MATH4822E FOURIER ANALYSIS AND APPLICATIONS
CHAPTER 4 CONVERGENCE OF FOURIER SERIES

5. Convergence of Fourier Series

We understand that a function f being *integrable* over the interval $[a, b]$ is to be interpreted in an elementary, that is, f is either continuous or have a finite number of discontinuities (which can be either either bounded or unbounded).

Definition. Let f be defined on $[a, b]$ with at most a finite number of discontinuities. Then f is said to be *absolutely integrable* if $|f(x)|$ is absolutely integrable over $[a, b]$. That is,

$$\int_a^b |f(x)|\, dx$$

exists.

We also recall that f being an absolutely integrable on $[a, b]$ must also be integrable there.

Lemma 5.1. *Let f be an absolutely integrable function on $[a, b]$. Then for each $\epsilon > 0$, there is piecewise continuous function g such that*

$$\int_a^b |f(x) - g(x)|\, dx \le \epsilon.$$

We skip the proof of this elementary lemma.

Theorem 5.2 (Riemann-Lebesgue Lemma)**.** *Let f be an absolutely integrable function on $[a, b]$. Then*

$$\lim_{m\to\infty} \int_a^b f(x)\, \cos mx\, dx = \lim_{m\to\infty} \int_a^b f(x)\, \sin mx\, dx = 0.$$

Proof. Let ϵ be given. Then the Lemma 5.1 asserts that there exists a piecewise continuous function g such that

$$\int_a^b |f(x) - g(x)|\, dx \le \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}\Big|\int_a^b f(x)\, \cos mx\; dx\Big| &\le \Big|\int_a^b \big(f(x) - g(x)\big)\, \cos mx\; dx\Big| + \Big|\int_a^b g(x) \cos mx\; dx\Big|\\ &\le \Big|\int_a^b \big(f(x) - g(x)\big)\; dx\Big| + \Big|\int_a^b g(x)\, \cos mx\, dx\Big|.\end{aligned}$$

Integration-by-parts yields

$$\int_a^b g(x)\, \cos mx\, dx = \frac{1}{m}\Big[g(x)\, \sin mx\Big]_a^b - \frac{1}{m}\int_a^b \sin mx\, g'(x)\; dx,$$

which is bounded, if we choose m to be sufficiently large. Hence

$$\Big|\int_a^b g(x) \cos mx\; dx\Big| < \frac{\epsilon}{2}.$$

for all m sufficiently large. Hence we deduce

$$\Big|\int_a^b f(x)\,\cos mx\;dx\Big| < \epsilon,$$

for all m sufficiently large. $\square$

Theorem 5.3 (Partial sum integral representation)**.** *Let f be periodic function of period 2π, and suppose that*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Let

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Then

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\;du$$

(*That is, an integral representation of the $n-$th partial sum*).

Proof. Let

$$s = \frac{1}{2} + \cos u + \cos 2u + \cdots + \cos nu.$$

Then

$$\begin{aligned}
2s\sin\frac{u}{2} &= \sin\frac{u}{2} + 2\sin\frac{u}{2}\,\cos u + 2\sin\frac{u}{2}\,\cos 2u + \cdots + 2\sin\frac{u}{2}\,\cos nu\\
&= \sin\frac{u}{2} + \Big(\sin\frac{3u}{2} - \sin\frac{u}{2}\Big) + \Big(\sin\frac{5u}{2} - \sin\frac{3u}{2}\Big) + \cdots + \Big(\sin\frac{(2n+1)u}{2} - \sin\frac{(2n-1)u}{2}\Big)\\
&= \sin\frac{(2n+1)u}{2}.
\end{aligned}$$

So

$$s = \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}.$$

On the other hand,

$$\int_{-\pi}^{\pi}\frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\;du = \int_{-\pi}^{\pi}\Big(\frac{1}{2} + \sum_{k=1}^{n}\cos ku\Big)\;du = \pi + 0 = \pi.$$

That is,

$$\begin{aligned}
1 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\;du\\
&= \frac{2}{\pi}\int_{0}^{\pi}\frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\;du,
\end{aligned}$$

since the integrand is even.

