MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

9. The Fourier Bessel series

9.1. **Bessel functions.** Let ν be a complex number. We call the differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)\, y = 0. \tag{9.1}$$

the *Bessel equation of order* ν. This equation has a *regular singular point* at $z = 0$. It is known that via the work of *Frobenius* that one can have power series of the form

$$y(z) = \sum_{k=0}^{\infty} c_k z^{\alpha+k},$$

where α is a parameter and c_k are coefficients to be determined. Substituting the above series into the Bessel equation

$$z^2 \frac{d^2y}{dz^2} + x\frac{dy}{dz} + (z^2 - \nu^2)\, y = 0. \tag{9.2}$$

and separating the coefficients yields:

$$\begin{aligned} 0 &= z^2 \frac{d^2y}{dz^2} + x\frac{dy}{dz} + (z^2 - \nu^2)\, y \\ &= \sum_{k=0}^{\infty} c_k(\alpha+k)(\alpha+k-1)z^{\alpha+k} + \sum_{k=0}^{\infty} c_k(\alpha+k)z^{\alpha+k} \\ &\qquad + (z^2-\nu^2)\cdot \sum_{k=0}^{\infty} c_k z^{\alpha+k} \\ &= \sum_{k=0}^{\infty} c_k[(\alpha+k)-\nu^2]z^{\alpha+k} + \sum_{k=0}^{\infty} c_k z^{\alpha+k+2} \\ &= c_0(\alpha^2-\nu^2)z^{\alpha} + \sum_{k=0}^{\infty} \big\{c_k[(\alpha+k)^2-\nu^2] + c_{k-2}\big\} z^{\alpha+k} \end{aligned}$$

The first term on the right side of the above expression is

$$c_0(\alpha^2 - \nu^2)z^{\alpha}$$

and the remaining are

$$\begin{aligned} c_1[(\alpha+1)^2-\nu^2]+0 &= 0\\ c_2[(\alpha+2)^2-\nu^2]+c_0 &= 0\\ c_3[(\alpha+3)^2-\nu^2]+c_1 &= 0\\ \cdots\cdots &\\ c_k[(\alpha+k)^2-\nu^2]+c_{k-2} &= 0 \qquad (9.3)\\ \cdots\cdots & \qquad (9.4) \end{aligned}$$

Hence a series solution could exist only if $\alpha=\pm\nu$. Hence a series solution could exist only if $\alpha=\pm\nu$. When $k>1$, then we require

$$c_k[(\alpha+k)^2-\nu^2]+c_{k-2}=0,$$

and this determines c_k in terms of c_{k-2} unless $\alpha-\nu=-2\nu$ or $\alpha+\nu=2\nu$ is an integer. Suppose we discard these exceptional cases for the moment, then it follows from (9.3) that

$$c_1=c_3=c_5=\cdots=c_{2k+1}=\cdots=0.$$

Thus we could express the coefficients c_{2k} in terms of

$$c_{2k}=\frac{(-1)^k c_0}{(\alpha-\nu+2)(\alpha-\nu+4)\cdots(\alpha-\nu+2k)\cdot(\alpha+\nu+2)(\alpha+\nu+4)\cdots(\alpha+\nu+2k)}.$$

If we now choose $\alpha=\nu$, then we obtain

$$c_0 z^{\nu}\Big[1+\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{k!(\nu+1)(\nu+2)\cdots(\nu+k)}\Big]. \qquad (9.5)$$

Alternatively, if we choose $\alpha=-\nu$, then we obtain

$$c_0'\, z^{-\nu}\Big[1+\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{k!(-\nu+1)(-\nu+2)\cdots(-\nu+k)}\Big]. \qquad (9.6)$$

9.2. **Gamma function.** We recall that

$$k!=k\times(k-1)\times\cdots 3\times 2\times 1. \qquad (9.7)$$

Euler was able to give a correct definition to $k!$ when k is not a positive integer. He invented the Euler-Gamma function in the year 1729 that solved the *interpolation problem* of finding a function that equals to k ! for every positive integer k but has meaning elsewhere. That is,

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,dt, \qquad x>0. \qquad (9.8)$$

So the integral will "converge" for all positive real x. Since

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1. \tag{9.9}$$

In fact, the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Re(z) > 0, \tag{9.10}$$

converges for all complex x with $\Re(x) > 0$. In fact, it can be shown that it is an analytic function in $\Re(x) > 0$. One the other hand, an integration-by-parts yields

$$\Gamma(x+1) = \int_0^\infty t^x\,e^{-t}\,dt = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1}\,e^{-t}\,dt = x\,\Gamma(x)$$

since $x > 0$. So one has

$$\Gamma(x+1) = x\,\Gamma(x), \tag{9.11}$$

and so for each positive integer n

$$\Gamma(n+1) = n\,\Gamma n = n(n-1)\Gamma(n-1) = n(n-1)\cdot 3\cdot 2\cdot 1\cdot \Gamma(1) = n\,!. \tag{9.12}$$

