the string. Since both ends of the string are fixed, so the boundary and initial conditions are given, respectively, by

$$u(0,t) = 0 = u(l,t), \qquad t \geq 0$$

$$\text{and} \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where f and g are continuous functions and vanish for $x = 0$, l. We now apply the method of Sturm-Liouville to

$$u = u(x,t) = \Phi(x)T(x)$$

to get

$$\Phi(x)\,T''(t) = a^2\Phi''\,(x)T(t).$$

That is,

$$\frac{\Phi''}{\Phi} = \frac{T''}{a^2T} = -\lambda.$$

Thus,

$$\Phi'' + \lambda\Phi = 0,$$
$$T'' + a^2\lambda T = 0,$$

subject to $u(0,t) = \Phi(0)T(t) = 0 = \Phi(l)T(t) = u(l,t)$ for all $t \geq 0$, that is, subject to $\Phi(0) = 0 = \Phi(l)$. We will assume λ is positive, so we write λ^2 instead:

$$\Phi'' + \lambda^2\Phi = 0, \qquad T'' + a^2\lambda^2 T = 0.$$

The general solution of the first equation is

$$\Phi(x) = c_1\cos\lambda x + c_2\sin\lambda x.$$

But the boundary condition gives

$$\Phi(0) = 0 = c_1\cos 0 + c_2\sin 0 = c_1,$$

that is, $c_1 = 0$, and so

$$0 = \Phi(l) = c_2 \sin \lambda l.$$

But $c_2 \neq 0$, so $\lambda = \dfrac{\pi k}{l}$. But the above analysis works for all λ_k, $k = 0, 1, 2, \ldots$, we obtain $\lambda_k = \dfrac{\pi k}{l}$, $k = 0, 1, 2, \ldots$ and

$$\Phi_k(x) = \sin \frac{\pi k x}{l}, \qquad k = 0, 1, 2, \ldots$$

Thus, the second differential equation gives

$$T_k(t) = A_k \cos(a\lambda_k t) + B_k \sin(a\lambda_k t), \qquad k = 0, 1, 2, \ldots$$

Hence

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t) \\ &= \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{a\pi k t}{l}\right) + B_k \sin\left(\frac{a\pi k t}{l}\right) \right] \sin\left(\frac{\pi k x}{l}\right). \end{aligned}$$

We now apply the initial condition to $u(x,\, t)$. So we require

$$\begin{aligned} f(x) = u(x,0) &= \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi k x}{l}\right) \\ \text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x,0) &= \sum_{k=0}^{\infty} B_k \left(\frac{a\pi k}{l}\right) \sin\left(\frac{\pi k x}{l}\right) \end{aligned}$$

are just the Fourier series of f and g with respect to $\{\sin \dfrac{\pi k x}{l}\}$. Thus

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l f(x) \sin \frac{\pi k x}{l}\, dx, \\ B_k &= \frac{2}{a\pi k} \int_0^l g(x) \sin \frac{\pi k x}{l}\, dx. \end{aligned}$$

8.4. **Remarks on separation of variables.** Here are some reasons that one wants to study linear second order ODEs

- Solve PDEs in $\mathbb{R}^3$ such as Laplace Eqn. ($\Delta^2\Phi = 0$), Helmholtz Eqn. [$(\Delta^2 + k^2)\Phi = 0$], etc.;
- *Separation of variables* of the PDEs under different curvilinear orthogonal coordinate systems giving various ODEs;
- *Boundary value* (Sturm-Liouville type)problems;
- Some of these linear ODEs are better understood (*Bessel* Eqn.) than the others (*Spheroidal Wave* Eqn.)
- Almost all of these Eqns are *ancient*.

Question: Under what $3D$ curvilinear orthogonal coordinate systems (u_1, u_2, u_3) do we have a solution of the (elliptic PDE) Helmholtz Eqn

$$(\Delta^2 + k^2)\Phi = 0,$$

to be solved by *separation of variables* of the form

$$\Phi(\mathbf{r}) = \Phi_1(u_1) \cdot \Phi_2(u_2) \cdot \Phi_3(u_3) \quad ?$$

Theorem 8.1 (Eisenhart (1934))**.** *There are precisely eleven curvilinear orthogonal coordinate systems in each of which the Helmholtz equation separates.*

("Separable Systems of Stäckel." Ann. Math. **35** (1934), 284–305.) (Morse & Feshbach, "Methods of Theoretical Physics I". NY: McGraw-Hill, pp. 125–

The Eleven Coordinate Systems:

(1) Cartesian	$x = x, \quad y = y, \quad z = z$	
(2) Cylindrical	$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$ $\rho \geqslant 0, \quad -\pi < \phi \leqslant \pi$	
(3) Spherical polar	$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi$ $z = r\cos\theta$ $r \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi, \quad -\pi < \phi \leqslant \pi$	
(4) Parabolic cylinder	$x = u^2 - v^2, \quad y = 2uv, \quad z = z$ $u \geqslant 0, \quad -\infty < v < \infty$	Half plane
(5) Elliptic cylinder	$x = f\cosh\xi\cos\eta, \quad y = f\sinh\xi\sin\eta$ $z = z$ $\xi \geqslant 0, \quad -\pi < \eta \leqslant \pi$	Infinite strip Plane with straight aperture
(6) Rotation-paraboloidal	$x = 2uv\cos\phi, \quad y = 2uv\sin\phi$ $z = u^2 - v^2$ $u \geqslant 0, \quad v \geqslant 0, \quad -\pi < \phi \leqslant \pi$	Half-line
(7) Prolate spheroidal	$x = \ell\sinh u\sin v\cos\phi$ $y = \ell\sinh u\sin v\sin\phi$ $z = \ell\cosh u\cos v$ $u \geqslant 0, \quad 0 \leqslant v \leqslant \pi, \quad -\pi < \phi \leqslant \pi$	Finite line segment Two half-lines
(8) Oblate spheroidal	$x = \ell\cosh u\sin v\cos\phi$ $y = \ell\cosh u\sin v\sin\phi$ $z = \ell\sinh u\cos v$ $u \geqslant 0, \quad 0 \leqslant v \leqslant \pi, \quad -\pi < \phi \leqslant \pi$	Circular plate (disc) Plane with circular aperture
(9) Paraboloidal	$x = \frac{1}{2}\ell(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)$ $y = 2\ell\cosh\alpha\cos\beta\sinh\gamma$ $z = 2\ell\sinh\alpha\sin\beta\cosh\gamma$ $\alpha \geqslant 0, \quad -\pi < \beta \leqslant \pi, \quad \gamma \geqslant 0$	Parabolic plate Plane with parabolic aperture
(10) Elliptic conal	$x = kr\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta, \quad y = (ik/k')r\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta$ $z = (1/k')r\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta$ $r \geqslant 0, \quad -2K < \alpha \leqslant 2K$ $\beta = K + iu, \quad 0 \leqslant u \leqslant 2K'$	Plane sector (including quarter-plane)
(11) Ellipsoidal	$x = k^2\ell\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma$ $y = (-k^2\ell/k')\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma$ $z = (i\ell/k')\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma$ α, β as in (10), $\quad \gamma = iK' + w, \quad 0 < w \leqslant K$	Elliptic plate Plane with elliptic aperture

FIGURE 2. (Arscott & Darai 1981 IMA Appl. Math.)

11 Curvilinear Orthogonal Coordinate systems

(1) Cartesian;
(2) Cylindrical;
(3) Spherical polar;
(4) Parabolic cylinderical;
(5) Elliptic cylinderical;
(6) Rotation paraboloidal;
(7) Prolate paraboloidal;
(8) Oblate paraboloidal;
(9) Paraboloidal;
(10) Elliptic conal;
(11) Ellipsoidal.

The Classification Table

Separation of Laplace and Helmholtz equations in various co-ordinate system

	Ordinary differential equations arising from separation of	
Co-ordinate system	Laplace equation	Helmholtz equation
(1) Cartesian	(Trivial)	(Trivial)
(2) Cylindrical	(Trivial)	Bessel's
(3) Spherical polar	Associated Legendre	Associated Legendre and spherical Bessel
(4) Parabolic cylinder	(Trivial)	Weber's
(5) Elliptic cylinder	(Trivial)	Mathieu's
(6) Rotation-paraboloidal	Bessel's	Confluent hypergeom (Tricomi's)
(7) Prolate spheroidal	Associated Legendre	Spheroidal wave
(8) Oblate spheroidal	Associated Legendre	Spheroidal wave
(9) Paraboloidal	Mathieu's	Whittaker–Hill
(10) Elliptic conal	Lamé's	Spherical Bessel and Lamé's
(11) Ellipsoidal	Lamé's	Ellipsoidal wave

FIGURE 3. (Arscott & Darai 1981 IMA Appl. Math.)

The Equation Table

Associated Legendre:	$(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left\{n(n+1) - \dfrac{m^2}{(1-x^2)}\right\} y = 0$
Bessel:	$x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-n^2)y = 0$
Spherical Bessel:	$x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + [x^2-n(n+1)]y = 0$
Weber's:	$\dfrac{d^2y}{dx^2} + (\lambda - \frac{1}{4}x^2)y = 0$
Confluent hypergeometric:	$x\dfrac{d^2y}{dx^2} + (\gamma - x)\dfrac{dy}{dx} - \alpha y = 0$
Mathieu's:	$\dfrac{d^2y}{dx^2} + (\lambda - 2q\cos 2x)y = 0$
Spheroidal wave:	$(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left\{\lambda - \dfrac{\mu^2}{(1-x^2)} + \gamma^2(1-x^2)\right\} y = 0$
Lamé's:	$\dfrac{d^2y}{dx^2} + [h - n(n+1)k^2\,\mathrm{sn}^2\,x]y = 0$
Whittaker–Hill:	$\dfrac{d^2y}{dx^2} + (a + b\cos 2x + c\cos 4x)y = 0$
Ellipsoidal wave:	$\dfrac{d^2y}{dx^2} - (a + bk^2\,\mathrm{sn}^2\,x + qk^4\,\mathrm{sn}^4\,x)y = 0$

FIGURE 4. (Arscott & Darai 1981 IMA Appl. Math.)