We rewrite

$$
\begin{aligned}
s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\Big[\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\cos kt\cdot\cos kx\,dt + \sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\sin kt\cdot\sin kx\,dt\Big]\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\,\frac{1}{2} + \sum_{k=1}^{n}(\cos kt\,\cos kx + \sin kt\,\sin kx)\Big]\,dt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\Big]\,dt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin(\frac{t-x}{2})}\,dt\\
&= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(u+x)\cdot\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \qquad \text{Let } u=t-x\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\cdot\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du
\end{aligned}
$$

since both $f(x+u)$ and $\dfrac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}$ have period 2π. □

Theorem 5.4. *Let f be absolutely integrable function with period 2π. Then at any continuity point where the right-hand and the left-hand derivatives exist, the Fourier series converges to $f(x)$. (In particular, the result holds if f has a derivative at x)*

Proof. Let x be a continuity point of f and

$$
\begin{aligned}
f'_+(x) &= \lim_{\substack{u\to 0\\ u>0}} \frac{f(x+u)-f(x+0)}{u}\\
f'_-(x) &= \lim_{\substack{u\to 0\\ u<0}} \frac{f(x+u)-f(x-0)}{u}
\end{aligned}
\tag{5.1}
$$

exist. We want to show that

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty} s_n(x)\\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}du.
\end{aligned}
$$

That is,

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\Big[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du - f(x)\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du\Big]\\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+u)-f(x)]\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du.
\end{aligned}
$$

Let

$$
\phi(u) = \frac{f(x+u)-f(x)}{2\sin\frac{u}{2}} = \frac{f(x+u)-f(x)}{u}\cdot\frac{u}{2\sin\frac{u}{2}} := \phi_1(u)\cdot\phi_2(u),
$$

where

$$
\phi_1(u) =: \frac{f(x+u)-f(x)}{u} \quad\text{and}\quad \phi_2(u) := \frac{u}{2\sin\frac{u}{2}}.
$$

We claim that ϕ is absolutely integrable over $[-\pi, \pi]$. That is,

$$\int_{-\pi}^{\pi} |\phi(u)|\, du < +\infty.$$

It is sufficient to show that $\phi_1(u)$ is absolutely integrable, since $\phi_2(u)$ is a bounded continuous function (being continuous) on $[-\pi, \pi]$. But then ϕ is a product of an absolutely integrable function and a bounded function, so it is also absolutely integrable. Now we need to verify that $\phi_1(u)$ is absolutely integrable. But it follows from (5.1) that both the left- and right-handed derivatives exist, so there is some $\delta > 0$ such that

$$|\phi_1(u)| = \left|\frac{f(x+u) - f(x)}{u}\right| \leq M$$

for some $M > 0$, on $[-\delta, \delta]$.

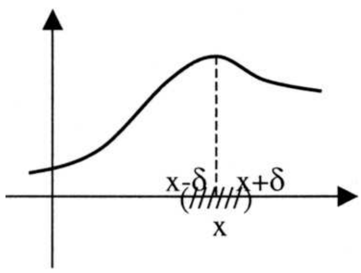

FIGURE 1

Since $f(x)$ and hence $f(x+u)$ are absolutely integrable, so we deduce that $\phi_1(u)$ (x is fixed) can have at most a finite number of discontinuities when $u \neq 0$. We conclude that it is absolutely integrable (with respect to u) on the interval $[-\delta, \delta]$. When u lies outside $[-\delta, \delta]$, we notice that

$$|\frac{1}{u}| \leq \frac{1}{\delta}$$

and that $f(x+u) - f(x)$ is clearly absolutely integrable (with respect to u). So $\phi(u)$ is a product of an absolutely integrable function and a bounded function (with respect to u) outside $[-\delta, \delta]$. So the $\phi_1(u)$ is absolutely integrable on $[-\pi, \pi]$. Hence $\phi(u)$ is absolutely integrable over $[-\pi, \pi]$. We deduce from the Riemann-Lebesgue Lemma (Theorem 5.2) that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \phi(u) \, \sin\big(n + \frac{1}{2}\big)u \; du = 0.$$

Rearranging the terms in the above equation yields the desired result. □

Theorem 5.5. *Let f be absolutely integrable function of period 2π. Then, at every point of discontinuity where f has a right-hand and left-hand derivatives, the Fourier series converges to*

$$\frac{f(x+0) + f(x-0)}{2}.$$

Proof. We aim to prove

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u)\, \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\, du = \frac{f(x+0) + f(x-0)}{2}.$$

It suffices to prove that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du=\frac{f(x+0)}{2} \tag{5.2}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{0} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du=\frac{f(x-0)}{2} \tag{5.3}$$

hold. We first note that

$$\frac{f(x+0)}{2}=\frac{1}{\pi}\int_0^{\pi} f(x+0)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du$$

Thus the (5.2) is equivalent to

$$\lim_{n\to\pi}\frac{1}{\pi}\int_0^{\pi}\Big[f(x+u)-\frac{f(x+0)}{2}\Big]\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du=0 \tag{5.4}$$

Again denote

$$\phi(u)=\frac{f(x+u)-f(x)}{2\sin\frac{u}{2}}=\frac{f(x+u)-f(x)}{u}\cdot\frac{u}{2\sin\frac{u}{2}}.$$

