MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

8. The Eigenfunction Method and its Applications to PDEs

8.1. **Linear partial differential equations.**

General description. Many mathematical physics problems lead to linear partial differential equations:

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial^2 u}{\partial t^2}, \tag{8.1}$$

$$P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu = \frac{\partial u}{\partial t}, \tag{8.2}$$

where P,R and Q are functions of x, and $u = u(x,t)$.

Here is a list of partial differential equations researchers often encounter.

(I) *Heat Flow in a Rod:*

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho, \ (K = \text{thermal conductivity},\ c = \text{heat capacity},\ \rho = \text{density})$$

(II) *Vibration String*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = T/\rho \qquad (T \text{ is Tension, } \rho \text{ is mass per unit length})$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{F(u,t)}{\rho} \qquad \text{(Forced vibration)}$$

(II) *Vibration of Rectangular Membrane*

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \qquad c^2 = T/\rho \quad (T = \text{Tension},\ \rho = \text{surface density}).$$

(III) Vibration of Circular Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big) + \frac{F(r,\theta,t)}{\rho} \qquad \text{(Forced Vibration)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big) \qquad \text{(independent of direction, that is, } \theta)$$

Boundary and initial value conditions. However, the solutions of these partial differential equations are subjected to *boundary conditions* :

$$
\begin{aligned}
\alpha\, u(a,\, t) + \beta\, \frac{\partial u(a,\, t)}{\partial x} &= 0 \\
\gamma\, u(b,\, t) + \delta\, \frac{\partial u(b,\, t)}{\partial x} &= 0,
\end{aligned}
\tag{8.3}
$$

for $t \geq 0$, $a \leq x \leq b$, where α, β, γ and δ are constants, and *initial condition*:

$$u(x, 0) = f(x), \tag{8.4}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \tag{8.5}$$

for $a \leq x \leq b$ where $f(x)$ and $g(x)$ are given continuous functions.

We note that the above boundary and initial conditions can be interpreted as:

$$
\begin{aligned}
\alpha \lim_{x \to a} u(x, t) + \beta \lim_{x \to a} \frac{\partial u}{\partial x}(x, t) &= 0, \\
\gamma \lim_{x \to b} u(x, t) + \delta \lim_{x \to b} \frac{\partial u}{\partial x}(x, t) &= 0,
\end{aligned}
$$

for $t \geq 0$, and

$$\lim_{t \to 0} u(x, t) = f(x), \qquad \lim_{t \to 0} \frac{\partial u}{\partial t}(x, t) = g(x),$$

for $a \leq x \leq b$. We also assume that $(\alpha,\, \beta) \neq (0,\, 0)$ and $(\gamma,\, \delta) \neq (0,\, 0)$.

8.2. **Separation of variables method.** The idea is to write

$$u = u(x, t) = \Phi(x)\, T(t)$$

where $\Phi(x)$ and $T(t)$ are functions of x and t only respectively. We further assume this $u(x,\, t)$ satisfies the boundary conditions wrote down above. We substitute u into

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

and this gives

$$P\Phi'' T + R\Phi' T + Q\, \Phi\, T = \Phi T'',$$

or, after dividing both sides by $u = \Phi \cdot T$

$$\frac{P\Phi'' + R\Phi' + Q\Phi}{\Phi} = \frac{T''}{T}.$$

We observe that the left-side of the above equation is a function of x only, and the right-side is a function of t only. We deduce both sides must be equal to the same constant $-\lambda$, say. Thus we obtain

$$P\,\Phi'' + R\,\Phi' + Q\,\Phi + \lambda\Phi = 0, \tag{8.6}$$

and

$$T'' + \lambda T = 0. \tag{8.7}$$

It can easily be verified that the above boundary conditions for Φ becomes

$$\begin{aligned} \alpha\,\Phi(a) + \beta\,\Phi'(a) &= 0 \\ \gamma\,\Phi(b) + \delta\,\Phi'(b) &= 0. \end{aligned} \tag{8.8}$$

The second order ODE (8.6) and the boundary condition (8.8) is called a *Sturm-Liouville boundary value problem.*

For this first equation in Φ above, we will indicate that the Strum-Liouville problem has an infinite set of solutions Φ and their corresponding positive λ, that is,

$$\Phi = \Phi_n(x), \qquad \lambda = \lambda_n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda_n \to +\infty$ (see later). In the second equation in T, then

$$T = T_n(t) = A_n \cos(\sqrt{\lambda_n t}) + B_n \sin(\sqrt{\lambda_n})t,$$

$n = 0, 1, 2, \ldots$. Since the PDE is linear, The superposition principle gives

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.9}$$

if the series converges and we can differentiate term-by-term twice. We note that the $\{\Phi_n\}$ are in fact orthogonal called the *eigenfunctions* and $\{\lambda_n\}$ are called the *eigenvalues* corresponding to the eigenfunctions.

Substitute the infinite sum of u from (8.36) into the PDEs and after rearranging yields

$$\begin{aligned} &P\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu - \frac{\partial^2 u}{\partial t^2} \\ &\qquad = P\sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial x^2} + R\sum_{k=0}^{\infty}\frac{\partial u_k}{\partial x} + Q\sum_{k=0}^{\infty} u_k - \sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial t^2} \\ &\qquad = \sum_{k=0}^{\infty}\Big(P\,\frac{\partial^2 u_n}{\partial x^2} + R\frac{\partial u_k}{\partial x} + Q\,u_k - \frac{\partial^2 u_k}{\partial t^2}\Big) = 0. \end{aligned}$$

The u in infinite sum (8.36) should satisfy the initial condition (8.4):

$$f(x) = u(x,\,0) = \sum_{k=0}^{\infty} T_k(0)\,\Phi_k(x),$$

and

$$g(x) = \frac{\partial u}{\partial t}(x, 0) = \sum_{k=0}^{\infty} T_k'(0)\, \Phi_k(x)$$

Thus the problem becomes to expand f and g by the orthogonal family of eigenfunctions $\{\Phi_n\}$:

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x), \qquad g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(t)$$

and requiring $T_k(0) = C_k$, $T_k'(0) = c_k$ for $k = 0, 1, 2, \ldots$ If $\lambda > 0$, then we could also work out the A_k and B_k for $T_k(t)$:

$$A_k = C_k, \qquad B_k = \frac{c_k}{\sqrt{\lambda_k}}, \qquad k = 0,\ 1,\ 2,\ \ldots$$

8.3. **An example of vibrating string.**

Example. Equation of Vibrating String

We consider a homogeneous string, stretched, and fastened at both ends ($x = 0$ and $x = l$). If the string is displaced by a small displacement and then released, then it will start to vibrate. Let $u(x, t)$ be the vertical displacement at the distance x and time t. We analyze the forces acting on a portion AB of the string: Then the difference of the tensions in the vertical direction is approximately

Figure 1.

measured by:

$$\begin{aligned}
T \cdot \Big(\sin(\phi + \Delta\phi) - \sin\phi\Big) &\approx T \cdot \Big(\frac{\sin(\phi + \Delta\phi)}{\cos(\phi + \Delta\phi)} - \frac{\sin\phi}{\cos\phi}\Big) \\
&= T \cdot \Big(\tan(\phi + \Delta\phi) - \tan\phi\Big) = T \cdot \Big(\frac{\partial u(x + \Delta x, t)}{\partial x} - \frac{\partial u}{\partial x}(x, t)\Big) \\
&= T \cdot \frac{\partial^2 u}{\partial x^2}\big(x + \theta\Delta x,\ t\big) \cdot \Delta x, \qquad 0 < \theta < 1.
\end{aligned}$$

Now the Newton's second law of motion ($F = ma$) gives

$$\underbrace{\rho\ \Delta x}_{\text{mass density}} \quad \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = \underbrace{T \cdot \frac{\partial^u}{\partial x^2}\ \Delta x}_{\text{force}}, \qquad \rho \text{ is mass per unit length}$$