Euler worked out that

$$\Big(\frac{1}{2}\Big)! = \frac{1}{2}\sqrt{\pi}. \tag{9.13}$$

So

$$\Big(5\frac{1}{2}\Big)! = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Big(\frac{1}{2}\Big)!. \tag{9.14}$$

In fact, the infinite integral will not only "converge" for all positive real x. That is, it is not just continuous and (real) differentiable function for all positive x, the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Re(z) > 0, \tag{9.15}$$

converges for all complex x with $\Re(x) > 0$. In fact, it can be shown that it is an analytic function in $\Re(x) > 0$.

We can now use the recursion formula (9.11) to "analytically continue" the $\Gamma(x)$ into the $\Re(x) > -1$, and then again to $\Re(x) > -2$, etc, and eventually into whole complex plane $\mathbb{C}$ except on the negative integers, since, for example,

$$(-5)! = \Gamma(-4) = (-4+1)(-4+2)(-4+3)(-4+4)\Gamma(0) = (-3)(-2)(-1)(0)\,\Gamma(0)$$

meaning that $\Gamma(0)$ would need to be infinity if the left side is to have a meaning. In fact, $\Gamma(x)$ is analytic in $\mathbb{C}$ except at the negative integers including 0 which are simple poles. So it is a *meromorphic function.*

One can even compute negative factorial:

$$\Gamma(1/2) = (-1/2)\,(-3/2)\,(-5/2)\,(-19/2)\,(-11/2)\,\Gamma(-11/2). \tag{9.16}$$

We mention that the Euler-Gamma function as an example of the *Mellin transform* defined by

$$M(f)(z) = \int_0^\infty t^{z-1} f(t)\,dt.$$

That is, the Gamma function is the Mellin transform of the $f(x) = e^x$.

9.3. **Bessel functions of first kind.**

We recall that we have assumed that ν is not an integer. Since c_0 and c_0' are arbitrary, so we choose them to be

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}, \qquad c_0' = \frac{1}{2^{-\nu}\Gamma(-\nu+1)} \tag{9.17}$$

so that the two series (9.5) and (9.6) can be written in the forms:

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{\nu+2k}}{\Gamma(\nu+k+1)\,k!}, \qquad J_{-\nu}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{-\nu+2k}}{\Gamma(-\nu+k+1)\,k!} \tag{9.18}$$

and both are called the *Bessel function of order* ν and $-\nu$ and both are *of the first kind.* In this case, we have $\{J_\nu,\, J_{-\nu}\}$ forms a *fundamental set* of the Bessel equation. That is,

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$$

is the general solution to the Bessel equation. More precisely, it can be shown that the Wronskian of J_ν and $J_{-\nu}$ is given by (G. N. Watson "A Treatise On The Theory Of Bessel Functions", pp. 42–43):

$$W(J_\nu,\, J_{-\nu}) = -\frac{2\sin\nu\pi}{\pi z}, \tag{9.19}$$

which is non-zero provided that ν is not equal to an integer and $z \neq 0$. This shows that the J_ν and $J_{-\nu}$ forms a *fundamental set of solutions.* One can also see that when ν is not an integer, then

$$J_\nu(x) = C\, J_{-\nu}(x)$$

would not hold for all x for any constant C. Just take $s \to 0$. In fact, when $\nu = n$ is an integer, then we can easily check that

$$J_{-n}(z) = (-1)^n J_n(z). \tag{9.20}$$

9.4. **Bessel functions of second kind.** We define

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \tag{9.21}$$

when ν is not an integer. It is easy to check that the wronskian of J_ν and Y_ν is non-zero, thus showing that $\{J_\nu,\, Y_\nu\}$ forms a fundamental set of the Bessel equation when when ν is not an integer. The case when ν is an integer n is defined by

$$Y_n(z) = \lim_{\nu\to n} \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}. \tag{9.22}$$

The idea behind is that both the numerator and denominator is undefined as ν tends to an integer. So we apply the Ĺ Hospital's rule to yields

$$\begin{aligned}
Y_n(x) &= \lim_{\nu\to n} \frac{(\partial/\partial\nu)\big(J_\nu(z)\cos\nu\pi - J_{-\nu}(z)\big)}{(\partial/\partial\nu)\sin\nu\pi}\\
&= \lim_{\nu\to n} \frac{\cos\nu\pi(\partial/\partial\nu)J_\nu(z) - \pi J_\nu(z)\sin\nu\pi - (\partial/\partial\nu)J_{-\nu}(z)}{\pi\cos\nu\pi}\\
&= \frac{(-1)^\nu(\partial/\partial\nu)J_\nu(z) - (\partial/\partial\nu)J_{-\nu}(z)}{\pi(-1)^\nu}\Big|_{\nu=n}
\end{aligned}$$

The Y_ν so defined is linearly independent with J_ν for *all* values of ν.

