MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

9. The Fourier Bessel series

9.1. **Bessel functions.** Let ν be a complex number. We call the differential equation

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2)\, y = 0. \tag{9.1}$$

the *Bessel equation of order* ν. This equation has a *regular singular point* at $z = 0$. It is known that via the work of *Frobenius* that one can have power series of the form

$$y(z) = \sum_{k=0}^{\infty} c_k z^{\alpha+k},$$

where α is a parameter and c_k are coefficients to be determined. Substituting the above series into the Bessel equation

$$z^2 \frac{d^2y}{dz^2} + x \frac{dy}{dz} + (z^2 - \nu^2)\, y = 0. \tag{9.2}$$

and separating the coefficients yields:

$$\begin{aligned}
0 &= z^2 \frac{d^2y}{dz^2} + x \frac{dy}{dz} + (z^2 - \nu^2)\, y \\
&= \sum_{k=0}^{\infty} c_k(\alpha+k)(\alpha+k-1) z^{\alpha+k} + \sum_{k=0}^{\infty} c_k(\alpha+k) z^{\alpha+k} \\
&\qquad + (z^2 - \nu^2) \cdot \sum_{k=0}^{\infty} c_k z^{\alpha+k} \\
&= \sum_{k=0}^{\infty} c_k[(\alpha+k) - \nu^2] z^{\alpha+k} + \sum_{k=0}^{\infty} c_k z^{\alpha+k+2} \\
&= c_0(\alpha^2 - \nu^2) z^{\alpha} + \sum_{k=0}^{\infty} \big\{ c_k[(\alpha+k)^2 - \nu^2] + c_{k-2} \big\} z^{\alpha+k}
\end{aligned}$$

The first term on the right side of the above expression is

$$c_0(\alpha^2 - \nu^2) z^{\alpha}$$

and the remaining are

$$
\begin{aligned}
c_1[(\alpha+1)^2-\nu^2]+0 &= 0\\
c_2[(\alpha+2)^2-\nu^2]+c_0 &= 0\\
c_3[(\alpha+3)^2-\nu^2]+c_1 &= 0\\
\cdots\cdots &\\
c_k[(\alpha+k)^2-\nu^2]+c_{k-2} &= 0 \qquad (9.3)\\
\cdots\cdots & \qquad (9.4)
\end{aligned}
$$

Hence a series solution could exist only if $\alpha = \pm\nu$. Hence a series solution could exist only if $\alpha = \pm\nu$. When $k > 1$, then we require

$$c_k[(\alpha+k)^2-\nu^2]+c_{k-2}=0,$$

and this determines c_k in terms of c_{k-2} unless $\alpha-\nu=-2\nu$ or $\alpha+\nu=2\nu$ is an integer. Suppose we discard these exceptional cases for the moment, then it follows from (9.3) that

$$c_1=c_3=c_5=\cdots=c_{2k+1}=\cdots=0.$$

Thus we could express the coefficients c_{2k} in terms of

$$c_{2k}=\frac{(-1)^k c_0}{(\alpha-\nu+2)(\alpha-\nu+4)\cdots(\alpha-\nu+2k)\cdot(\alpha+\nu+2)(\alpha+\nu+4)\cdots(\alpha+\nu+2k)}.$$

If we now choose $\alpha=\nu$, then we obtain

$$c_0 z^{\nu}\Big[1+\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{k!(\nu+1)(\nu+2)\cdots(\nu+k)}\Big]. \tag{9.5}$$

Alternatively, if we choose $\alpha=-\nu$, then we obtain

$$c_0'\, z^{-\nu}\Big[1+\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{k!(-\nu+1)(-\nu+2)\cdots(-\nu+k)}\Big]. \tag{9.6}$$

9.2. **Gamma function.** We recall that

$$k! = k\times(k-1)\times\cdots 3\times 2\times 1. \tag{9.7}$$

Euler was able to give a correct definition to $k!$ when k is not a positive integer. He invented the Euler-Gamma function in the year 1729 that solved the *interpolation problem* of finding a function that equals to k ! for every positive integer k but has meaning elsewhere. That is,

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,dt, \qquad x>0. \tag{9.8}$$

So the integral will "converge" for all positive real x. Since

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1. \tag{9.9}$$

In fact, the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Re(z) > 0, \tag{9.10}$$

converges for all complex x with $\Re(x) > 0$. In fact, it can be shown that it is an analytic function in $\Re(x) > 0$. One the other hand, an integration-by-parts yields

$$\Gamma(x+1) = \int_0^\infty t^x\,e^{-t}\,dt = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1}\,e^{-t}\,dt = x\,\Gamma(x)$$

since $x > 0$. So one has

$$\Gamma(x+1) = x\,\Gamma(x), \tag{9.11}$$

and so for each positive integer n

$$\Gamma(n+1) = n\,\Gamma n = n(n-1)\Gamma(n-1) = n(n-1)\cdot 3\cdot 2\cdot 1\cdot\Gamma(1) = n\,!. \tag{9.12}$$