Dividing both sides by Δx gives

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad a^2 = \frac{T}{\rho}$$

which is the equation for *free vibration* of the string. Since both ends of the string are fixed, so the boundary and initial conditions are given, respectively, by

$$u(0,t) = 0 = u(l,t), \qquad t \geq 0$$

$$\text{and} \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where f and g are continuous functions and vanish for $x = 0$, l. We now apply the method of Sturm-Liouville to

$$u = u(x,t) = \Phi(x)T(x)$$

to get

$$\Phi(x)\, T''(t) = a^2 \Phi''\,(x) T(t).$$

That is,

$$\frac{\Phi''}{\Phi} = \frac{T''}{a^2 T} = -\lambda.$$

Thus,

$$\begin{aligned} \Phi'' + \lambda\Phi &= 0, \\ T'' + a^2\lambda T &= 0, \end{aligned}$$

subject to $u(0,t) = \Phi(0)T(t) = 0 = \Phi(l)T(t) = u(l,t)$ for all $t \geq 0$, that is, subject to $\Phi(0) = 0 = \Phi(l)$. We will assume λ is positive, so we write λ^2 instead:

$$\Phi'' + \lambda^2\Phi = 0, \qquad T'' + a^2\lambda^2 T = 0.$$

The general solution of the first equation is

$$\Phi(x) = c_1 \cos \lambda x + c_2 \sin \lambda x.$$

But the boundary condition gives

$$\Phi(0) = 0 = c_1 \cos 0 + c_2 \sin 0 = c_1,$$

that is, $c_1 = 0$, and so

$$0 = \Phi(l) = c_2 \sin \lambda l.$$

But $c_2 \neq 0$, so $\lambda = \dfrac{\pi k}{l}$. But the above analysis works for all λ_k, $k = 0, 1, 2, \ldots$, we obtain $\lambda_k = \dfrac{\pi k}{l}$, $k = 0, 1, 2, \ldots$ and

$$\Phi_k(x) = \sin \frac{\pi k x}{l}, \qquad k = 0, 1, 2, \ldots$$

Thus, the second differential equation gives

$$T_k(t) = A_k \cos(a\lambda_k t) + B_k \sin(a\lambda_k t), \qquad k = 0,\, 1,\, 2,\, \ldots$$

Hence

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t) \\ &= \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{a\pi kt}{l}\right) + B_k \sin\left(\frac{a\pi kt}{l}\right) \right] \sin\left(\frac{\pi kx}{l}\right). \end{aligned}$$

We now apply the initial condition to $u(x,\, t)$. So we require

$$\begin{aligned} & f(x) = u(x,0) = \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi kx}{l}\right) \\ \text{and} \qquad & g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} B_k \Big(\frac{a\pi k}{l}\Big) \sin\left(\frac{\pi kx}{l}\right) \end{aligned}$$

are just the Fourier series of f and g with respect to $\{\sin \dfrac{\pi kx}{l}\}$. Thus

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l f(x) \, \sin \frac{\pi kx}{l} \; dx, \\ B_k &= \frac{2}{a\pi k} \int_0^l g(x) \, \sin \frac{\pi kx}{l} \; dx. \end{aligned}$$

8.4. **Remarks on separation of variables.** The *Helmholtz equation* named after the German physicist *Hermann von Helmholtz* refers to second order (elliptic) partial differential equations of the form:

$$(\Delta^2 + k^2)\Phi = 0, \tag{8.10}$$

where k is a constant. If $k = 0$, then it reduces to the *Laplace equations.* In this discussion, we shall restrict ourselves in the Euclidean space $\mathbb{R}^3$. One of the most powerful theories developed in solving linear PDEs is the *the method of separation of variables.* For example, the *wave equation*

$$\Big(\Delta^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\Psi(\mathbf{r},\, t) = 0, \tag{8.11}$$

can be solved by assuming $\Psi(\mathbf{r},\, t) = \Phi(\mathbf{r}) \cdot T(t)$ where $T(t) = e^{i\omega t}$. This yields

$$\Big(\Delta^2 - \frac{\omega^2}{c^2}\Big)\Phi(\mathbf{r}) = 0, \tag{8.12}$$

which is a Helmholtz equation. The questions now is under what $3-$dimensional coordinate system $(u_1,\, u_2,\, u_3)$ do we have a solution that is in the separation of variables form

$$\Phi(\mathbf{r}) = \Phi_1(u_1) \cdot \Phi_2(u_2) \cdot \Phi_3(u_3) \quad ? \tag{8.13}$$

Eisenhart, L. P. ("Separable Systems of Stäckel." Ann. Math. 35, 284-305, 1934) determines via a certain *Stäckel determinant* is fulfilled (see e.g. Morse, P. M. and Feshbach, H. "Methods of Theoretical Physics, Part I". New York: McGraw-Hill, pp. 125–126, 271, and 509–510, 1953).

Theorem 8.1 (Eisenhart (1934))**.** *There are a total of eleven curvilinear coordinate systems in which the Helmholtz equation separates.*

Each of the curvilinear coordinate is characterized by *quadrics.* That is, surfaces defined by

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + lz + J = 0. \tag{8.14}$$

One can visit http://en.wikipedia.org/wiki/Quadric for some of the quadric surfaces. Curvilinear coordinate systems are formed by putting relevant orthogonal quadric surfaces. Wikipedia contains quite a few of these pictures. We list the eleven coordinate systems here:

Co-ordinate Systems

Name of system	Transformation formulae	Degenerate Surfaces
(1) Cartesian	$x = x, \quad y = y, \quad z = z$	
(2) Cylindrical	$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$ $\rho \geq 0, \qquad -\pi < \phi \leq \pi$	
(3) Spherical polar	$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,$ $z = r\cos\theta$ $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad -\pi \leq \phi \leq \pi$	
(4) Parabolic cylinder	$x = u^2 - v^2, \quad y = 2uv,$ $z = z$ $u \geq 0, \qquad -\infty < v < +\infty$	Half-plane
(5) Elliptic cylinder	$x = f\cosh\xi\cos\eta, y = f\sinh\xi\sin\eta,$ $z = z$ $\xi \geq 0, -\infty < \eta < +\infty$	Infinite strip; Plane with straight aperture
(6) Rotation paraboloidal	$x = 2uv\cos\phi, \quad 2uv\sin\phi,$ $z = u^2 - v^2$ $u,\ v \geq 0, \qquad -\pi < \phi < \pi$	Half-line
(7) Prolate spheroidal	$x = \ell\sinh u\sin v\cos\phi,$ $y = \ell\sinh u\sin v\sin\phi,$ $z = \ell\cosh u\cos v,$ $u \geq 0,\ 0 \leq v \leq \pi,\ -\pi < \phi \leq \pi$	Finite line ; segment Two half-lines
(8) Oblate spheroidal	$x = \ell\cosh u\sin v\cos\phi,$ $y = \ell\cosh u\sin v\sin\phi,$ $z = \ell\sinh u\cos v,$ $u \geq 0,\ 0 \leq v \leq \pi,\ -\pi < \phi \leq \pi$	Circular plate (disc); Plane with circular aperture
(9) Paraboloidal	$x = \frac{1}{2}\ell(\cosh 2\alpha + 2\cos 2\beta - \cosh 2\gamma,$ $y = 2\ell\cosh\alpha\cos\beta\sinh\gamma,$ $z = 2\ell\sinh\alpha\sin\beta\cosh\gamma,$ $\alpha,\ \gamma \geq 0,\ -\pi < \beta \leq \pi$	Parabolic plate; Plane with parabolic aperture
(10) Elliptic conal	$x = kr\mathrm{sn}\alpha\mathrm{sn}\beta;$ $y = (ik/k')r\mathrm{cn}\alpha\mathrm{cn}\beta,$ $z = (1/k')r\mathrm{dn}\alpha\mathrm{dn}\beta;$ $r \geq 0, \quad -2K < \alpha \leq 2K,$ $\beta = K + iu, \quad 0 \leq u \leq 2K'$	Plane sector; Including quarter plane
(11) Ellipsoidal	$x = k^2\ell\mathrm{sn}\alpha\mathrm{sn}\beta\mathrm{sn}\gamma,$ $y = (-K^2\ell/k')\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\,\mathrm{cn}\gamma,$ $z = (i\ell/k')\mathrm{dn}\alpha\,\mathrm{dn}\,\beta\mathrm{cn}\,\gamma;$ $\alpha,\ \beta$ as in (10), $\gamma = iK' + w, 0 < w \leq K$	Elliptic plate; Plane with elliptic aperture