In particular, we obtain

$$\begin{aligned}
Y_n(x) &= \frac{-1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\Big(\frac{x}{2}\Big)^{2k-n}\\
&\quad + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}\Big[2\log\frac{x}{2} - \psi(k+1) - \psi(k+n+1)\Big]\\
&= \frac{2}{\pi}J_n(x)\Big(\log\frac{x}{2} + \gamma\Big) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{x}{2}\Big)^{-n+2k}\\
&\quad - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{k!(n+k)!}\Big(\sum_{j=1}^{n+k}\frac{1}{j} + \sum_{j=1}^{k}\frac{1}{j}\Big)
\end{aligned}$$

for $|\arg x| < \pi$ and $n = 0, 1, 2, \cdots$ with the understanding that we set the sum to be 0 when $n = 0$. Here the $\psi(z) = \Gamma'(x)/\Gamma(x)$, $\gamma = 0.57721566$ is the Euler constant. We note that the function is unbounded when $x = 0$.

When $n = 0$, we have

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi} J_0(x) \Big(\log \frac{x}{2} + \gamma \Big) \\ &\quad - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \Big(\frac{x}{2}\Big)^{2k} \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\Big) \end{aligned}$$

9.5. **Recurrence Formulae for J_ν.** We consider arbitrary complex ν.

$$\begin{aligned} \frac{d}{dx} x^\nu J_\nu(x) &= \frac{d}{dx} \frac{(-1)^k x^{2\nu+2k}}{2^{\nu+2k}\, k!\Gamma(\nu+k+1)} \\ &= \frac{d}{dx} \frac{(-1)^k x^{2\nu-1+2k}}{2^{\nu-1+2k}\, k!\Gamma(\nu+k)} \\ &= x^\nu J_{\nu-1}(x). \end{aligned} \tag{9.23}$$

But the left side can be expanded and this yields

$$x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x). \tag{9.24}$$

Similarly,

$$\frac{d}{dx} x^{-\nu} J_\nu(x) = -x^{-\nu} J_{\nu+1}(x). \tag{9.25}$$

and this yields

$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x) \tag{9.26}$$

Subtracting and adding the above recurrence formulae yield

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \tag{9.27}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x). \tag{9.28}$$

Replacing ν by $-\nu$ in (9.25) yields

$$\frac{d}{dx} x^\nu J_{-\nu}(x) = -x^\nu J_{-\nu+1}(x). \tag{9.29}$$

9.6. **Recurrence Formulae for Y_ν.** Suppose ν is not an integer. It follows from (9.23) and (9.29) that

$$\begin{aligned}\frac{d}{dx}x^\nu Y_\nu(x) &= \frac{d}{dx}x^\nu \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}\\ &= x^\nu \frac{J_{\nu-1}(x)\cos\nu\pi + J_{-\nu+1}(x)}{\sin\nu\pi}\\ &= -x^\nu \frac{J_{\nu-1}(x)\cos\nu\pi + J_{-\nu+1}(x)}{\sin\nu\pi}\\ &= x^\nu \frac{J_{\nu-1}(x)\cos(\nu-1)\pi - J_{-\nu+1}(x)}{\sin(\nu-1)\pi}\\ &= x^\nu Y_{\nu-1}(x).\end{aligned}$$

Let us now replace ν by $-\nu$ in (9.23) to obtain

$$\frac{d}{dx}x^{-\nu}\, J_\nu(x) = x^{-\nu} J_{-\nu-1}(x),$$

Combining this together with (9.25), yields, similarly,

$$\frac{d}{dx}x^{-\nu}Y_\nu(x) = -x^{-\nu}Y_{\nu+1}(x).$$

9.7. **Generating Function for J_n.** Jacobi in 1836 gave

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} t^k J_k(z). \tag{9.30}$$

Many of the forumulae derived above can be obtained from this expression.