8.5. **Sturm-Liouville Boundary Value Problems.** We assume that the function P in (8.1) does not vanish. As a result, we can rewrite the equation (8.1) into a *self-adjoint form*:

Lemma 8.2. *The equation*

$$P\Phi'' + R\Phi' + Q\Phi = -\lambda\Phi \tag{8.10}$$

can be written in the form

$$(p\Phi')' + q\Phi = -\lambda r\Phi,$$

where p, q and r are continuous functions of x on $[a, b]$, p is positive and has a continuous derivative, and r is positive.

Proof. Multiply the equation (8.10) on both sides by r. Then we require:

$$\begin{aligned} rP\Phi'' + rR\Phi' + rQ\Phi &= -\lambda r\Phi, \\ p\Phi'' + p'\Phi' + q\Phi &= -\lambda r\Phi \end{aligned}$$

to be the same. So it is sufficient that we require

$$\begin{array}{rccc} & p' = rR & \text{and} & p = rP, \\ \text{that is, if} & \dfrac{p'}{p} = \dfrac{R}{P} & \text{and} & r = \dfrac{p}{P}, \\ \text{or equivalently} & p = e^{\int \frac{R}{P}} & \text{and} & r = \dfrac{1}{P}e^{\int \frac{R}{P}} \end{array}$$

which is well-defined since P is positive, and the conclusion for p, q and r thus follow. □

Lemma 8.3. *Let*

$$L(\Phi) = \frac{d}{dx}\Big(p\frac{d\Phi}{dx}\Big) + q\,\Phi. \tag{8.11}$$

Then for any two twice differentiable functions Φ and Ψ, we have

$$\Phi L(\Psi) - \Psi L(\Phi) = \frac{d}{dx}\big(p(\Phi\Psi' - \Phi'\Psi)\big). \tag{8.12}$$

Proof. Direct verification. □

Lemma 8.4. *If Φ and Ψ satisfy the boundary condition*

$$\begin{aligned} \alpha\Phi(a) + \beta\Phi'(a) &= 0, \\ \gamma\Phi(b) + \delta\Phi'(b) &= 0 \end{aligned} \tag{8.13}$$

then

$$\Phi\Psi' - \Phi'\Psi\Big|_{x=a} = 0 = \Phi\Psi' - \Phi'\Psi\Big|_{x=b}. \tag{8.14}$$

Proof. The equations

$$\alpha\Phi(a) + \beta\Phi'(a) = 0,$$
$$\alpha\Psi(a) + \beta\Psi'(a) = 0$$

have non-trivial solutions for α and β if and only if

$$\begin{vmatrix} \Phi(a) & \Phi'(a) \\ \Psi(a) & \Psi'(a) \end{vmatrix} = 0.$$

Similar argument gives the condition at $x = b$. □

Lemma 8.5. *Let L be the Sturm-Loiuville operator given in (8.11) and*

$$L(\Phi) = -\lambda r\Phi,$$
$$L(\Psi) = -\lambda r\Psi$$

and both Φ and Ψ satisfy the same boundary condition in (8.13) at $x = a$ and b. Then Φ and Ψ are orthogonal on $[a, b]$ with respect to the function r (called the orthogonal weight function).

Proof. Since

$$\Psi\, L(\Phi) - \Phi\, L(\Psi) = (\mu - \lambda)\, r\, \Phi\, \Psi.$$

But (8.12) gives

$$\begin{aligned} \frac{d}{dx}\, p\, (\Phi\Psi' - \Phi'\Psi) &= \Psi\, L(\Phi) - \Phi\, L(\Psi) \\ &= (\mu - \lambda)\, r\, \Phi\, \Psi. \end{aligned}$$

The (8.14) now gives

$$0 = p(\Phi\Psi' - \Phi'\Psi)\Big|_{x=a}^{x=b} = (\mu - \lambda) \int_a^b r\, \Phi\, \Psi\; dx.$$

This proves that Φ and Ψ are orthogonal with respect to the weight function r over $[a, b]$. □

Lemma 8.6. *Let L be the Sturm-Loiuville operator given in (8.11) and*

$$L(\Phi) = -\lambda\, r\, \Phi.$$

Then λ must be real.