Laplace & Helmholtz

Coordinate system	Laplace Equation	Helmholtz equation
(1) Cartesian	(Trivial)	(Trivial)
(2) Cylinderical	(Trivial)	Bessel
(3) Spherical polar	Associated Legender	Associated Legender
(4) Parabolic cylinder	(Trivial)	Weber
(5) Elliptic cylinder	(Trivial)	Mathieu
(6) Rotation-paraboloidal	Bessel	Confluent hypergeometric
(7) Prolate spheroidal	Associated Legender	Spheroidal wave
(8) Prolate spheroidal	Associated Legender	Spheroidal wave
(9) Paraboloidal	Mathieu	Whittaker-Hill
(10) Elliptic conal	Lame	Spherical Bessel, Lame
(11) Ellipsoidal	Lame	Ellipsoidal

(1) **Associated Legendre**:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\Big\{n(n+1)-\frac{m^2}{(1-x^2)}\Big\}y=0$$

(2) **Bessel**:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

(3) **Spherical Bessel**:

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-n(n+1))y=0$$

(4) **Weber**:

$$\frac{d^2y}{dx^2}+(\lambda-\frac{1}{4}x^2)y=0$$

(5) **Confluent hypergeometric**:

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0$$

(6) **Mathieu**:

$$\frac{d^2y}{dx^2}+(\lambda-2q\cos 2x)y=0$$

(7) **Spheroidal wave**:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\Big\{\lambda-\frac{\mu^2}{(1-x^2)}+\gamma^2(1-x^2)\Big\}y=0$$

(8) **Lame**:

$$\frac{d^2y}{dx^2}+\big(h-n(n+1)k^2\mathrm{sn}^2x\big)y=0$$

(9) **Whittaker-Hill**:

$$\frac{d^2y}{dx^2}+(a+b\cos 2x+c\cos 4x)y=0$$

(10) **Ellipsoidal wave**:

$$\frac{d^2y}{dx^2}+\big(a+bk^2\,\mathrm{sn}^2x+qk^4\mathrm{sn}^4x\big)y=0$$

The $\mathrm{sn}\, x$ is called the *Jacobian sine elliptic function.*

8.5. **Sturm-Liouville Boundary Value Problems.** We assume that the function P in (8.1) does not vanish. As a result, we can rewrite the equation (8.1) into a *self-adjoint form*:

Lemma 8.2. *The equation*

$$P\Phi'' + R\Phi' + Q\Phi = -\lambda\Phi \tag{8.15}$$

can be written in the form

$$(p\Phi')' + q\Phi = -\lambda r\Phi,$$

where p, q and r are continuous functions of x on $[a, b]$, p is positive and has a continuous derivative, and r is positive.

Proof. Multiply the equation (8.15) on both sides by r. Then we require:

$$\begin{aligned} rP\Phi'' + rR\Phi' + rQ\Phi &= -\lambda r\Phi, \\ p\Phi'' + p'\Phi' + q\Phi &= -\lambda r\Phi \end{aligned}$$

to be the same. So it is sufficient that we require

$$\begin{aligned} & p' = rR \qquad \text{and} \qquad p = rP, \\ \text{that is, if} \qquad & \frac{p'}{p} = \frac{R}{P} \qquad \text{and} \qquad r = \frac{p}{P}, \\ \text{or equivalently} \qquad & p = e^{\int \frac{R}{P}} \qquad \text{and} \qquad r = \frac{1}{P}e^{\int \frac{R}{P}} \end{aligned}$$

which is well-defined since P is positive, and the conclusion for p, q and r thus follow. ☐

Lemma 8.3. *Let*

$$L(\Phi) = \frac{d}{dx}\Big(p\frac{d\Phi}{dx}\Big) + q\,\Phi. \tag{8.16}$$

Then for any two twice differentiable functions Φ and Ψ, we have

$$\Phi L(\Psi) - \Psi L(\Phi) = \frac{d}{dx}\big(p(\Phi\Psi' - \Phi'\Psi)\big). \tag{8.17}$$

Proof. Direct verification. ☐

Lemma 8.4. *If Φ and Ψ satisfy the boundary condition*

$$\begin{aligned} \alpha\Phi(a) + \beta\Phi'(a) &= 0, \\ \gamma\Phi(b) + \delta\Phi'(b) &= 0 \end{aligned} \tag{8.18}$$

then

$$\Phi\Psi' - \Phi'\Psi\Big|_{x=a} = 0 = \Phi\Psi' - \Phi'\Psi\Big|_{x=b}. \tag{8.19}$$

Proof. The equations

$$\alpha\Phi(a) + \beta\Phi'(a) = 0,$$
$$\alpha\Psi(a) + \beta\Psi'(a) = 0$$

have non-trivial solutions for α and β if and only if

$$\begin{vmatrix} \Phi(a) & \Phi'(a) \\ \Psi(a) & \Psi'(a) \end{vmatrix} = 0.$$

Similar argument gives the condition at $x = b$. □

Lemma 8.5. *Let L be the Sturm-Loiuville operator given in (8.16) and*

$$L(\Phi) = -\lambda r\Phi,$$
$$L(\Psi) = -\lambda r\Psi$$

and both Φ and Ψ satisfy the same boundary condition in (8.18) at $x = a$ and b. Then Φ and Ψ are orthogonal on $[a,\, b]$ with respect to the function r (called the orthogonal weight function).

Proof. Since

$$\Psi\, L(\Phi) - \Phi\, L(\Psi) = (\mu - \lambda)\, r\, \Phi\, \Psi.$$

But (8.17) gives

$$\begin{aligned} \frac{d}{dx}\, p\,(\Phi\Psi' - \Phi'\Psi) &= \Psi\, L(\Phi) - \Phi\, L(\Psi) \\ &= (\mu - \lambda)\, r\, \Phi\, \Psi. \end{aligned}$$

The (8.19) now gives

$$0 = p(\Phi\Psi' - \Phi'\Psi)\Big|_{x=a}^{x=b} = (\mu - \lambda) \int_a^b r\, \Phi\, \Psi\; dx.$$

This proves that Φ and Ψ are orthogonal with respect to the weight function r over $[a,\, b]$. □

Lemma 8.6. *Let L be the Sturm-Loiuville operator given in (8.16) and*

$$L(\Phi) = -\lambda\, r\, \Phi.$$

Then λ must be real.

Proof. For suppose λ is complex with $\lambda = \mu + i\nu$, $\nu \neq 0$ and $\Phi = \phi + i\psi$. Then

$$\big(p(\phi' + i\psi')\big)' + q(\phi + i\nu) = -(\mu + i\nu)\, r\, (\phi + i\psi).$$

Taking complex conjugate of this equation gives

$$\big(p(\phi' - i\psi')\big)' + q(\phi - i\nu) = -(\mu - i\nu)\, r\, (\phi - i\psi).$$

which implies that $\phi - i\nu = \overline{\Phi}$ is an eigenvector and $\overline{\lambda} = \mu - i\nu$ is the corresponding eigenvalue. If we now follow the argument used in Lemma 8.5, then we obtain

$$\int_a^b r\, \Phi\, \overline{\Phi}\; dx = \int_a^b r\, (\phi^2 + \psi^2)\; dx > 0$$

contradicting that Φ and $\overline{\Phi}$ are orthogonal. $\square$

Theorem 8.7. *If $r > 0$, $q \leq 0$ and if the boundary conditions imply that*

$$p\, \Phi\, \Phi' \,\Big|_a^b \leq 0, \tag{8.20}$$

then all the eigenvalues of the boundary value problem for

$$(p\Phi')' + q\, \Phi = -\lambda\, r\, \Phi$$

are non-negative.

Proof. We multiply both sides of

$$(p\Phi')' + q\Phi = -\lambda r \Phi.$$

by Φ and integrate both sides of the resulted equation over $(a,\, b)$ to get

$$p\, \Phi'\, \Phi\, \Big|_{x=a}^{x=b} - \int_a^b p\, \Phi'^2\, dx + \int_a^b q\, \Phi^2\; dx = -\lambda \int_a^b r\, \Phi^2\; dx.$$

It follows from the hypothesis that $\lambda \geq 0$. In addition, $\lambda = 0$ only if $q \equiv 0$, $\Phi' \equiv 0$ over $[a,\, b]$. That is, if Φ is a constant eigenvector. $\square$

Remark. The assumption (8.20) includes

(1) $\Phi(a) = 0 = \Phi(b)$,
(2) $\Phi'(a) = 0 = \Phi'(b)$,
(3) $\Phi'(a) - h\Phi(a) = 0$, $\Phi'(b) + H\Phi(b) = 0$, where h and H are non-negative constants.