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} c_k(z) t^k \tag{9.31}$$

for $0 < |t| < \infty$. We multiply the power series

$$e^{\frac{zt}{2}} = 1 + \frac{(z/2)}{1!}t + \frac{(z/2)^2}{2!}t^2 + \cdots \tag{9.32}$$

and

$$e^{-\frac{z}{2t}} = 1 - \frac{(z/2)}{1!}t^{-1} + \frac{(z/2)^2}{2!}t^{-2} - \cdots \tag{9.33}$$

and comparing the coefficients of t^k yield

$$c_n(z) = J_n(z), \quad n = 0,\, 1,\, \cdots \tag{9.34}$$

$$c_n(z) = (-1)^n J_{-n}(z), \quad n = -1,\, -2,\, \cdots . \tag{9.35}$$

Thus

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = J_0(z) + \sum_{k=1}^{+\infty} J_k[t^k + (-1)^k t^{-k}]. \tag{9.36}$$

9.8. **Bessel Functions of half-integer Orders.** It follows from the definition of Bessel function that

$$\begin{aligned} J_{1/2}(x) &= \frac{\sqrt{x}}{\sqrt{2}\Gamma(\frac{3}{2})}\Big(1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots\Big) \\ &= \frac{1}{\sqrt{2x}\Gamma(\frac{3}{2})}\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big) \\ &= \frac{1}{\sqrt{2x}\Gamma(\frac{3}{2})}\sin x \end{aligned}$$

On the other hand,

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\int_0^\infty e^{-x}\frac{dx}{\sqrt{x}} = \int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}.$$

Hence

$$J_{\frac{1}{2}}(x) = \Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x$$

Similarly, one can check that

$$J_{-\frac{1}{2}}(x) = \Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x, \tag{9.37}$$

Moreover,

$$J_{n+\frac{1}{2}}(z) = (-1)^n\sqrt{\frac{2}{\pi}}z^{n+\frac{1}{2}}\Big(\frac{d}{z\,dz}\Big)^n\frac{\sin z}{z}, \qquad n = 0,\, 1,\, 2,\, \cdots . \tag{9.38}$$

9.9. **Lommel's Polynomials.** Iterating the recurrence formula

$$J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z) - J_{\nu-1} \tag{9.39}$$

with respect to ν a number of times give

$$J_{\nu+k}(x) = P(1/x)J_\nu(x) - Q(1/x)J_{\nu-1}(x). \tag{9.40}$$

Lommel (1871) [See Watson, pp. 294–295] found that

$$J_{\nu+k}(x) = R_{k,\nu}(x)J_\nu(x) - R_{k-1,\nu+1}(x)J_{\nu-1}(x). \tag{9.41}$$

Hence, in the special case when $\nu = n + \frac{1}{2}$, we have

$$J_{n+\frac{1}{2}}(x) = (-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}\Big(\frac{d}{x\,dx}\Big)^n\frac{\sin x}{x}, \qquad n = 0,\, 1,\, 2,\, \cdots . \tag{9.42}$$

Thus applying a recurrence formula and using the Lommel polynomials yield

$$J_{n+\frac{1}{2}}(x) = R_{n,\nu}(x)J_{\frac{1}{2}}(x) - R_{n-1,\nu+1}J_{-\frac{1}{2}}(x) \tag{9.43}$$

That is, we have

$$J_{n+\frac{1}{2}}(x) = R_{n,\frac{1}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x - R_{n-1,\frac{3}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x, \tag{9.44}$$

and similarly,

$$(-1)^nJ_{-n-\frac{1}{2}}(x) = R_{n,\frac{1}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\cos x + R_{n-1,\frac{3}{2}}(x)\Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\sin x. \tag{9.45}$$

9.10. **Some formulae for Lommel's polynomials.** For each fixed ν, the Lommel polynomials are given by

$$R_{n\,\nu}(z) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k)!(\nu)_{n-k}}{k!(n-2k)!(\nu)_k}\left(\frac{2}{z}\right)^{n-2k} \tag{9.46}$$

where the $[x]$ means the largest integer not exceeding x. Lommel is a German who is one of the main mathematicians made a major contribution to Bessel functions.

"Pythagoras' Theorem" for Bessel Function. These Lommel polynomials have remarkable properties. Since

$$J_{-\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}}\cos z, \qquad J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}}\sin z \tag{9.47}$$

and $\sin^2 x + \cos^2 x = 1$; we now have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = 2(-1)^n\frac{R_{2n,\,\frac{1}{2}-n}(z)}{\pi z}. \tag{9.48}$$

That is, we have

$$J^2_{n+\frac{1}{2}}(z) + J^2_{-n-\frac{1}{2}}(z) = \frac{2}{\pi z}\sum_{k=0}^{n}\frac{(2z)^{2n-2k}(2n-k)!(2n-2k)!}{[(n-k)!]^2 k!}. \tag{9.49}$$

A few special cases are

(1) $J^2_{\frac{1}{2}}(z) + J^2_{-\frac{1}{2}}(z) = \dfrac{2}{\pi z}$;

(2) $J^2_{\frac{3}{2}}(z) + J^2_{-\frac{3}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{1}{z^2}\Big)$;