Euler worked out that

$$\Big(\frac{1}{2}\Big)! = \frac{1}{2}\sqrt{\pi}. \tag{9.13}$$

So

$$\Big(5\frac{1}{2}\Big)! = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Big(\frac{1}{2}\Big)!. \tag{9.14}$$

In fact, the infinite integral will not only "converge" for all positive real x. That is, it is not just continuous and (real) differentiable function for all positive x, the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Re(z) > 0, \tag{9.15}$$

converges for all complex x with $\Re(x) > 0$. In fact, it can be shown that it is an analytic function in $\Re(x) > 0$.

We can now use the recursion formula (9.11) to "analytically continue" the $\Gamma(x)$ into the $\Re(x) > -1$, and then again to $\Re(x) > -2$, etc, and eventually into whole complex plane $\mathbb{C}$ except on the negative integers, since, for example,

$$(-5)! = \Gamma(-4) = (-4+1)(-4+2)(-4+3)(-4+4)\Gamma(0) = (-3)(-2)(-1)(0)\,\Gamma(0)$$

meaning that $\Gamma(0)$ would need to be infinity if the left side is to have a meaning. In fact, $\Gamma(x)$ is analytic in $\mathbb{C}$ except at the negative integers including 0 which are simple poles. So it is a *meromorphic function.*

One can even compute negative factorial:

$$\Gamma(1/2) = (-1/2)\,(-3/2)\,(-5/2)\,(-19/2)\,(-11/2)\,\Gamma(-11/2). \tag{9.16}$$

We mention that the Euler-Gamma function as an example of the *Mellin transform* defined by

$$M(f)(z) = \int_0^\infty t^{z-1} f(t)\, dt.$$

That is, the Gamma function is the Mellin transform of the $f(x) = e^x$.

9.3. **Bessel functions of first kind.** We recall that we have assumed that ν is not an integer. Since c_0 and c_0' are arbitrary, so we choose them to be

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}, \qquad c_0' = \frac{1}{2^{-\nu}\Gamma(-\nu+1)} \tag{9.17}$$

so that the two series (9.5) and (9.6) can be written in the forms:

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{\nu+2k}}{\Gamma(\nu+k+1)\, k!}, \qquad J_{-\nu}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{-\nu+2k}}{\Gamma(-\nu+k+1)\, k!} \tag{9.18}$$

and both are called the *Bessel function of order* ν and $-\nu$ and both are *of the first kind.* In this case, we have $\{J_\nu,\, J_{-\nu}\}$ forms a *fundamental set* of the Bessel equation. That is,

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$$

is the general solution to the Bessel equation. More precisely, it can be shown that the Wronskian of J_ν and $J_{-\nu}$ is given by (G. N. Watson "A Treatise On The Theory Of Bessel Functions", pp. 42–43):

$$W(J_\nu,\, J_{-\nu}) = -\frac{2\sin\nu\pi}{\pi z}, \tag{9.19}$$

which is non-zero provided that ν is not equal to an integer and $z \neq 0$. This shows that the J_ν and $J_{-\nu}$ forms a fundamental set of solutions. One can also see that when ν is not an integer, then

$$J_\nu(x) = C\, J_{-\nu}(x)$$

would not hold for all x for any constant C. Just take $s \to 0$. In fact, when $\nu = n$ is an integer, then we can easily check that

$$J_{-n}(z) = (-1)^n J_n(z). \tag{9.20}$$

9.4. **Bessel functions of second kind.** We define

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \tag{9.21}$$

when ν is not an integer. It is easy to check that the wronskian of J_ν and Y_ν is non-zero, thus showing that $\{J_\nu,\, Y_\nu\}$ forms a fundamental set of the Bessel equation when when ν is not an integer. The case when ν is an integer n is defined by

$$Y_n(z) = \lim_{\nu\to n} \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}. \tag{9.22}$$

The idea behind is that both the numerator and denominator is undefined as ν tends to an integer. So we apply the Ĺ Hospital's rule to yields

$$\begin{aligned}
Y_n(x) &= \lim_{\nu\to n} \frac{(\partial/\partial\nu)\big(J_\nu(z)\cos\nu\pi - J_{-\nu}(z)\big)}{(\partial/\partial\nu)\sin\nu\pi}\\
&= \lim_{\nu\to n} \frac{\cos\nu\pi(\partial/\partial\nu)J_\nu(z) - \pi J_\nu(z)\sin\nu\pi - (\partial/\partial\nu)J_{-\nu}(z)}{\pi\cos\nu\pi}\\
&= \frac{(-1)^\nu(\partial/\partial\nu)J_\nu(z) - (\partial/\partial\nu)J_{-\nu}(z)}{\pi(-1)^\nu}\Big|_{\nu=n}
\end{aligned}$$

The Y_ν so defined is linearly independent with J_ν for *all* values of ν.