Proof. For suppose λ is complex with $\lambda = \mu + i\nu$, $\nu \neq 0$ and $\Phi = \phi + i\psi$. Then

$$\big(p(\phi' + i\psi')\big)' + q(\phi + i\nu) = -(\mu + i\nu)\, r\, (\phi + i\psi).$$

Taking complex conjugate of this equation gives

$$\big(p(\phi' - i\psi')\big)' + q(\phi - i\nu) = -(\mu - i\nu)\, r\, (\phi - i\psi).$$

which implies that $\phi - i\nu = \overline{\Phi}$ is an eigenvector and $\overline{\lambda} = \mu - i\nu$ is the corresponding eigenvalue. If we now follow the argument used in Lemma 8.5, then we obtain

$$\int_a^b r\, \Phi\, \overline{\Phi}\, dx = \int_a^b r\, (\phi^2 + \psi^2)\, dx > 0$$

contradicting that Φ and $\overline{\Phi}$ are orthogonal. □

Theorem 8.7. *If $r > 0$, $q \leq 0$ and if the boundary conditions imply that*

$$p\, \Phi\, \Phi' \Big|_a^b \leq 0, \tag{8.15}$$

then all the eigenvalues of the boundary value problem for

$$(p\Phi')' + q\, \Phi = -\lambda\, r\, \Phi$$

are non-negative.

Proof. We multiply both sides of

$$(p\Phi')' + q\Phi = -\lambda r \Phi.$$

by Φ and integrate both sides of the resulted equation over (a, b) to get

$$p\, \Phi'\, \Phi \Big|_{x=a}^{x=b} - \int_a^b p\, \Phi'^2\, dx + \int_a^b q\, \Phi^2\, dx = -\lambda \int_a^b r\, \Phi^2\, dx.$$

It follows from the hypothesis that $\lambda \geq 0$. In addition, $\lambda = 0$ only if $q \equiv 0$, $\Phi' \equiv 0$ over $[a, b]$. That is, if Φ is a constant eigenvector. □

Remark. The assumption (8.15) includes

(1) $\Phi(a) = 0 = \Phi(b)$,
(2) $\Phi'(a) = 0 = \Phi'(b)$,
(3) $\Phi'(a) - h\Phi(a) = 0$, $\Phi'(b) + H\Phi(b) = 0$, where h and H are non-negative constants.

8.6. **Existence of eigenvalues.**

Theorem 8.8. *The Sturm-Liouville boundary value problem*

$$\begin{aligned}(p\Phi')' + q\Phi &= -\lambda r\Phi,\\ \alpha\Phi(a) + \beta\Phi'(a) &= 0,\\ \gamma\Phi(b) + \delta\Phi'(b) &= 0\end{aligned}$$

where p, q, r *are continuous functions of* x *on* $[a, b]$, p *and* r *are positive and* p' *is continuous, has infinitely many eigenfunction solutions and the corresponding eigenvalues* λ_n, *such that* $\lambda_1 < \lambda_2 < \ldots$, *and* $\lambda_n \to \infty$ *as* $n \to +\infty$.

Moreover, each eigenfunction corresponding to its eigenvalue, λ_n, *say, has exactly* $n-1$ *zeros in the open interval* (a, b).

The proof of the above theorem, which depends on Green's functions, is beyond the scope of this course. Interested students can consult Chapter 10 of Folland's book.

Remark. The above problem is commonly called the *regular Sturm-Liouville* boundary value problems. The *singular Sturm-Liouville* boundary value problems may include situation where the function p may vanish at one or both endpoints of $[a, b]$, the weight $r(x)$ may vanish or be unbounded at one or both endpoints of $[a, b]$. Besides, the interval $[a, b]$ may be unbounded, that is, $a = -\infty$ and/or $b = +\infty$.

We consider some examples that illustrate the above theorem. The examples are taken from Folland pages 91–93.

Example. We are given the differential equation

$$y''(x) + \lambda y(x) = 0,$$

with boundary condition

$$\begin{aligned}\alpha y(0) - y'(0) &= 0,\\ \gamma y(\ell) - y'(\ell) &= 0.\end{aligned} \tag{8.16}$$

Suppose $\lambda = 0$. Then the solution to the differential equation $y''(x) = 0$ is $y = cx+d$. The boundary condition at $x = 0$ and ℓ gives

$$\alpha d = c, \quad \text{and} \quad \gamma(c\ell + d) = c$$

respectively. Thus, $\gamma = \alpha/(\alpha\ell+1)$. We may therefore choose $y = x+\alpha$. Suppose now that $\lambda \neq 0$, then the Lemma 8.6 asserts that λ must be real. Thus, it remains to consider $\lambda = \nu^2$ for some real $\nu > 0$ or $\lambda = (i\mu)^2 = -\mu^2$ for some real $\mu > 0$. Suppose $\lambda = \nu^2$. The boundary condition (8.16) implies, without loss of generality, that the general solution can be written as

$$\begin{aligned}y(x) &= c\cos\nu x + d\sin\nu x\\ &= \nu\cos\nu\, x + \alpha\sin\nu\, x\end{aligned}$$

since $\alpha c = \alpha y(0) = y'(0) = \nu d$. Thus $d = c\alpha/\nu$. Hence we have discard the c above. But then the boundary condition (8.16) at $x = \ell$ implies that

$$-\nu^2 \sin \nu\ell + \alpha\nu \cos \nu\ell = \beta(\nu \cos \nu\ell + \alpha \sin \nu\ell),$$

or

$$\tan \nu\ell = \frac{(\alpha - \beta)\nu}{\alpha\beta + \nu^2}.$$

On the other hand, if $\lambda = -\nu^2$ or $\nu = i\mu$, and noting that $\tan ix = i \tanh x$, so that the above equation would become

$$\tan \mu\ell = \frac{(\alpha - \beta)\mu}{\alpha\beta - \mu^2}$$

instead. It is clear that the above equations for ν or μ does not admit nice closed form solutions unless $\alpha = \beta$.