(3) $J^2_{\frac{5}{2}}(z) + J^2_{-\frac{5}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{3}{z^2} + \dfrac{9}{z^4}\Big)$;

(4) $J^2_{\frac{7}{2}}(z) + J^2_{-\frac{7}{2}}(z) = \dfrac{2}{\pi z}\Big(1 + \dfrac{6}{z^2} + \dfrac{45}{z^4} + \dfrac{225}{z^6}\Big)$

9.11. **Asymptotics of Bessel functions.** The substitution

$$y = \frac{z}{\sqrt{x}}$$

transforms the Bessel equation

$$y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

into the form

$$z''(x) + \big(1 + \rho\big)z(x) = 0, \tag{9.50}$$

where $\rho = \rho(x) = m/x^2$, and $m = \frac{1}{4} - \nu^2$. It is clear that ρ becomes very small for large x. That is, the equation (9.50) is "close to" to

$$z''(x) + z(x) = 0,$$

indicating that the solutions to (9.50) are close to

$$z = z(x) = A\,\sin(x + \omega), \qquad A,\ \omega \text{ constants.}$$

It is reasonable to assume that a solution to (9.50) assumes the form

$$z = z(x) = \alpha\,\sin(x + \delta) \tag{9.51}$$

where both $\alpha = \alpha(x)$ and $\delta = \delta(x)$ are functions of x, they should converge to some definite limits as $x \to +\infty$. We shall justify the claim below. We also make another assumption that

$$z'(x) = \alpha\, \cos(x+\delta), \tag{9.52}$$

in order to establish the above claim. Differentiating (9.52) yields

$$z''(x) = \alpha'\, \cos(x+\delta) - \alpha(1+\delta')\, \sin(x+\delta).$$

Combining this with

$$z''(x) = -(1+\rho)z(x) = -\alpha(1+\delta)\sin(x+\delta)$$

gives,

$$\tan(x+\delta) = \frac{\alpha'}{\alpha(\delta'-\rho)}. \tag{9.53}$$

On the other hand, differentiating (9.51) yields

$$z'(t) = \alpha' \sin(x+\delta) + \alpha(1+\delta')\cos(x+\delta).$$

Comparing it with (9.52) yields

$$\tan(x+\delta) = -\frac{\alpha\,\delta'}{\alpha'}. \tag{9.54}$$

Multiplying (9.53) and (9.54) yields,

$$\tan^2(x+\delta) = -\frac{\delta'}{\delta'-\rho},$$

and from which we deduce

$$\delta' = \rho\, \sin^2(x+\delta).$$

We deduce from (9.54) that

$$\frac{\alpha'}{\alpha} = -\frac{\delta'}{\tan(x+\delta')} = -\rho\sin(x+\delta)\cos(x+\delta). \tag{9.55}$$

Moreover, it follows from (9.51) and (9.52) that the function $\alpha(x)$ must be non-vanishing throughout. For this would mean the $z(x)$ and $z'(z)$ would simultaneous vanish at a same point, implying that z be identically zero solution. A contradiction. Besides, the existence of δ can be asserted via solving the (9.51) and (9.52) with initial condition given there also. Once the δ is found, the α can also be retrieved from (9.55) and initial condition again from (9.51) and (9.52).

But $\rho = \rho(x) = m/x^2$ and so

$$\delta(x) = \delta(b) - \int_x^b \delta'(t)\ dt = \delta(b) - m\int_x^b \frac{\sin^2(t+\delta)}{t^2}\ dt.$$

We now take limit $x \to \infty$ and observe that the improper integral exists. Therefore, $\delta(b)$ must also exist as $b \to \infty$. Let

$$\lim_{b\to\infty} \delta(b) = \omega$$

then

$$\delta(x) = \omega - m\int_x^\infty \frac{\sin^2(t+\delta)}{t^2}\ dt.$$

However,

$$0 < \int_x^\infty \frac{\sin^2(t+\delta)}{t^2}\ dt < \int_x^\infty \frac{1}{t^2}\ dt = \frac{1}{x},$$

so that

$$0 < mx\int_x^\infty \frac{\sin^2(t+\delta)}{t^2}\ dt < m.$$

We deduce that

$$\delta(x) = \omega + \frac{\eta(x)}{x}$$

where

$$\eta(x) := -mx\int_x^\infty \frac{\sin^2(t+\delta)}{t^2}\ dt$$

is a bounded function.