8.6. Existence of eigenvalues.

Theorem 8.8. *The Sturm-Liouville boundary value problem*

$$\begin{aligned}(p\Phi')' + q\Phi &= -\lambda r\Phi,\\ \alpha\Phi(a) + \beta\Phi'(a) &= 0,\\ \gamma\Phi(b) + \delta\Phi'(b) &= 0\end{aligned}$$

where p, q, r are continuous functions of x on $[a, b]$, p and r are positive and p' is continuous, has infinitely many eigenfunction solutions and the corresponding eigenvalues λ_n, such that $\lambda_1 < \lambda_2 < \dots$, and $\lambda_n \to \infty$ as $n \to +\infty$.

Moreover, each eigenfunction corresponding to its eigenvalue, λ_n, say, has exactly $n-1$ zeros in the open interval (a, b).

The proof of the above theorem, which depends on Green's functions, is beyond the scope of this course. Interested students can consult Chapter 10 of Folland's book.

Remark. The above problem is commonly called the *regular Sturm-Liouville* boundary value problems. The *singular Sturm-Liouville* boundary value problems may include situation where the function p may vanish at one or both endpoints of $[a, b]$, the weight $r(x)$ may vanish or be unbounded at one or both endpoints of $[a, b]$. Besides, the interval $[a, b]$ may be unbounded, that is, $a = -\infty$ and/or $b = +\infty$.

We consider some examples that illustrate the above theorem. The examples are taken from Folland pages 91–93.

Example. We are given the differential equation

$$y''(x) + \lambda y(x) = 0,$$

with boundary condition

$$\begin{aligned}\alpha y(0) - y'(0) &= 0,\\ \gamma y(\ell) - y'(\ell) &= 0.\end{aligned} \tag{8.21}$$

Suppose $\lambda = 0$. Then the solution to the differential equation $y''(x) = 0$ is $y = cx + d$. The boundary condition at $x = 0$ and ℓ gives

$$\alpha d = c, \quad \text{and} \quad \gamma(c\ell + d) = c$$

respectively. Thus, $\gamma = \alpha/(\alpha\ell + 1)$. We may therefore choose $y = x + \alpha$. Suppose now that $\lambda \neq 0$, then the Lemma 8.6 asserts that λ must be real. Thus, it remains to consider $\lambda = \nu^2$ for some real $\nu > 0$ or $\lambda = (i\mu)^2 = -\mu^2$ for some real $\mu > 0$. Suppose $\lambda = \nu^2$. The boundary condition (8.21) implies, without loss of generality, that the general solution can be written as

$$\begin{aligned}y(x) &= c\cos\nu x + d\sin\nu x\\ &= \nu\cos\nu\, x + \alpha\sin\nu\, x\end{aligned}$$

since $\alpha c = \alpha y(0) = y'(0) = \nu d$. Thus $d = c\alpha/\nu$. Hence we have discard the c above. But then the boundary condition (8.21) at $x = \ell$ implies that

$$-\nu^2 \sin \nu\ell + \alpha\nu \cos \nu\ell = \beta(\nu \cos \nu\ell + \alpha \sin \nu\ell),$$

or

$$\tan \nu\ell = \frac{(\alpha - \beta)\nu}{\alpha\beta + \nu^2}.$$

On the other hand, if $\lambda = -\nu^2$ or $\nu = i\mu$, and noting that $\tan ix = i \tanh x$, so that the above equation would become

$$\tan \mu\ell = \frac{(\alpha - \beta)\mu}{\alpha\beta - \mu^2}$$

instead. It is clear that the above equations for ν or μ does not admit nice closed form solutions unless $\alpha = \beta$.

Case I: $\alpha = 1$, $\beta = -1$, and $\ell = \pi$. We plot the curves of $\tan \pi\nu$ and $\dfrac{2\nu}{\nu^2 - 1}$ respectively. The graph shows that there are infinitely many positive solutions ν_n increasing to infinity. In fact, the graphical method shows that

$$\lambda_n = \nu_n^2 \approx (n-1)^2,$$

for large n. There is no intersection for the second case when $\lambda < 0$. The corresponding eigen-functions are

$$y_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x.$$

Case II: $\alpha = 1$, $\beta = 4$, and $\ell = \pi$. Then we plot the curves of

$$\tan \pi\nu = \frac{-3\nu}{4 + \nu^2}.$$

It turns out that apart from $\lambda_n = \nu^2 \approx n^2$ for positive n, the equation, for $\lambda = -\mu^2$,

$$\tan \pi\nu = \frac{3\nu}{\mu^2 - 4}.$$

admit one positive solution ν_0. Thus, the eigen-functions are

$$y_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x, \quad n \geq 1,$$

and

$$y_0(x) = \mu_0 \cosh \mu_0 x + \sinh \mu_0 x.$$

8.7. **Eigen-function expansions.** We quote without proof the following results.

Theorem 8.9. *Let f be continuous on $[a,\, b]$ with piecewise smooth f' such that it satisfies the Sturm-Liouville boundary value problem with boundary condition*

$$\begin{aligned} \alpha f(a) + \beta f'(a) &= 0, \\ \gamma f(b) + \delta f'(b) &= 0. \end{aligned}$$

Then the Fourier series of f with respect to the eigenfunctions, $\{\Phi_n\}$, that is,

$$f(x) \sim c_0\Phi_0(x) + c_1\Phi_1(x) + \dots$$

and

$$\int_a^b r\Phi_n^2(x)\ dx = 1, \qquad c_n = \int_a^b rf(x)\Phi_n(x)\ dx,$$

$n = 0, 1, 2, \dots,$ *converges to $f(x)$ absolutely and uniformly.*

Moreover

Theorem 8.10. *If f is only a piecewise smooth on $[a,\, b]$ instead in the last theorem, then the Fourier series of f with respect to the eigenfunctions $\{\Phi_n\}$ converges for $a < x < b$ to the value of $f(x)$ at every point of continuity, and to the value*

$$\frac{f(x+0) + f(x-0)}{2}$$

at every point of jump discontinuity.

As for the completeness of the system $\{\Phi_n\}$ for a square integrable function $f(x)$ over $[a,\, b]$, we quote

Theorem 8.11. *Let $\{\Phi_n\}$ be the orthogonal eigenfunctions derived from the Sturm-Liouville boundary value problem above. Then*

$$\int_a^b r(x)f(x)^2\ dx = \sum_{k=0}^{\infty} c_k^2\|\sqrt{r}\Phi_k(x)\|^2 = \sum_{k=0}^{\infty} c_k^2,$$

holds for every square integrable function $f(x)$ over $[a,\, b]$. That is, the system $\{\Phi_n\}$ is complete with respect to the weight function $r(x)$.

8.8. **Solutions to PDEs.**

Theorem 8.12. *Let $u(x,\, t)$, be a continuous solution of*

$$P\, u_{xx} + R\, u_x + Q\, u = u_{tt}$$

in $a \le x \le b$, $t \ge 0$ with $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ bounded on $[a,\, b] \times [0,\, t_0]$ for every t_0, and it satisfies the boundary value problem

$$\alpha u(a,t) + \beta u_x(a,t) = 0, \qquad \gamma u(b,t) + \delta u_x(b,t) = 0$$

and the initial condition

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Then

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\, \Phi_n(x),$$

where $\{\Phi_n\}$ are eigenfunctions associated with the boundary value problem. The function T_n can be found from solving the second initial conditions

$$T_n'' + \lambda_n T_n = 0, \qquad n = 0,\, 1,\, 2,\, \ldots \tag{8.22}$$

subject to

$$T_n(0) = C_n, \qquad T_n'(0) = c_n, \qquad n = 0,\, 1,\, 2,\, \ldots$$

where C_n, c_n are, respectively, the Fourier series coefficients of $f(x)$ and $g(x)$.

Remark. We note that the above result does not claim that the expansion for $u(x,\, t)$ necessarily be a solution to the boundary value problem, even if the Fourier series for $f(x)$ and $g(x)$ (for them to be sufficiently smooth to) converge. This is because the series for $u(x,\, t)$ would need to converge uniformly after being differentiated with respect to x twice.