In particular, we obtain

$$\begin{aligned}
Y_n(x) &= \frac{-1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\Big(\frac{x}{2}\Big)^{2k-n}\\
&\quad + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}\Big[2\log\frac{x}{2} - \psi(k+1) - \psi(k+n+1)\Big]\\
&= \frac{2}{\pi}J_n(x)\Big(\log\frac{x}{2} + \gamma\Big) - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\Big(\frac{x}{2}\Big)^{-n+2k}\\
&\quad - \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}\Big(\sum_{j=1}^{n+k}\frac{1}{j} + \sum_{j=1}^{k}\frac{1}{j}\Big)
\end{aligned}$$

for $|\arg x| < \pi$ and $n = 0, 1, 2, \cdots$ with the understanding that we set the sum to be 0 when $n = 0$. Here the $\psi(z) = \Gamma'(x)/\Gamma(x)$, $\gamma = 0.57721566$ is the Euler constant. We note that the function is unbounded when $x = 0$.

When $n = 0$, we have

$$\begin{aligned} Y_0(x) = \frac{2}{\pi} J_0(x)\Big(\log\frac{x}{2} + \gamma\Big) \\ - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\Big(\frac{x}{2}\Big)^{2k}\Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\Big) \end{aligned}$$

9.5. **Recurrence Formulae for J_ν.** We consider arbitrary complex ν.

$$\begin{aligned} \frac{d}{dx} x^\nu J_\nu(x) &= \frac{d}{dx}\frac{(-1)^k x^{2\nu+2k}}{2^{\nu+2k}\, k!\Gamma(\nu+k+1)} \\ &= \frac{d}{dx}\frac{(-1)^k x^{2\nu-1+2k}}{2^{\nu-1+2k}\, k!\Gamma(\nu+k)} \\ &= x^\nu J_{\nu-1}(x). \end{aligned} \tag{9.23}$$

But the left side can be expanded and this yields

$$xJ'_\nu(x) + \nu J_\nu(x) = xJ_{\nu-1}(x). \tag{9.24}$$

Similarly,

$$\frac{d}{dx} x^{-\nu} J_\nu(x) = -x^{-\nu} J_{\nu+1}(x). \tag{9.25}$$

and this yields

$$xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x) \tag{9.26}$$

Subtracting and adding the above recurrence formulae yield

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \tag{9.27}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x). \tag{9.28}$$

Replacing ν by $-\nu$ in (9.25) yields

$$\frac{d}{dx} x^\nu J_{-\nu}(x) = -x^\nu J_{-\nu+1}(x). \tag{9.29}$$

9.6. **Recurrence Formulae for Y_ν.** Suppose ν is not an integer. It follows from (9.23) and (9.29) that

$$\begin{aligned}\frac{d}{dx}x^\nu Y_\nu(x) &= \frac{d}{dx}x^\nu \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}\\ &= x^\nu \frac{J_{\nu-1}(x)\cos\nu\pi + J_{-\nu+1}(x)}{\sin\nu\pi}\\ &= -x^\nu \frac{J_{\nu-1}(x)\cos\nu\pi + J_{-\nu+1}(x)}{\sin\nu\pi}\\ &= x^\nu \frac{J_{\nu-1}(x)\cos(\nu-1)\pi - J_{-\nu+1}(x)}{\sin(\nu-1)\pi}\\ &= x^\nu Y_{\nu-1}(x).\end{aligned}$$

Let us now replace ν by $-\nu$ in (9.23) to obtain

$$\frac{d}{dx}x^{-\nu}\, J_\nu(x) = x^{-\nu} J_{-\nu-1}(x),$$

Combining this together with (9.25), yields, similarly,

$$\frac{d}{dx}x^{-\nu} Y_\nu(x) = -x^{-\nu} Y_{\nu+1}(x).$$

9.7. **Generating Function for J_n.** Jacobi in 1836 gave

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} t^k J_k(z). \tag{9.30}$$

Many of the forumulae derived above can be obtained from this expression.