On the other hand, it follows from (9.55) that

$$\log\alpha(x) = \log\alpha(b) + m\int_x^b \frac{\sin(t+\delta)\cos(t+\delta)}{t^2}\ dt.$$

We similarly deduce that $\lim_{b\to\infty}\alpha(b) := A \neq 0$ exists. Hence

$$\log\alpha(x) = \log A + m\int_x^\infty \frac{\sin(t+\delta)\cos(t+\delta)}{t^2}\ dt.$$

Hence

$$\log\alpha(x) = \log A + \frac{\phi(x)}{x}$$

where the

$$\phi(x) = mx \int_x^\infty \frac{\sin(t+\delta)\cos(t+\delta)}{t^2}\ dt$$

is again bounded, and whence

$$\alpha(x) = A\exp\Big(\frac{\phi(x)}{x}\Big).$$

We apply the Taylor expansion with one them ($e^t = 1 + te^{\theta t}$ for some $0 < \theta < 1$) to obtain

$$\exp\Big(\frac{\phi(x)}{x}\Big) = 1 + \frac{\phi(x)}{x}\exp\Big(\frac{\theta\,\phi(x)}{x}\Big).$$

Hence we can write

$$\exp\Big(\frac{\theta\,\phi(x)}{x}\Big) = 1 + \frac{\xi(x)}{x},$$

and so

$$\alpha(x) = A\Big(1 + \frac{\xi(x)}{x}\Big).$$

We finally deduce

$$z(x) = \alpha\sin(x+\delta) = A\Big(1+\frac{\xi(x)}{x}\Big)\sin\Big(x+\omega+\frac{\eta(x)}{x}\Big).$$

We can further write, by Taylor expansion that

$$\begin{aligned}\sin\Big(x+\omega+\frac{\eta(x)}{x}\Big) &= \sin(x+\omega) + \frac{\eta(x)}{x}\cos\Big(x+\omega+\theta\frac{\eta(x)}{x}\Big)\\ &:= \sin(x+\omega) + \frac{\zeta(x)}{x}\end{aligned}$$

Clearly, the $\zeta(x)$ is a bounded function. This in turn means that

$$z(x) = A\Big(1+\frac{\xi(x)}{x}\Big)\Big(\sin(x+\omega)+\frac{\zeta(x)}{x}\Big) \tag{9.56}$$

$$= A\sin(x+\omega) + \frac{r(x)}{x} \tag{9.57}$$

where $r(x)$ is a bounded function, so that

$$z(x) \to A\sin(x+\omega)$$

as $x \to \infty$. Hence

$$y(x) = A\Big[\frac{\sin(x+\omega)}{x}\Big] + \frac{r(x)}{x\sqrt{x}}, \qquad x \to \infty$$

One can also deduce

$$z'(x) = A\left(1 + \frac{\xi(x)}{x}\right)\cos\left(x + \omega + \frac{\eta(x)}{x}\right) \tag{9.58}$$

$$= A\cos(x+\omega) + \frac{s(x)}{x} \tag{9.59}$$

where $s(x)$ again is a bounded function as $x \to \infty$.

We quote without proof the following more precise asymptotic formulae

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) + \frac{r_\nu(x)}{x\sqrt{x}},$$

and

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \frac{\rho_\nu(x)}{x\sqrt{x}},$$

where both $r_\nu(x)$ and $\rho_\nu(x)$ are bounded functions as $x \to +\infty$. In fact, one can do better and even on the complex plane:

Theorem 9.1. *For* $|\arg z| < \pi$*, we have*

$$J_\nu(z) \sim \Big[\cos\big(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\big)\sum_{k=0}^{\infty}\frac{(-1)^k(\nu,\,2k)}{(2z)^{2k}} - \sin\big(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\big)\sum_{k=0}^{\infty}\frac{(-1)^k(\nu,\,2k+1)}{(2z)^{2k+1}}\Big]$$

We note that the notation means

$$(v,\,n) = (-1)^n\frac{(\frac{1}{2}-\nu)_n(\frac{1}{2}+\nu)_n}{n!} = \frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})}.$$

That is, one can have as many terms of accuracy as one wants.

Remark. In fact the above asymptotic formulae were derived by using complex analytic contour integral method. We briefly introducing this. We define the *Bessel function of the third kind*, the *Hankel functions* are defined by $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ and $H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$. Then Hankel found

Theorem 9.2 (Hankel 1869). *Let* $\Re(\nu) > -\frac{1}{2}$*. Then*

$$H_\nu^{(1)}(x) = \Big(\frac{2}{\pi x}\Big)^{\frac{1}{2}}\frac{e^{i[x-\nu\pi/2-\pi/4]}}{\Gamma(\nu+1/2)}\int_0^{\infty\cdot\exp(i\beta)} e^{-u}u^{\nu-\frac{1}{2}}\Big(1+\frac{iu}{2x}\Big)^{\nu-\frac{1}{2}}\,du, \tag{9.60}$$

where $|\beta| < \pi/2$ *and* $-\frac{1}{2}\pi + \beta < \arg x < \frac{3}{2} + \beta$.