Case I: $\alpha = 1$, $\beta = -1$, and $\ell = \pi$. We plot the curves of $\tan \pi\nu$ and $\dfrac{2\nu}{\nu^2 - 1}$ respectively. The graph shows that there are infinitely many positive solutions ν_n increasing to infinity. In fact, the graphical method shows that

$$\lambda_n = \nu_n^2 \approx (n-1)^2,$$

for large n. There is no intersection for the second case when $\lambda < 0$. The corresponding eigen-functions are

$$y_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x.$$

Case II: $\alpha = 1$, $\beta = 4$, and $\ell = \pi$. Then we plot the curves of

$$\tan \pi\nu = \frac{-3\nu}{4 + \nu^2}.$$

It turns out that apart from $\lambda_n = \nu^2 \approx n^2$ for positive n, the equation, for $\lambda = -\mu^2$,

$$\tan \pi\nu = \frac{3\nu}{\mu^2 - 4}.$$

admit one positive solution ν_0. Thus, the eigen-functions are

$$y_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x, \quad n \geq 1,$$

and

$$y_0(x) = \mu_0 \cosh \mu_0 x + \sinh \mu_0 x.$$

8.7. **Eigen-function expansions.** We quote without proof the following results.

Theorem 8.9. *Let f be continuous on $[a, b]$ with piecewise smooth f' such that it satisfies the Sturm-Liouville boundary value problem with boundary condition*

$$\alpha f(a) + \beta f'(a) = 0,$$
$$\gamma f(b) + \delta f'(b) = 0.$$

Then the Fourier series of f with respect to the eigenfunctions, $\{\Phi_n\}$, that is,

$$f(x) \sim c_0\Phi_0(x) + c_1\Phi_1(x) + \dots$$

and

$$\int_a^b r\Phi_n^2(x)\ dx = 1, \qquad c_n = \int_a^b rf(x)\Phi_n(x)\ dx,$$

$n = 0, 1, 2, \dots,$ *converges to $f(x)$ absolutely and uniformly.*

Moreover

Theorem 8.10. *If f is only a piecewise smooth on $[a, b]$ instead in the last theorem, then the Fourier series of f with respect to the eigenfunctions $\{\Phi_n\}$ converges for $a < x < b$ to the value of $f(x)$ at every point of continuity, and to the value*

$$\frac{f(x+0) + f(x-0)}{2}$$

at every point of jump discontinuity.

As for the completeness of the system $\{\Phi_n\}$ for a square integrable function $f(x)$ over $[a, b]$, we quote

Theorem 8.11. *Let $\{\Phi_n\}$ be the orthogonal eigenfunctions derived from the Sturm-Liouville boundary value problem above. Then*

$$\int_a^b r(x)f(x)^2\ dx = \sum_{k=0}^{\infty} c_k^2 \|\sqrt{r}\Phi_k(x)\|^2 = \sum_{k=0}^{\infty} c_k^2,$$

holds for every square integrable function $f(x)$ over $[a, b]$. That is, the system $\{\Phi_n\}$ is complete with respect to the weight function $r(x)$.

8.8. **Solutions to PDEs.**

Theorem 8.12. *Let $u(x,\, t)$, be a continuous solution of*

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

in $a \le x \le b$, $t \ge 0$ with $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ bounded on $[a,\, b] \times [0,\, t_0]$ for every t_0, and it satisfies the boundary value problem

$$\alpha u(a,t) + \beta u_x(a,t) = 0, \qquad \gamma u(b,t) + \delta u_x(b,t) = 0$$

and the initial condition

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Then

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\, \Phi_n(x),$$

where $\{\Phi_n\}$ are eigenfunctions associated with the boundary value problem. The function T_n can be found from solving the second initial conditions

$$T_n'' + \lambda_n T_n = 0, \qquad n = 0,\, 1,\, 2,\, \ldots \tag{8.17}$$

subject to

$$T_n(0) = C_n, \qquad T_n'(0) = c_n, \qquad n = 0,\, 1,\, 2,\, \ldots$$

where C_n, c_n are, respectively, the Fourier series coefficients of $f(x)$ and $g(x)$.