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} c_k(z) t^k \tag{9.31}$$

for $0 < |t| < \infty$. We multiply the power series

$$e^{\frac{zt}{2}} = 1 + \frac{(z/2)}{1!}t + \frac{(z/2)^2}{2!}t^2 + \cdots \tag{9.32}$$

and

$$e^{-\frac{z}{2t}} = 1 - \frac{(z/2)}{1!}t^{-1} + \frac{(z/2)^2}{2!}t^{-2} - \cdots \tag{9.33}$$

Multiplying the two series and comparing the coefficients of t^k yield

$$c_n(z) = J_n(z), \quad n = 0, 1, \cdots \tag{9.34}$$

$$c_n(z) = (-1)^n J_{-n}(z), \quad n = -1, -2, \cdots . \tag{9.35}$$

Thus

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = J_0(z) + \sum_{k=1}^{+\infty}[t^k + (-1)^k t^{-k}]J_k(z). \tag{9.36}$$

9.8. **Bessel Functions of half-integer Orders.** It follows from the definition of Bessel function that

$$\begin{aligned} J_{1/2}(x) &= \frac{\sqrt{x}}{\sqrt{2}\Gamma(\frac{3}{2})}\Big(1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots\Big) \\ &= \frac{1}{\sqrt{2x}\Gamma(\frac{3}{2})}\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big) \\ &= \frac{1}{\sqrt{2x}\Gamma(\frac{3}{2})}\sin x \end{aligned}$$

On the other hand,

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\int_0^\infty e^{-x}\frac{dx}{\sqrt{x}} = \int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}.$$

Hence

$$J_{\frac{1}{2}}(x) = \Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x$$

Similarly, one can check that

$$J_{-\frac{1}{2}}(x) = \Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x, \tag{9.37}$$

Moreover,

$$J_{n+\frac{1}{2}}(z) = (-1)^n\sqrt{\frac{2}{\pi}}z^{n+\frac{1}{2}}\Big(\frac{d}{z\,dz}\Big)^n\frac{\sin z}{z}, \qquad n = 0, 1, 2, \cdots . \tag{9.38}$$

9.9. **Lommel's Polynomials.** Iterating the recurrence formula

$$J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z) - J_{\nu-1} \tag{9.39}$$

with respect to ν a number of times give

$$J_{\nu+k}(x) = P(1/x)J_\nu(x) - Q(1/x)J_{\nu-1}(x). \tag{9.40}$$

Lommel (1871) [See Watson, pp. 294–295] found that

$$J_{\nu+k}(x) = R_{k,\nu}(x)J_\nu(x) - R_{k-1,\nu+1}(x)J_{\nu-1}(x). \tag{9.41}$$

Hence, in the special case when $\nu = n + \frac{1}{2}$, we have

$$J_{n+\frac{1}{2}}(x) = (-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}\Big(\frac{d}{x\,dx}\Big)^n\frac{\sin x}{x}, \qquad n = 0,\ 1,\ 2,\ \cdots. \tag{9.42}$$

Thus applying a recurrence formula and using the Lommel polynomials yield

$$J_{n+\frac{1}{2}}(x) = R_{n,\nu}(x)J_{\frac{1}{2}}(x) - R_{n-1,\nu+1}J_{-\frac{1}{2}}(x) \tag{9.43}$$

That is, we have

$$J_{n+\frac{1}{2}}(x) = R_{n,\frac{1}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x - R_{n-1,\frac{3}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x, \tag{9.44}$$

and similarly,

$$(-1)^nJ_{-n-\frac{1}{2}}(x) = R_{n,\frac{1}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x + R_{n-1,\frac{3}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x. \tag{9.45}$$

9.10. **Some formulae for Lommel's polynomials.** For each fixed ν, the Lommel polynomials are given by

$$R_{n\,\nu}(z) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k)!(\nu)_{n-k}}{k!(n-2k)!(\nu)_k}\left(\frac{2}{z}\right)^{n-2k} \tag{9.46}$$

where the $[x]$ means the largest integer not exceeding x. Lommel is a German who is one of the main contributors to Bessel functions.

"Pythagoras' Theorem" for Bessel Function. These Lommel polynomials have remarkable properties. Since

$$J_{-\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}}\cos z, \qquad J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}}\sin z \tag{9.47}$$

and $\sin^2 x + \cos^2 x = 1$; we now have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = 2(-1)^n\frac{R_{2n,\,\frac{1}{2}-n}(z)}{\pi z}. \tag{9.48}$$

That is, we have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = \frac{2}{\pi z}\sum_{k=0}^{n} \frac{(2z)^{2n-2k}(2n-k)!(2n-2k)!}{[(n-k)!]^2 k!}. \tag{9.49}$$

A few special cases are

(1) $J^2_{\frac{1}{2}}(z) + J^2_{-\frac{1}{2}}(z) = \dfrac{2}{\pi z}$;

(2) $J^2_{\frac{3}{2}}(z) + J^2_{-\frac{3}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{1}{z^2}\Big)$;

(3) $J^2_{\frac{5}{2}}(z) + J^2_{-\frac{5}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{3}{z^2} + \dfrac{9}{z^4}\Big)$;

(4) $J^2_{\frac{7}{2}}(z) + J^2_{-\frac{7}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{6}{z^2} + \dfrac{45}{z^4} + \dfrac{225}{z^6}\Big)$

To be continued ...