from there we have found

Theorem 9.3. *For* $-\pi < \arg x < 2\pi$,

$$H_\nu^{(1)}(x) = e^{i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\Big[\sum_{m=0}^{p-1}\frac{(\frac{1}{2}-\nu)_m(\frac{1}{2}+\nu)_m}{(2ix)^m m!} + R_p^{(1)}(x)\Big], \tag{9.61}$$

and for $-2\pi < \arg x < \pi$,

$$H_\nu^{(2)}(x) = e^{-i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\Big[\sum_{m=0}^{p-1}\frac{(\frac{1}{2}-\nu)_m(\frac{1}{2}+\nu)_m}{(2ix)^m m!} + R_p^{(2)}(x)\Big], \tag{9.62}$$

where

$$R_p^{(1)}(x) = O(x^{-p}) \qquad \textit{and} \qquad R_p^{(2)}(x) = O(x^{-p}), \tag{9.63}$$

as $x \to +\infty$, *uniformly in* $-\pi+\delta < \arg x < 2\pi-\delta$ *and* $-2\pi+\delta < \arg x < \pi-\delta$ *respectively.*

The two expansions are valid simultaneously in $-\pi < \arg x < \pi$. From here, one can derive Bessel's asymptote stated in the Theorem 9.1 above.

9.12. **Zeros of Bessel functions.** There is a wealth of information about the zeros of Bessel functions (of all kinds) of real order ν. The most comprehensive treatment on the zeros can be found from G. N. Watson's "A Treatise on the Theory of Bessel Functions", Cambridge University Press 1922 (2nd ed. 1944) and A. Erdélyi (Ed) Bateman Manuscript Project: Higher Transcendental Functions, Vol. II, 1953.

Theorem 9.4. *Let* ν *be a real number. Then the zeros*

(i) *of* $J_\nu(x)$ *are located close to the zeros of* $\sin(x+\omega)$, *i.e., close to* $k_n = n\pi - \omega$ $n \ge 1$, *where* $\omega = \frac{\nu\pi}{2} - \frac{\pi}{4}$.
(ii) *of* $J_\nu(x)$ *and the zeros of* $J_{\nu+1}(x)$ *separate each other;*
(iii) *of* $J_\nu(x)$ *and the zeros of* $J'_\nu(x)$ *separate each other;*
(iv) *of* $J'_\nu(x)$ *is an infinite set and are positive;*

Proof. We first comment on the part (i). It would be difficult to give a vigorous proof about the asymptotic locations of the large zeros of the Bessel functions without using complex analytic methods. In fact, the idea being to work on the reminders of the asymptotic expansions of the $J_\nu(x)$ given in the Theorem 9.1 above. Thus we will only be indicative that it follows from (9.56) that the zeros situate near

$$k\pi - \frac{\nu\pi}{2} + \frac{\pi}{4}, \qquad k \to \infty.$$

One can argue that there is one such zero near the above locations. For suppose that there are two distinct zeros cluster there, then it follow from Rolle's theorem that $J'_\nu(x)$ to have a zero in between the two zeros of $J_\nu(x)$. But this would contradict of the (9.58) and hence to $J'_\nu(x)$. Hence there can be at most one zero close to $k\pi - \frac{\nu\pi}{2} + \frac{\pi}{4}$. As a by-product we easily see that the $J'_\nu(x)$ has infinitely many zeros. Also since the $J'_\nu(x)/x^\nu$ is an even functions, so there are as many negative zeros as with the positive zeros. We recall from the formulae (9.25)

$$\frac{d}{dx}x^{-\nu}J_\nu(x) = -x^{-\nu}J_{\nu+1}(x).$$

and (9.23) that

$$\frac{d}{dx} x^{\nu+1} J_{\nu+1}(x) = x^{\nu+1} J_\nu(x).$$

Hence we deduce $J_{\nu+1}(x)$ must have a zero between any two consecutive zeros of $J_\nu(x)$, and that $J_\nu(x)$ must also have a zero between any two consecutive zeros of $J_{\nu+1}(x)$. Thus the zeros of the two functions interlace as asserted. This proves (ii). This also establishes (iii) and also (iv). □

Remark. We note that $J_\nu(x)$ and $J_{\nu+1}(x)$ cannot share a common non-zero zero, for otherwise, this would mean that both $J_\nu(x)$ and $J'_\nu(x)$ would share that common zero. but then the the $J_\nu(x) \equiv 0$ by the uniqueness of differential equations, or from the power series consideration.