Remark. We note that the above result does not claim that the expansion for $u(x,\, t)$ necessarily be a solution to the boundary value problem, even if the Fourier series for $f(x)$ and $g(x)$ (for them to be sufficiently smooth to) converge. This is because the series for $u(x,\, t)$ would need to converge uniformly after being differentiated with respect to x twice.

Proof. We multiply both sides of the PDE by

$$r = \frac{1}{P}\, e^{\int_{x_0}^{x} \frac{R}{P}\, dx} = \frac{p}{P},$$

Then we can re-write the above PDE into the form

$$p\, \frac{\partial^2 u}{\partial x^2} + p'\, \frac{\partial u}{\partial x} + q\, u = r\, \frac{\partial^2 u}{\partial t^2},$$

which is in a "self-adjoint form":

$$L(u) = \frac{\partial}{\partial x}\Big(p \frac{\partial u}{\partial x}\Big) + q\, u = r\, \frac{\partial^2 u}{\partial t^2},$$

so that we can apply the Lemma 8.3 to write

$$L(u) = r\, \frac{\partial^2 u}{\partial t^2} \tag{8.18}$$

where we also have

$$L(\Phi_n) = -\lambda_n\, r\, \Phi_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots \tag{8.19}$$

We apply the Theorem 8.10 that for x in $[a,\, b]$ and for every (fixed) $t \geq 0$, the $u(x,\, t)$ can be expanded in a Fourier series in terms of the eigenfunction solutions $\{\phi_n\}$ of the form

$$u(x,\, t) = T_0(t)\Phi_0(x) + T_1(t)\Phi_1(x) + \cdots = \sum_{n=0}^{\infty} T_n(t)\, \Phi_n(x)$$

where the Fourier coefficients T_n (with t fixed) for $u(x,\, t)$ (t fixed) are given by

$$T_n(t) = \int_a^b u(x,\, t)\, \Phi_n(x)\, r(x)\, dx, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots . \tag{8.20}$$

But then the (8.19) implies that

$$r\, \Phi_n(x) = -\frac{1}{\lambda_n}\, L(\Phi_n),$$

and so

$$T_n(t) = \int_a^b u(x,\, t)\, \Big(-\frac{1}{\lambda_n}\, L(\Phi_n)\Big)\ dx = -\frac{1}{\lambda_n} \int_a^b u(x,\, t)\ L(\Phi_n)\ dx$$

and the Lagrange identity

$$uL(\Phi) - \Phi L(u) = \frac{d}{dx}\big(p(u\Phi' - \Phi\, u')\big)$$

yields

$$T_n(t) = -\frac{1}{\lambda_n} \int_a^b \Phi_n(t)\, L(u(x,\, t))\ dx + \frac{1}{\lambda_n}\Big[p\Big(\Phi_n \frac{\partial u}{\partial x} - \Phi_n'\, u\Big)\Big]_a^b$$

for which the second term vanishes according to the Lemma 8.4. Thus

$$T_n(t) = -\frac{1}{\lambda_n} \int_a^b \Phi_n(t)\, L(u(x,\, t))\ dx = -\frac{1}{\lambda_n} \int_a^b \frac{\partial^2 u}{\partial t^2}\, \Phi_n(x)\, r\ \ dx.$$

Let us differentiate the (8.20) with respect to t twice yields

$$T_n''(t) = \int_a^b \frac{\partial^2 u(x,\, t)}{\partial t^2}\, \Phi_n(x)\, r(x)\, dx,$$

which is identical to the equation (8.17). The differentiation under the integral sigh with respect to t is justified because of the assumption of the boundedness of the first and second partial derivatives with respect to t. This establishes the equation (8.17). On the other hand, since $u(x,\, t)$ is continuous, so

$$\lim_{t\to 0} T_n(t) = \lim_{t\to 0} \int_a^b u(x,\, t)\, \Phi_n(x)\, r(x)\, dx = \int_a^b f(x)\, \Phi_n(x)\, r(x)\, dx := C_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots .$$

where C_n is the Fourier coefficient of $f(x)$ (with respect to the system $\{\Phi_n\}$). But the $T_n(t)$ is continuous, so

$$T_n(0) = C_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots$$

as required. Similarly, we have

$$T_n'(0) = c_n, \qquad n = 0, 1, 2, 3, \cdots$$

where the c_n are the Fourier coefficients for the $g(x)$. $\square$

A problem of working with a general orthogonal (orthonormal) system $\{\Phi_n\}$ for a boundary value problem is to make sure that the functions $f(x)$ and $g(x)$ in initial condition

$$u(x, 0) = f(x), \qquad \frac{\partial f}{\partial t} = g(x),$$

can be expanded in as Fourier series in terms of $\{\Phi_n\}$. That is, suppose we have