Proof. We multiply both sides of the PDE by

$$r = \frac{1}{P}\, e^{\int_{x_0}^{x} \frac{R}{P}\, dx} = \frac{p}{P},$$

Then we can re-write the above PDE into the form

$$p\, \frac{\partial^2 u}{\partial x^2} + p'\, \frac{\partial u}{\partial x} + q\, u = r\, \frac{\partial^2 u}{\partial t^2},$$

which is in a "self-adjoint form":

$$L(u) = \frac{\partial}{\partial x}\Big(p \frac{\partial u}{\partial x}\Big) + q\, u = r\, \frac{\partial^2 u}{\partial t^2},$$

so that we can apply the Lemma 8.3 to write

$$L(u) = r\,\frac{\partial^2 u}{\partial t^2} \tag{8.23}$$

where we also have

$$L(\Phi_n) = -\lambda_n\, r\,\Phi_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots \tag{8.24}$$

We apply the Theorem 8.10 that for x in $[a,\, b]$ and for every (fixed) $t \geq 0$, the $u(x,\, t)$ can be expanded in a Fourier series in terms of the eigenfunction solutions $\{\phi_n\}$ of the form

$$u(x,\, t) = T_0(t)\Phi_0(x) + T_1(t)\Phi_1(x) + \cdots = \sum_{n=0}^{\infty} T_n(t)\,\Phi_n(x)$$

where the Fourier coefficients T_n (with t fixed) for $u(x,\, t)$ (t fixed) are given by

$$T_n(t) = \int_a^b u(x,\, t)\,\Phi_n(x)\,r(x)\,dx, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots. \tag{8.25}$$

But then the (8.24) implies that

$$r\,\Phi_n(x) = -\frac{1}{\lambda_n}\,L(\Phi_n),$$

and so

$$T_n(t) = \int_a^b u(x,\, t)\left(-\frac{1}{\lambda_n}\,L(\Phi_n)\right)\,dx = -\frac{1}{\lambda_n}\int_a^b u(x,\, t)\;L(\Phi_n)\;dx$$

and the Lagrange identity

$$uL(\Phi) - \Phi L(u) = \frac{d}{dx}\big(p(u\Phi' - \Phi\, u')\big)$$

yields

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \Phi_n(t)\,L(u(x,\, t))\;\,dx + \frac{1}{\lambda_n}\Big[p\Big(\Phi_n\frac{\partial u}{\partial x} - \Phi_n'\, u\Big)\Big]_a^b$$

for which the second term vanishes according to the Lemma 8.4. Thus

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \Phi_n(t)\,L(u(x,\, t))\;\,dx = -\frac{1}{\lambda_n}\int_a^b \frac{\partial^2 u}{\partial t^2}\,\Phi_n(x)\,r\;\,dx.$$

Let us differentiate the (8.25) with respect to t twice yields

$$T_n''(t) = \int_a^b \frac{\partial^2 u(x,\, t)}{\partial t^2}\,\Phi_n(x)\,r(x)\,dx,$$

which is identical to the equation (8.22). The differentiation under the integral sigh with respect to t is justified because of the assumption of the boundedness of the first and second partial derivatives with respect to t. This establishes the equation (8.22). On the other hand, since $u(x,\, t)$ is continuous, so

$$\lim_{t\to 0} T_n(t) = \lim_{t\to 0} \int_a^b u(x,\, t)\, \Phi_n(x)\, r(x)\, dx = \int_a^b f(x)\, \Phi_n(x)\, r(x)\, dx := C_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots .$$

where C_n is the Fourier coefficient of $f(x)$ (with respect to the system $\{\Phi_n\}$). But the $T_n(t)$ is continuous, so

$$T_n(0) = C_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots$$

as required. Similarly, we have

$$T_n'(0) = c_n, \qquad n = 0,\, 1,\, 2,\, 3,\, \cdots$$

where the c_n are the Fourier coefficients for the $g(x)$. □

A problem of working with a general orthogonal (orthonormal) system $\{\Phi_n\}$ for a boundary value problem is to make sure that the functions $f(x)$ and $g(x)$ in initial condition

$$u(x,\, 0) = f(x), \qquad \frac{\partial f}{\partial t} = g(x),$$

can be expanded in as Fourier series in terms of $\{\Phi_n\}$. That is, suppose we have

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi_k(x),$$

and

$$g(x) = \sum_{k=0}^{\infty} c_k \Phi_k(x),$$

then they need to match with the earlier

$$f(x) = u(x, 0) = \sum_{k=0}^{\infty} A_k \sin\Big(\frac{\pi k x}{l}\Big)$$

$$\text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x, 0) = \sum_{k=0}^{\infty} B_k \Big(\frac{a\pi k}{l}\Big) \sin\Big(\frac{\pi k x}{l}\Big)$$

where

$$A_k = \frac{2}{l} \int_0^l f(x)\, \sin \frac{\pi k x}{l}\, dx,$$

$$B_k = \frac{2}{a\pi k} \int_0^l g(x)\, \sin \frac{\pi k x}{l}\, dx.$$

That is, we require

$$A_k = C_k, \qquad B_k = c_k/\sqrt{\lambda_k}$$

for $k \geq 0$. So the above requirement would be true if both f and g are known to be "sufficiently smooth", which is rarely the case in reality. Then the problem is if the coefficients A_k and B_k decline sufficiently fast that guarantee the convergence as well as about term-by-term differentiation of the two $x-$derivatives of u. In any case, The above theorems show that if a physical problem has any solution at all, then its solution from the above separation of variables method and Sturm-Liouville method would lead to a solution. So we use the term *generalised solution* to the boundary value problem even if the solution found by the above does not satisfy some of the above requirements (of the boundary value problem). We call *exact solution* for a genuine (real) solution. We discuss below if the generalised solution is still of some use.

Theorem 8.13. *Let*

$$u(x,\, t) \sim \sum_{k=0}^{\infty} T_k(t)\Phi_k(x)$$

be either the exact or the generalised solution of equation (8.1), that satisfies the boundary condition (8.3) and initial condition (8.4). Suppose $f_m(x)$ and $g_m(x)$ converge to $f(x)$ and $g(x)$ respectively, in the mean, as $m \to \infty$, that is,

$$\lim_{m\to\infty} \int_a^b [f_m(x) - f(x)]^2\, r\ dx = 0 = \lim_{m\to\infty} \int_a^b [g_m(x) - g(x)]^2\, r\ dx. \tag{8.26}$$

Suppose that

$$u_m(x,\, t) = \sum_{k=0}^{\infty} T_{m\,n}(t)\, \Phi_n(x)$$

is either the exact or the generalised solution of equation (8.1), that satisfies the boundary condition (8.3) and new initial condition

$$u_m(x,\, t) = f_m(x), \qquad \frac{\partial u_m(x,\, 0)}{\partial t} = g_m(x), \qquad m \geq 0$$

then $u_m(x,\, t)$ converges to $u(x,\, t)$ in the mean, as $m \to \infty$.

Proof. So the idea is to compare the Fourier coefficients of the $T_n(x)$ and $T_{n\,m}(x)$. Recall that the C_n, c_n are respectively, the Fourier coefficients of $f(x)$ and $g(x)$ for the boundary value problem

$$T_n'' + \lambda_n\, T_n = 0, \qquad T_n(0) = C_n, \quad T_n'(0) = c_n; \qquad n \geq 0$$

and the $C_{n\,m}$, $c_{n,\,m}$ are respectively, the Fourier coefficients of $f(x)$ and $g(x)$ for the boundary value problem

$$T_{m\,n}'' + \lambda_n T_{m\,n} = 0, \qquad T_{m\,n}(0) = C_{m\,n}, \quad T_{m\,n}'(0) = c_{m\,n}; \qquad n \geq 0$$

for each $m \geq 0$. We also know that

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \qquad \lim_{n\to\infty} \lambda_n = +\infty.$$

and that all perhaps except a finite number of them can be negative. Suppose $\lambda_n \leq 0$ when $n \leq N$ and $\lambda_n > 0$ for $n > N$. Then,

$$T_n(x) = \begin{cases} \frac{1}{2}\Big(C_n + \dfrac{c_n}{\sqrt{-\lambda_n}}\Big)\, e^{\sqrt{-\lambda_n} t} + \dfrac{1}{2}\Big(C_n - \dfrac{c_n}{\sqrt{-\lambda_n}}\Big)\, e^{-\sqrt{-\lambda_n} t}, & n \leq N \\ C_n \cos\sqrt{\lambda_n} t + \dfrac{c_n}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n} t, & n > N. \end{cases} \tag{8.27}$$

and similarly,

$$T_{m\,n}(x) = \begin{cases} \frac{1}{2}\Big(C_{m\,n} + \dfrac{c_{m\,n}}{\sqrt{-\lambda_n}}\Big)\, e^{\sqrt{-\lambda_n} t} + \dfrac{1}{2}\Big(C_{m\,n} - \dfrac{c_{m\,n}}{\sqrt{-\lambda_n}}\Big)\, e^{-\sqrt{-\lambda_n} t}, & n \leq N \\ C_{m\,n} \cos\sqrt{\lambda_n} t + \dfrac{c_{m\,n}}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n} t, & n > N. \end{cases} \tag{8.28}$$