Theorem 9.5. *Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the positive zeros of $J_\nu(x)$. Then*

(i) *$J_\nu(x) > 0$ over $(0,\, \lambda_1)$,*
(ii) *and that $J_\nu(x)$ changes signs alternatively from negative to positive between $(\lambda_1,\, \lambda_2)$ to $(\lambda_2,\, \lambda_3)$ and then from positive to negative from $(\lambda_2,\, \lambda_3)$ to $(\lambda_3,\, \lambda_4)$, etc,*
(iii) *the function $xJ'_\nu(x) - HJ_\nu(x)$, for each real H, alternately negative and positive at $x = \lambda_1,\, \lambda_2,\, \lambda_3,\, \cdots$.*

Proof. Part (i) follows from consideration of the power series of $J_\nu(x)$ for positive ν. Part (ii) follows since $J_\nu(\lambda_k) = 0$ but $J'_\nu(\lambda_k) \neq 0$. Hence $J'_\nu(\lambda_1) < 0$ because of (i). Thus, $J'_\nu(\lambda_2) > 0$, $J'_\nu(\lambda_3) < 0$, etc. Let $x = \lambda_k$. Then

$$xJ'_\nu(x) - HJ_\nu(x)\Big|_{x=\lambda_k} = \lambda_k J'_\nu(\lambda_k) \neq 0,$$

so that $xJ'_\nu(x) - HJ_\nu(x)$ alternates its signs on the sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. This also shows that $xJ'_\nu(x) - HJ_\nu(x)$ has infinitely many positive zeros. □

9.13. **Orthogonality.** We have

Theorem 9.6. *Let $\nu > -1$. Suppose λ and κ are two non-negative roots of $J_\nu(x)$. Then $J_\nu(\lambda x)$ and $z = J_\nu(\kappa x)$ are orthogonal with respect to the weight x over $(0,\, 1)$. That is, we have*

$$\int_0^1 J_\nu(\lambda x) J_\nu(\kappa x)\; x\, dx = 0. \tag{9.64}$$

Proof. Then it is easy to check that $y = J_\nu(\lambda x)$ and $z = J_\nu(\kappa x)$ satisfy the differential equations

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0$$

and

$$x^2 z'' + xz' + (\kappa^2 x^2 - \nu^2) z = 0$$

respectively, or, after dividing both sides by x:

$$xy'' + y' - \frac{\nu^2}{x}\, y = -\lambda^2 x\, y$$

and

$$xz'' + z' - \frac{\nu^2}{x}\, z = -\kappa^2 x\, z.$$

Let us multiply the first equation by z and the second equation by y. Then we subtract the two equations to obtain

$$x(yz'' - zy'') + (yz' - zy') = (\lambda^2 - \kappa^2)xyz,$$

or

$$x\frac{d}{dx}(yz' - zy') + (yz' - zy') = (\lambda^2 - \kappa^2)xyz,$$

so that

$$\frac{d}{dx}[x(yz' - zy')] = (\lambda^2 - \kappa^2)xyz.$$

We note from the series expansions of Bessel functions that

$$xyz = x^{2\nu+1}\Phi(x),$$

for some continuous (in fact even differentiable/analytic) function $\Phi(x)$. Hence we can find a constant $M > 0$ so that

$$|xyz = |x^{2\nu+1}\Phi(x)| \le Mx^{2\nu+1}, \qquad 0 \le x \le 1.$$

But $2\nu + 1 > -1$ so that the integrand

$$\begin{aligned}[x(yz' - zy')]\Big|_{x=0}^{x=1} &= (\lambda^2 - \kappa^2)\int_0^1 xyz\, dx \\ &= (\lambda^2 - \kappa^2)\int_0^1 x^{2\nu+2}\Phi(x)\, dx\end{aligned}$$

exists. But the left-side of the above equation

$$[x(yz' - zy')]\Big|_{x=0}^{x=1} = \kappa J_\nu(\lambda)J_\nu'(\kappa) - \lambda J_\nu(\kappa)J_\nu'(\lambda) - 0 = 0 - 0$$

obviously vanishes. However, the factor $(\lambda^2 - \kappa^2) \ne 0$ does not vanish on the right hand side. So we deduce

$$\int_0^1 J_\nu(\lambda x)J_\nu(\kappa x)\; x\, dx = 0,$$

as required. □

Remark. We note that the above argument also works if we replace the positive zeros of λ_k of $J_\nu(x)$ by the zeros λ_k' of $J_\nu'(x)$. A slightly more lengthy argument can also show that the zero sequence ξ_k of $xJ_\nu(x) - HJ_\nu'(x)$ again works. That are, both the $\{J_\nu(\lambda_k' x)\}$ and $\{xJ_\nu(\xi_k x) - HJ_\nu'(\xi_k x)\}$ are orthogonal families.

To be continued ...