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x),$$

and

$$g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(x),$$

then they need to match with the earlier

$$f(x) = u(x, 0) = \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi k x}{l}\right)$$

$$\text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x, 0) = \sum_{k=0}^{\infty} B_k \left(\frac{a\pi k}{l}\right) \sin\left(\frac{\pi k x}{l}\right)$$

where

$$A_k = \frac{2}{l} \int_0^l f(x) \sin\frac{\pi k x}{l}\, dx,$$

$$B_k = \frac{2}{a\pi k} \int_0^l g(x) \sin\frac{\pi k x}{l}\, dx.$$

That is, we require

$$A_k = C_k, \qquad B_k = c_k/\sqrt{\lambda_k}$$

for $k \geq 0$. So the above requirement would be true if both f and g are known to be "sufficiently smooth", which is rarely the case in reality. Then the problem is if the coefficients A_k and B_k decline sufficiently fast that guarantee the convergence as well as about term-by-term differentiation of the two $x-$derivatives of u. In any case, The above theorems show that if a physical problem has any solution at all, then its solution from the above separation of variables method and Sturm-Liouville method would lead to a solution. So we use the term *generalised solution* to the boundary value problem even if the solution found by the above does not satisfy some of the above requirements (of the boundary value problem). We call *exact solution* for a genuine (real) solution. We discuss below if the generalised solution is still of some use.

Theorem 8.13. *Let*

$$u(x,\,t)\sim\sum_{k=0}^{\infty}T_k(t)\Phi_k(x)$$

be either the exact or the generalised solution of equation (8.1), that satisfies the boundary condition (8.3) and initial condition (8.4). Suppose $f_m(x)$ and $g_m(x)$ converge to $f(x)$ and $g(x)$ respectively, in the mean, as $m\to\infty$, that is,

$$\lim_{m\to\infty}\int_a^b [f_m(x)-f(x)]^2\,r\;dx=0=\lim_{m\to\infty}\int_a^b [g_m(x)-g(x)]^2\,r\;dx. \tag{8.21}$$

Suppose that

$$u_m(x,\,t)=\sum_{k=0}^{\infty}T_{m\,n}(t)\,\Phi_n(x)$$

is either the exact or the generalised solution of equation (8.1), that satisfies the boundary condition (8.3) and new initial condition

$$u_m(x,\,t)=f_m(x),\qquad \frac{\partial u_m(x,\,0)}{\partial t}=g_m(x),\qquad m\geq 0$$

then $u_m(x,\,t)$ converges to $u(x,\,t)$ in the mean, as $m\to\infty$.

Proof. So the idea is to compare the Fourier coefficients of the $T_n(x)$ and $T_{n\,m}(x)$. Recall that the C_n, c_n are respectively, the Fourier coefficients of $f(x)$ and $g(x)$ for the boundary value problem

$$T_n''+\lambda_n\,T_n=0,\qquad T_n(0)=C_n,\quad T_n'(0)=c_n;\qquad n\geq 0$$

and the $C_{n\,m}$, $c_{n,\,m}$ are respectively, the Fourier coefficients of $f(x)$ and $g(x)$ for the boundary value problem

$$T_{m\,n}''+\lambda_nT_{m\,n}=0,\qquad T_{m\,n}(0)=C_{m\,n},\quad T_{m\,n}'(0)=c_{m\,n};\qquad n\geq 0$$

for each $m\geq 0$. We also know that

$$\lambda_0<\lambda_1<\lambda_2<\cdots\qquad \lim_{n\to\infty}\lambda_n=+\infty.$$

and that all perhaps except a finite number of them can be negative. Suppose $\lambda_n\leq 0$ when $n\leq N$ and $\lambda_n>0$ for $n>N$. Then,

$$T_n(x)=\begin{cases}\frac{1}{2}\Big(C_n+\dfrac{c_n}{\sqrt{-\lambda_n}}\Big)\,e^{\sqrt{-\lambda_n}t}+\dfrac{1}{2}\Big(C_n-\dfrac{c_n}{\sqrt{-\lambda_n}}\Big)\,e^{-\sqrt{-\lambda_n}t}, & n\leq N\\ C_n\cos\sqrt{\lambda_n}t+\dfrac{c_n}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t, & n>N.\end{cases} \tag{8.22}$$

and similarly,

$$T_{m\,n}(x)=\begin{cases}\frac{1}{2}\Big(C_{m\,n}+\dfrac{c_{m\,n}}{\sqrt{-\lambda_n}}\Big)\,e^{\sqrt{-\lambda_n}t}+\dfrac{1}{2}\Big(C_{m\,n}-\dfrac{c_{m\,n}}{\sqrt{-\lambda_n}}\Big)\,e^{-\sqrt{-\lambda_n}t}, & n\leq N\\ C_{m\,n}\cos\sqrt{\lambda_n}t+\dfrac{c_{m\,n}}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t, & n>N.\end{cases} \tag{8.23}$$