Notice that the assumption (8.26) implies that

$$0 = \lim_{m\to\infty} \int_a^b [f_m(x) - f(x)]^2\, r\, dx = \lim_{m\to\infty} \sum_{n=0}^{\infty} (C_n - C_{m\,n})^2 \tag{8.29}$$

and

$$0 = \lim_{m\to\infty} \int_a^b [g_m(x) - g(x)]^2\, r\, dx = \lim_{m\to\infty} \sum_{n=0}^{\infty} (c_n - c_{m\,n})^2, \tag{8.30}$$

so that

$$\lim_{m\to\infty} C_{m\,n} = C_n, \qquad \lim_{m\to\infty} c_{m\,n} = c_n.$$

We easily deduce from above that

$$\lim_{m\to\infty} (T_{m\,n} - T_n) = 0, \qquad n \geq 0. \tag{8.31}$$

We actually know more when $n > N$:

$$\begin{aligned} (T_{m\,n} - T_n)^2 &= \Big[(C_n - C_{m\,n}) \cos\sqrt{\lambda_n} t + \frac{(c_n - c_{m\,n})}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n} t\Big]^2 \\ &\leq 2\Big[(C_m - C_{n\,n})^2 + \Big(\frac{(c_n - c_{m\,n})}{\sqrt{\lambda_n}}\Big)^2\Big] \end{aligned} \tag{8.32}$$

But then we deduce from (8.29), (8.30), (8.32) and (8.31) that

$$\begin{aligned}\int_a^b [u_m(x,\,t)-u(x,\,t)]^2\, r\ dx &= \int_a^b \sum_{n=0}^{\infty}(T_{m\,n}(t)-T_n(t))^2\Phi_n(x)^2\, r\ dx\\ &= \sum_{n=0}^{\infty}(T_n-T_{m\,n})^2\int_a^b \Phi_n(x)^2\, r\ dx\\ &= \sum_{n=0}^{\infty}(T_n-T_{m\,n})^2\cdot 1\\ &= \sum_{n=0}^{N}(T_n-T_{m\,n})^2+\sum_{n=N+1}^{\infty}(T_n-T_{m\,n})^2\\ &\longrightarrow 0,\end{aligned}$$

as $m\to\infty$, which is what we desire to prove. □

Remark. Suppose we choose f_m and g_m above as the $m-$partial sums of the $f(x)$ and $g(x)$ respectively, then the solutions u_m are *exact solutions* to the (8.1) subject to the conditions (8.3) and the corresponding (8.4). Thus, if $u(x,\,t)$ is either the exact or generalised solutions to (8.1) subject to the (8.3) and (8.4) is the limit of $u_m(x,\,t)$, as $f_m\to f$ and $g_m\to g$ either uniformly or in the mean.

8.9. **Remarks on Forced vibrations.** We have instead

$$P\frac{\partial^2 u}{\partial x^2}+R\frac{\partial u}{\partial x}+Qu=\frac{\partial^2 u}{\partial t^2}+F(x,\,t), \tag{8.33}$$

subject to the same (8.3) and (8.4), where the $F(x,\,t)$ stands for an external force. The same technique would lead to

$$L(u)=r\frac{\partial^2 u}{\partial t^2}+rF(x,t).$$

Then the Sturm-Liouville problem would give

$$L(\Phi)=-\lambda r\,\Phi.$$

We also have

$$T_n(t)=\int_a^b u(x,\,t)\,\Phi_n(x)\,r(x)\,dx,\qquad n=0,\,1,\,2,\,3,\,\cdots. \tag{8.34}$$

A similar argument used earlier give

$$T_n(t)=-\frac{1}{\lambda_n}\int_a^b \Phi_n(t)\,L(u(x,\,t))\ dx=-\frac{1}{\lambda_n}\int_a^b \frac{\partial^2 u}{\partial t^2}\,\Phi_n(x)\,r\ \ dx-\frac{1}{\lambda_n}\int_a^b r\,F(x,\,t)\,\Phi_n(x)\ dx$$

instead. Suppose

$$F(x,\, t) = \sum_{k=0}^{\infty} F_k(t)\Phi_k(x)$$

where

$$F_k(t) = \sum_{k=0}^{\infty} \int_a^b r\, F_k(x,\, t)\Phi_k(x)\ dx, \qquad n = 0,\, 1,\, 2, \cdots,$$

and this gives

$$T_k = -\frac{1}{\lambda_k} T_k'' - \frac{1}{\lambda_k} F_k,$$

which is

$$T_k'' + \lambda_k T_k + F_k = 0, \qquad n = 0,\, 1,\, 2,\, 3, \cdots.$$

8.10. **Remarks on Heat equations.** Recall that the heat equation assumes the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a = K/c\rho,\ (K = \text{thermal conductivity},\ c = \text{heat capacity},\ \rho = \text{density})$$

subject to the boundary condition (8.3) but initial condition becomes only the

$$u(x,\, 0) = f(x).$$

So we try

$$u(x,\, t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x), \tag{8.35}$$

where the Φ_k proceed as before, while the T_k satisfies

$$T_k'(t) + \lambda_k T_k(t) = 0, \quad T_k(0) = C_n, \qquad k = 0,\, 1,\, 2, \cdots$$

and the C_k are the Fourier coefficients of $f(x)$. Since the solution here is given by

$$T_k(t) = C_k\, e^{-\lambda_k t}, \qquad k \geq 0,$$

so that any generalised solution for the heat equation would become exact since the convergence of the series (8.38) is absolute and uniform and so can be differentiated any number of times.

Application to PDEs.

Completing the free vibrating string. We continue with our discussion of the vibrating string equation (§8.3) started at the beginning of this chapter:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad a^2 = \frac{T}{\rho}$$

the boundary and initial conditions are given, respectively, by

$$u(0,t) = 0 = u(l,t), \qquad t \geq 0$$
$$\text{and} \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where f and g are continuous functions and vanish for $x = 0,\, l$. The orthogonal system is given by $\{\Phi_n\}$:

$$\Phi = \Phi_n(x), \qquad \lambda = \lambda_n, \qquad n = 0, 1, 2, \ldots$$

with the corresponding eigenvalues $\lambda_n \to +\infty$. The T, then

$$T = T_n(t) = A_n \cos(\sqrt{\lambda_n t}) + B_n \sin(\sqrt{\lambda_n})t,$$

$n = 0,\, 1,\, 2,\, \ldots$. Since the PDE is linear, the superposition principle gives

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} T_k(t)\Phi_k(x) \tag{8.36}$$

$$= \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{a\pi k t}{l}\right) + B_k \sin\left(\frac{a\pi k t}{l}\right)\right] \sin\left(\frac{\pi k x}{l}\right). \tag{8.37}$$

We also recall that

$$f(x) = u(x,0) = \sum_{k=0}^{\infty} A_k \sin\left(\frac{\pi k x}{l}\right)$$
$$\text{and} \qquad g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} B_k \left(\frac{a\pi k}{l}\right) \sin\left(\frac{\pi k x}{l}\right)$$

are just the Fourier series of f and g with respect to $\{\sin \frac{\pi k x}{l}\}$. Thus

$$A_k = \frac{2}{l} \int_0^l f(x) \sin \frac{\pi k x}{l}\, dx,$$
$$B_k = \frac{2}{a\pi k} \int_0^l g(x) \sin \frac{\pi k x}{l}\, dx.$$