Since $f'_+(x)$ exists, then we argue as in the proof of Theorem 5.4 that the ratio $\dfrac{f(x+u)-f(x+0)}{u}$ is (bounded and) absolutely integrable when $u\to 0$ and $u>0$. Similarly, the ratio is also absolutely integrable on $[-\pi,\,\pi]$. Finally, we again argue as in Theorem 5.4 that $\phi(u)$ is a product of the absolutely integrable function above and the bounded function $\dfrac{u}{2\sin(\frac{u}{2})}$ on $[-\pi,\,\pi]$. Hence $\phi(u)$ is absolutely integrable on $[-\pi,\,\pi]$. We then use the Riemann-Lebesgue Lemma to show that (5.4) holds. A similar argument gives (5.3). We omit the details here. □

Theorem 5.6. *Let f be an absolutely integrable function of period 2π which is piecewise smooth on $[a,b]$. Then for all $x\in(a,b)$, the Fourier series converges to $f(x)$ at points of continuity, and to*

$$\frac{f(x+0)+f(x-0)}{2}$$

at points of discontinuity.

Proof. Since f is piecewise smooth on $[a,b]$, so the left-hand and right-hand derivatives exist and equal for each $x\in(a,b)$ except at at most a finite number of points, where either f has a "corner" or is discontinuous. If f has a "corner" at x, then it is easy to see that it has both left-hand and right-hand derivatives at x. In fact, one has, via the *mean value theorem* that

$$\lim_{\substack{u\to 0\\ u>0}}\frac{f(x+u)-f(x)}{u}=\lim_{\substack{u\to 0\\ u>0}} f'(\xi)=f'(x+0),$$

for $x<\xi<x+u$. The left-hand limit case can be dealt with similarly. Hence one can apply the Theorem 5.4 to arrive at the desired conclusion.

Suppose that f has a discontinuity at x. Then the mean value theorem again implies

$$\lim_{\substack{u\to 0\\ u>0}}\frac{f(x+u)-f(x+0)}{u}=\lim_{\substack{u\to 0\\ u>0}} f'(\zeta)=f'(x+0),$$

for $x < \zeta < x + u$. The discontinuity case follows from Theorem 5.5. $\square$

Remark. The above argument does not apply to the end points of the interval since only one-sided limits exist. However, one can extend f periodically beyond the 2π period so that the above theorem applies at these end points.

Theorem 5.7. *The Fourier series of a continuous, piecewise smooth function of f of period 2π converges to f absolutely and uniformly.*

Proof. Since f is continuous, piecewise smooth function of period 2π, so f' exists everywhere except at the "corner" of f, and is bounded on $[a, b]$. So the products $f'(x)\sin nx$ and $f'(x)\cos nx$ are absolutely integrable. Thus

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx &= \frac{1}{n\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\, dx \\ &= 0 - \frac{1}{n}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\, dx \\ &= -\frac{1}{n}\, b'_n \end{aligned}$$

where b'_n is the Fourier coefficient of $\sin nx$ of $f'(x)$. That is,

$$b'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\ \sin nx\ dx.$$

Similarly,

$$b_n = \frac{1}{n}\, a'_n = \frac{1}{n}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\, dx.$$

Since $f'(x)$ is bounded (except at a finite number of points) and so it is square integrable on $[-\pi, \pi]$. By the Bessel inequality,

$$\infty > \int_a^b f'(x)^2\, dx \geq \sum_{n=1}^{\infty} |a'_n|^2 + |b'_n|^2.$$

So

$$\sum_{n=1}^{\infty} {a'_n}^2 + {b'_n}^2$$

also converges. Notice that

$$0 \leq (|a'_k| - \frac{1}{k})^2 = |a'_k|^2 - \frac{2}{k}|a'_k| + \frac{1}{k^2}$$

and

$$0 \leq (|b'_k| - \frac{1}{k})^2 = |b'_k|^2 - \frac{2}{k}|b'_k| + \frac{1}{k^2}.$$

Hence,

$$\frac{|a'_k|}{k} + \frac{|b'_k|}{k} \leq \frac{1}{2}({a'_k}^2 + {b'_k}^2) + \frac{1}{k^2},$$

so

$$\sum_{k=1}^{\infty} |a_k| + |b_k| = \sum_{k=1}^{\infty} \frac{|a'_k|}{k} + \frac{|b'_k|}{k}$$

converges absolutely. We deduce

$$\Big|\sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx\Big| < \sum_{k=1}^{\infty} |a_k| + |b_k|.$$

By Weierstrass $M-$test, the Fourier series of f converges absolutely and uniformly to $f(x)$ for all x. $\square$

To be continued ...