Notice that the assumption (8.21) implies that

$$0=\lim_{m\to\infty}\int_a^b [f_m(x)-f(x)]^2\, r\ dx=\lim_{m\to\infty}\sum_{n=0}^{\infty}(C_n-C_{m\,n})^2 \tag{8.24}$$

and

$$0=\lim_{m\to\infty}\int_a^b [g_m(x)-g(x)]^2\, r\ dx=\lim_{m\to\infty}\sum_{n=0}^{\infty}(c_n-c_{m\,n})^2, \tag{8.25}$$

so that

$$\lim_{m\to\infty} C_{m\,n}=C_n, \qquad \lim_{m\to\infty} c_{m\,n}=c_n.$$

We easily deduce from above that

$$\lim_{m\to\infty}(T_{m\,n}-T_n)=0, \qquad n\ge 0. \tag{8.26}$$

We actually know more when $n>N$:

$$\begin{aligned}(T_{m\,n}-T_n)^2&=\Big[(C_n-C_{m\,n})\cos\sqrt{\lambda_n}t+\frac{(c_n-c_{m\,n})}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t\Big]^2\\ &\le 2\Big[(C_m-C_{n\,n})^2+\Big(\frac{(c_n-c_{m\,n})}{\sqrt{\lambda_n}}\Big)^2\Big]\end{aligned} \tag{8.27}$$

But then we deduce from (8.24), (8.25), (8.27) and (8.26) that

$$\begin{aligned}\int_a^b [u_m(x,\,t)-u(x,\,t)]^2\, r\ dx&=\int_a^b\sum_{n=0}^{\infty}(T_{m\,n}(t)-T_n(t))^2\Phi_n(x)^2\, r\ dx\\ &=\sum_{n=0}^{\infty}(T_n-T_{m\,n})^2\int_a^b\Phi_n(x)^2\, r\ dx\\ &=\sum_{n=0}^{\infty}(T_n-T_{m\,n})^2\cdot 1\\ &=\sum_{n=0}^{N}(T_n-T_{m\,n})^2+\sum_{n=N+1}^{\infty}(T_n-T_{m\,n})^2\\ &\longrightarrow 0,\end{aligned}$$

as $m\to\infty$, which is what we desire to prove. □

Remark. Suppose we choose f_m and g_m above as the $m-$partial sums of the $f(x)$ and $g(x)$ respectively, then the solutions u_m are *exact solutions* to the (8.1) subject to the conditions (8.3) and the corresponding (8.4). Thus, if $u(x,\,t)$ is either the exact or generalised solutions to (8.1) subject to the (8.3) and (8.4) is the limit of $u_m(x,\,t)$, as $f_m\to f$ and $g_m\to g$ either uniformly or in the mean.

8.9. **Remarks on Forced vibrations.** We have instead

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial^2 u}{\partial t^2} + F(x,\, t), \tag{8.28}$$

subject to the same (8.3) and (8.4), where the $F(x,\, t)$ stands for an external force. The same technique would lead to

$$L(u) = r\frac{\partial^2 u}{\partial t^2} + rF(x, t).$$

Then the Sturm-Liouville problem would give

$$L(\Phi) = -\lambda r\,\Phi.$$

We also have

$$T_n(t) = \int_a^b u(x,\, t)\,\Phi_n(x)\,r(x)\,dx, \qquad n = 0,\, 1,\, 2,\, 3, \cdots. \tag{8.29}$$

A similar argument used earlier give

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \Phi_n(t)\,L(u(x,\, t))\;\,dx = -\frac{1}{\lambda_n}\int_a^b \frac{\partial^2 u}{\partial t^2}\,\Phi_n(x)\,r\;\,dx - \frac{1}{\lambda_n}\int_a^b r\,F(x,\, t)\,\Phi_n(x)\;\,dx$$

instead. Suppose

$$F(x,\, t) = \sum_{k=0}^{\infty} F_k(t)\Phi_k(x)$$

where

$$F_k(t) = \sum_{k=0}^{\infty}\int_a^b r\,F_k(x,\, t)\Phi_k(x)\;\,dx, \qquad n = 0,\, 1,\, 2, \cdots,$$

and this gives

$$T_k = -\frac{1}{\lambda_k}T_k'' - \frac{1}{\lambda_k}F_k,$$

which is

$$T_k'' + \lambda_k T_k + F_k = 0, \qquad n = 0,\, 1,\, 2,\, 3, \cdots.$$

8.10. **Remarks on Heat equations.** Recall that the heat equation assumes the form

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho,\ (K = \text{thermal conductivity},\ c = \text{heat capacity},\ \rho = \text{density})$$

subject to the boundary condition (8.3) but initial condition becomes only the

$$u(x,\, 0) = f(x).$$

So we try

$$u(x,\, t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.30}$$

where the Φ_k proceed as before, while the T_k satisfies

$$T_k'(t) + \lambda_k T_k(t) = 0, \quad T_k(0) = C_n, \qquad k = 0,\, 1,\, 2, \cdots$$

and the C_k are the Fourier coefficients of $f(x)$. Since the solution here is given by

$$T_k(t) = C_k\, e^{-\lambda_k t}, \qquad k \geq 0,$$

so that any generalised solution for the heat equation would become exact since the convergence of the series (8.30) is absolute and uniform and so can be differentiated any number of times.

To be continued ...