We note that each u_k can be rewritten into the compact form

$$u_k(x,\,t) = H_n \,\sin\left(\frac{a\pi kt}{\ell} + \alpha_k\right)\,\sin\frac{\pi nx}{\ell}$$

where

$$H_k = \sqrt{A_k^2 + B_k^2}, \quad \sin\alpha_k = \frac{A_k}{H_k}, \quad \cos\alpha_k = \frac{B_k}{H_k}.$$

We see that the *amplitude* of each u_k is given by

$$\max_t |u_k(x,\,t)| = H_k\left|\sin\frac{\pi nx}{\ell}\right|.$$

Therefore the amplitude depends only on the x. This is called a *standing wave.* In particular, we see that the string *remains fixed* at the positions

$$x = 0,\ \frac{\ell}{k},\ \frac{2\ell}{k},\ \frac{3\ell}{k},\ \cdots,\ \frac{(k-1)\ell}{k},\ \ell$$

at *all time.* These points are called *nodes.* As a result, the nodes divide the segment of length ℓ into k smaller segments each with both end points fixed. Because of the sign change of the sine, two adjacent segments are on opposite sides from the horizontal position with the largest displacements occurs at the mid-point of each sub-segment. These points of largest displacements causing the largest amplitudes are called the *anti-nodes.*

The total effect of the vibration of the strings is given by the superimposed solution (8.36). The *fundamental mode*

$$\omega_1 = \frac{a\pi}{\ell} = \frac{\pi}{\ell}\sqrt{\frac{T}{\rho}},$$

with its corresponding *period*

$$\tau_1 = \frac{2\pi}{\omega_1} = 2\ell\sqrt{\frac{\rho}{T}},$$

while the other vibration modes of the string, i.e., the *harmonics* (or *overtones*) are the subsequent frequencies

$$\omega_k = \frac{a\pi k}{\ell} = \frac{\pi k}{\ell}\sqrt{\frac{T}{\rho}},$$

with its corresponding *periods*

$$\tau_k = \frac{2\pi}{\omega_k} = \frac{2\ell}{k}\sqrt{\frac{\rho}{T}},$$

which gives the “background timber or colour” of the sound.

Remark. If we hold fix the mid-point ($x = \ell/2$) of the string, then the fundamental mode, as well as the subsequent modes of odd frequencies disappear (since they cannot move) leaving only the even frequencies with periods divisible by two, that are, the even frequencies. The fundamental mode becomes the first even frequency with period $\ell/2$ from the original series. Similarly, one can kill off the frequencies which are multiples of three, etc.

Heat (diffusion) equations with both end points of the rod held at zero temperature. We consider heat flow in a heat conducting rod of length ℓ made with uniform material and is insulated on the round surface throughout, and subject to temperature maintained at zero degree at both ends. That is, let $u(x,\,t)$ be the temperature distribution function of the rod at the position x and at time t. We solve

$$\frac{\partial u}{\partial t} = a^2\,\frac{\partial^2 u}{\partial x^2}, \qquad a^2 = K/c\rho \tag{8.38}$$

where c is the *heat capacity*, ρ is the *density* (per unit length) of the rod, and K is the *thermal conductivity proportionality constant* of the rod material, is subject to the boundary condition

$$u(0,\,t) = 0 = u(\ell,\,t) \tag{8.39}$$

and the initial condition (initial temperature distribution)

$$u(x,\,0) = f(x) \tag{8.40}$$

We substitute the following proposed separation of variables form solution

$$u(x,\,t) = \Phi(x)\,T(t)$$

into (8.38) and after dividing the resulting equation by $\Phi(x)\,T(t)$, we obtain

$$\frac{\Phi''}{\Phi} = \frac{T'}{a^2\,T} = -\lambda^2$$

where the right-hand side is a constant. We deduce

$$\Phi''(x) + \lambda^2\Phi(x) = 0, \tag{8.41}$$

and

$$T'(t) + a^2\lambda^2\,T(t) = 0. \tag{8.42}$$

We solve the equation (8.41) together with the boundary condition

$$\Phi(0) = 0 = \Phi(\ell)$$

since $T(t)$ cannot be identically zero for all t. This implies that

$$\begin{aligned}\Phi(x) &= C_1 \cos\lambda x + C_2 \sin\lambda x \\ &= C_2 \sin\lambda_k x\end{aligned}$$

where $\lambda_n = \pi k/\ell$, for $k = 1, 2, 3, \cdots$ so that

$$\Phi_k(x) = \sin\Big(\frac{\pi k x}{\ell}\Big), \qquad k = 1, 2, 3, \cdots$$

As a result, we deduce that

$$T_k(t) = A_k\, e^{-a^2\lambda_k^2 t} = A_k \exp\Big(-\frac{a^2\pi^2k^2}{\ell^2}\,t\Big), \qquad k = 1, 2, 3, \cdots$$

Hence

$$u(x,\, t) = \sum_{k=1}^{\infty} A_k \sin\Big(\frac{\pi k x}{\ell}\Big)\, e^{-\frac{a^2\pi^2k^2}{\ell^2}t}. \tag{8.43}$$

We now require the $u(x,\, t)$ to satisfy the initial condition, that is

$$U(x,\ 0) = \sum_{k=1}^{\infty} A_k \sin\Big(\frac{\pi k x}{\ell}\Big) = f(x)$$

meaning that

$$A_k = \frac{1}{\ell}\int_0^{\ell} f(x) \sin\Big(\frac{\pi k x}{\ell}\Big)\, dx, \qquad k = 1, 2, 3, \cdots$$

Because of the exponential factor for T_k, the series (8.43) so obtained always converge uniformly in x for every fixed t. Moreover, the same also applies to the $x-$derivatives of $u(x,\, t)$. Hence the series satisfies the (8.38). It only remains to justify the convergence of the Fourier series to $f(x)$.

Heat (diffusion) equations with both end points of the rod held at constant temperatures. That is, everything is the same as in the previous case with the exception that both ends of the rod are held at constant, but otherwise arbitrary, temperatures. So the boundary condition (8.39) becomes

$$u(0,\, t) = A, \qquad u(\ell,\, t) = B \tag{8.44}$$

where A and B are arbitrary. The initial condition $u(x,\, 0) = f(x)$ remains the same. We also recall that

$$u(x,\, t) = \sum_{k=1}^{\infty} T_k(t)\, \sin\Big(\frac{\pi k x}{\ell}\Big), \tag{8.45}$$

but

$$T_k(t) = \frac{2}{\ell}\int_0^{\ell} u(x,\, t)\ \sin\Big(\frac{\pi k x}{\ell}\Big)\, dx. \tag{8.46}$$

Integration of (8.46) by parts twice yields

$$
\begin{aligned}
T_n(t) &= \frac{2}{\ell}\left\{ -\frac{\ell}{\pi k}\cos\left(\frac{\pi k x}{\ell}\right)u\Big|_0^\ell + \left(\frac{\ell}{\pi k}\right)^2 \sin\left(\frac{\pi k x}{\ell}\right) u_x\Big|_0^\ell - \left(\frac{\ell}{\pi k}\right)^2 \int_0^\ell \sin\left(\frac{\pi k x}{\ell}\right) u_{xx}\,dx \right\} \\
&= \frac{2}{\pi k}\big(A + (-1)^{k+1}B\big) + 0 - \frac{2\ell}{a^2\pi^2k^2}\int_0^\ell \sin\left(\frac{\pi k x}{\ell}\right) u_t\,dx. \qquad (8.47)
\end{aligned}
$$

Differentiating (8.46) with respect to t yields directly

$$T_k'(t) = \frac{2}{\ell}\int_0^\ell \sin\left(\frac{\pi k x}{\ell}\right) u_t\,dx.$$

We deduce from (8.47) that

$$T_k(t) = \frac{2}{\pi k}\big(A + (-1)^{k+1}B\big) - \frac{\ell^2}{a^2\pi^2k^2}\,T_k'(t).$$

One can easily check that the solution of this non-homogeneous first-order differential equation

$$T_k'(t) + \frac{a^2\pi^2k^2}{\ell^2}\,T_k(t) = \frac{2a^2\pi k}{\ell^2}\big(A + (-1)^{k+1}B\big).$$

is given by

$$T_k(t) = A_k \exp\left(-\frac{a^2\pi^2k^2}{\ell^2}\,t\right) + \frac{2}{\pi k}\big(A + (-1)^{k+1}B\big).$$

We deduce from the initial condition that

$$f(x) = u(x,\,0) = \sum_{k=1}^\infty T_k(0)\,\sin\left(\frac{\pi k x}{\ell}\right).$$

But then

$$\frac{2}{\ell}\int_0^\ell u(x,\,0)\,\sin\left(\frac{\pi k x}{\ell}\right)\,dx = T_k(0) = A_k + \frac{2}{\pi k}\big(A + (-1)^{k+1}B\big)$$

or

$$A_k = \frac{2}{\ell}\int_0^\ell f(x)\,\sin\left(\frac{\pi k x}{\ell}\right)\,dx - \frac{2}{\pi k}\big(A + (-1)^{k+1}B\big).$$

Thus this gives a complete solution to the problem.

Heat (diffusion) equations with both end points of the rod are at variable temperatures. We assume the initial condition remains unchanged. Suppose the boundary values are given by

$$u(0,\, t) = \phi(t), \qquad u(\ell,\, 0) = \psi(t) \tag{8.48}$$

where the functions $\phi(t)$ and $\psi(t)$ are certain functions of t. We leave the reader to check that one has

$$T_k'(t) + \frac{a^2\pi^2k^2}{\ell^2}\, T_k(t) = \frac{2a^2\pi k}{\ell^2}\big(\phi(t) + (-1)^{k+1}\psi(t)\big).$$

This non-homogenerous first order differential equation is slightly more tedious to solve by the integration factor method. We have, skipping some details,

$$T_k(t) = A_k \exp\Big(-\frac{a^2\pi^2k^2}{\ell^2}\, t\Big) + \frac{2a^2\pi k}{\ell^2} \exp\Big(-\frac{a^2\pi^2k^2}{\ell^2}\, t\Big) \int_0^\ell \exp\Big(\frac{a^2\pi^2k^2}{\ell^2}\, t\Big)\big(\phi(t) + (-1)^{k+1}\psi(t)\big)\, dt.$$

The coefficients A_k can be calculated via the a similar procedure as in the "constant temperatures" case above.

Heat (diffusion) equations with both end points of the rod exchange heat with surrounding medium. Suppose u_0 is the temperature of the surrounding medium, and H is the *constant of emissivity* which measures the rate of heat exchanges between of both ends of the rod with the surrounding medium. Then the initial condition that $u(x,\, 0) = f(x)$ $(0 \le x \le \ell)$ remains unchanged, but the boundary condition becomes

$$\frac{\partial u}{\partial x} - h(u - u_0)\Big|_{x=0} = 0, \qquad \frac{\partial u}{\partial x} - h(u - u_0)\Big|_{x=\ell} = 0$$

where $h = H/K$. Without loss of generality, we may assume $u_0 = 0$. Hence we have

$$\frac{\partial u}{\partial x} - hu\Big|_{x=0} = 0, \qquad \frac{\partial u}{\partial x} - hu\Big|_{x=\ell} = 0 \tag{8.49}$$

Separation of variables method $(u(x,\, t) = \Phi(x)T(t))$ will lead to solving

$$\Phi''(x) = -\lambda\Phi(x), \qquad T'(t) = -a^2\lambda t(t)$$

with boundary condition

$$\Phi'(0) - h\,\Phi(0) = 0, \tag{8.50}$$

$$\Phi'(\ell) + h\,\Phi(\ell) = 0. \tag{8.51}$$

It follows from the Theorem 8.7 that all eigen-values λ_k are non-negative. So let us write

$$\Phi''(x) = -\lambda^2\Phi(x), \qquad T'(t) = -a^2\lambda^2 t(t). \tag{8.52}$$

Applying the boundary condition (8.50) to the solution

$$\Phi(x) = C_1 \cos\lambda x + C_2 \sin\lambda x,$$

yields

$$\begin{aligned}
&C_2\lambda - hC_1 = 0,\\
&\lambda(-C_1 \sin\lambda\ell + C_2\cos\lambda\ell) + h(C_1\cos\lambda\ell + C_2\sin\lambda\ell) = 0.
\end{aligned}$$

We deduce

$$\frac{C_1}{\lambda} = \frac{C_2}{h},$$

so that

$$\tan\lambda\ell = \frac{2\lambda h}{\lambda^2 - h^2}.$$

Without loss of generality again, we may assume that $C_1 = \lambda_k$ and $C_2 = h$ for positive λ_k. Besides, we may discard any zero eigenvalue since this would violate the boundary condition. It is clear that we obtain infinitely many positive λ_k and their corresponding eigenfunctions are given by

$$\Phi_k(x) = \lambda_k\cos\lambda_k x + h\sin\lambda_k x, \qquad k = 1,\, 2,\, 3, \cdots.$$

Thus,

$$T_k(t) = A_k \exp\big(-a^2\lambda_k^2 t\big), \qquad k = 1,\, 2,\, 3, \cdots,$$

and

$$u(x,\, t) = \sum_{k=1}^{\infty} A_k(\lambda_k\cos\lambda_k x + h\sin\lambda_k x)\, \exp\big(-a^2\lambda_k^2 t\big).$$

The initial condition implies

$$u(x) = \sum_{k=1}^{\infty} A_k(\lambda_k\cos\lambda_k x + h\sin\lambda_k x)$$

and so

$$A_k = \frac{\displaystyle\int_0^\ell f(x)\,\Phi_k(x)\; dx}{\displaystyle\int_0^\ell \Phi_k^2(x)\; dx}, \qquad k = 1,\, 2,\, 3, \cdots,$$

We cannot calculate the numerator unless we know of the $f(x)$. However, we can still simplify the above denominator. Multiplying $\Phi_k(x)$ to both sides of the equation $\Phi_k'' + \lambda_k^2\Phi = 0$ yields $\lambda_k^2\Phi_k^2 = -\Phi_k\Phi_k''$. We deduce

$$\lambda_k^2 \int_0^\ell \Phi_k^2\, dx = -\Phi_k\Phi_k'\Big|_{x=0}^{x=\ell} + \int_0^\ell {\Phi_k'}^2\, dx \tag{8.53}$$

Combining $\Phi_k(x)$ and

$$\Phi_k'(x) = -\lambda_k^2 \sin\lambda_k x + h\lambda_k x$$

gives

$$\lambda_k^2\Phi_k(x)^2 + {\Phi_k'}^2 = \lambda_k^4 + h^2\lambda_k^2.$$

Integrating both sides over $[0,\, \ell]$ yields

$$\lambda_k^2 \int_0^\ell \Phi_k^2\, dx + \int_0^\ell {\Phi_k'}^2\, dx = (\lambda_k^4 + h^2\lambda_k^2)\ell.$$

Putting this into (8.53) yields

$$2\lambda_k^2 \int_0^\ell \Phi_k^2\, dx = (\lambda_k^4 + h^2\lambda_k^2)\ell - \Phi_k\Phi_k'\Big|_{x=0}^{x=\ell}.$$

We apply the boundary conditions (8.50) to (8.53) to obtain

$$\Phi_k^2(\lambda_k^2 + h^2) = \lambda_k^2\Phi_k^2 + h^2{\Phi_k}^2 = \lambda_k^4 + h^2\lambda_k^2,$$

or, after simplifying

$$\Phi_k^2 = \lambda_n^2.$$

at both $x = 0$ and $x = \ell$. In fact, the boundary conditions (8.50) also implies

$$(\Phi_k\Phi_k')\Big|_{x=0}^{x=\ell} = \Phi_k(\ell)\Phi_k'(\ell) - \Phi_k(0)\Phi_k'(0) = h\big(\Phi_k^2(\ell) - \Phi_k^2(0)\big) = -2h\,\lambda_k^2.$$

Thus, we deduce

$$2\lambda_k^2 \int_0^\ell \Phi_k^2\, dx = (\lambda_k^4 + h^2\lambda_k^2)\ell + 2h\,\lambda_k^2$$

and hence

$$\int_0^\ell \Phi_k^2\, dx = \frac{(\lambda_k^2 + h^2)\ell + 2h}{2}.$$

As a result, we obtain

$$A_k = \frac{2\displaystyle\int_0^\ell f(x)\,\Phi_k(x)\, dx}{(\lambda_k^2 + h^2)\ell + 2h}, \qquad k = 1,\, 2,\, 3, \cdots,$$