MATH4822E FOURIER ANALYSIS AND APPLICATIONS
CHAPTER 5 CONVERGENCE OF FOURIER SERIES

5. Convergence of Fourier Series

We understand that a function f being *integrable* over the interval $[a,\, b]$ is to be interpreted in an elementary, that is, f is either continuous or have a finite number of discontinuities (which can be either either bounded or unbounded).

Definition. Let f be defined on $[a,\, b]$ with at most a finite number of discontinuities. Then f is said to be *absolutely integrable* if $|f(x)|$ is absolutely integrable over $[a,\, b]$. That is,

$$\int_a^b |f(x)|\, dx$$

exists.

We also recall that f being an absolutely integrable on $[a,\, b]$ must also be integrable there. Note that we require stronger integrability than the Riemann integrability since a Riemann integrable function may have an infinite number of discontinuities, while we allow for only a finite number of discontinuities for our absolute integrability. In this case, absolute integrability implies integrability.

Here is an example of an unbounded absolute integrable function.

Example. Consider $f(x) = -\ln|2\sin(\dfrac{x}{2})|$ on the real axis.

It becomes positive infinite when $x = 2k\pi$, $k = 0, \pm1, \pm2, \ldots$, but is otherwise an even continuous function. On the other hand, it is easy to see that f has period 2π:

$$f(x+2\pi) = -\ln\left|2\sin(\frac{x+2\pi}{2})\right| = -\ln\left|2\sin(\frac{x}{2}+\pi)\right| = -\ln\left|2\sin(\frac{x}{2})\right| = f(x).$$

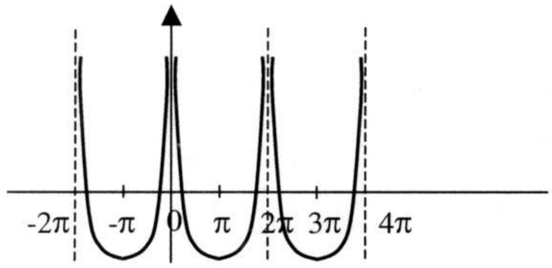

Figure 1

We show that f is absolutely integrable. Since f is periodic, so it suffice to show f to be absolutely integrable over the interval $[0,\, a]$, where $a < \frac{\pi}{3}$ (since $f(\frac{\pi}{3}) = 0$). Let $\epsilon > 0$. Then

$$\begin{aligned}\int_\epsilon^a \Big| -\ln\big|2\sin(\frac{x}{2})\big|\Big| &= -\int_\epsilon^a \ln\Big(2\sin(\frac{x}{2})\Big)\,dx\\ &= -x\ln\Big(2\sin(\frac{x}{2})\Big)\Big|_\epsilon^a + \int_\epsilon^a \frac{x\cos\frac{x}{2}}{2\sin\frac{x}{2}}\,dx\\ &= \Big[-a\ln(2\sin\frac{x}{2}) + \epsilon\ln(2\sin\frac{\epsilon}{2})\Big] + \frac{1}{2}\int_\epsilon^a x\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx.\end{aligned}$$

Since

$$\epsilon\ln(2\sin\frac{\epsilon}{2}) \sim \epsilon\ln\epsilon \to 0$$

as $\epsilon \to 0$, and

$$x\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} \sim \frac{x\cdot 1}{\frac{x}{2}} = 2$$

as $x \to 0$, so the last integral exists, and f is therefore absolutely integrable over $[-\pi,\,\pi]$.

Lemma 5.1. *Let f be an absolutely integrable function on $[a,\,b]$. Then for each $\epsilon > 0$, there is piecewise continuous function g such that*

$$\int_a^b |f(x) - g(x)|\,dx \le \epsilon.$$

We skip the proof of this elementary lemma.

Remark.
1. It is known that the product of an absolutely integrable function and a bounded integrable function is again absolutely integrable.
2. If f and g are continuous, piecewise smooth functions on $[a,\,b]$, then if f' and g' are absolutely integrable, we have

$$\int_a^b f(x)\,g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx.$$

Theorem 5.2 (Riemann-Lebesgue Lemma)**.** *Let f be an absolutely integrable function on $[a,\,b]$. Then*

$$\lim_{m\to\infty}\int_a^b f(x)\,\cos mx\,dx = \lim_{m\to\infty}\int_a^b f(x)\,\sin mx\,dx = 0.$$

Proof. Let ϵ be given. Then the Lemma 5.1 asserts that there exists a piecewise continuous function g such that

$$\int_a^b |f(x) - g(x)|\,dx \le \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}\Big|\int_a^b f(x)\,\cos mx\,dx\Big| &\le \Big|\int_a^b \big(f(x) - g(x)\big)\,\cos mx\,dx\Big| + \Big|\int_a^b g(x)\cos mx\,dx\Big|\\ &\le \Big|\int_a^b \big(f(x) - g(x)\big)\,dx\Big| + \Big|\int_a^b g(x)\,\cos mx\,dx\Big|.\end{aligned}$$

Integration-by-parts yields

$$\int_a^b g(x)\,\cos mx\,dx = \frac{1}{m}\Big[g(x)\,\sin mx\Big]_a^b - \frac{1}{m}\int_a^b \sin mx\,g'(x)\,dx,$$

which is bounded, if we choose m to be sufficiently large (the g' is continuous and hence absolutely integrable). Hence

$$\Big| \int_a^b g(x) \cos mx \; dx \Big| < \frac{\epsilon}{2}.$$

for all m sufficiently large. Hence we deduce

$$\Big| \int_a^b f(x) \cos mx \; dx \Big| < \epsilon,$$

for all m sufficiently large. □

Remark. We note that this lemma implies that the Fourier coefficients of an absolutely integrable function tend to zero as $m \to \infty$.

Theorem 5.3 (Partial sum integral representation)**.** *Let f be periodic function of period 2π, and suppose that*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Let

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Then

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})} \; du$$

(*That is, an integral representation of the $n-$th partial sum*).

Proof. Let

$$s = \frac{1}{2} + \cos u + \cos 2u + \cdots + \cos nu.$$

Then

$$\begin{aligned} 2s \sin\frac{u}{2} &= \sin\frac{u}{2} + 2\sin\frac{u}{2}\,\cos u + 2\sin\frac{u}{2}\,\cos 2u + \cdots + 2\sin\frac{u}{2}\,\cos nu \\ &= \sin\frac{u}{2} + \Big(\sin\frac{3u}{2} - \sin\frac{u}{2}\Big) + \Big(\sin\frac{5u}{2} - \sin\frac{3u}{2}\Big) + \cdots + \Big(\sin\frac{(2n+1)u}{2} - \sin\frac{(2n-1)u}{2}\Big) \\ &= \sin\frac{(2n+1)u}{2}. \end{aligned}$$

So

$$s = \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}.$$

On the other hand,

$$\int_{-\pi}^{\pi} \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})} \; du = \int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos ku\Big) \; du = \pi + 0 = \pi.$$

That is,

$$\begin{aligned}
1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\, du \\
&= \frac{2}{\pi}\int_{0}^{\pi} \frac{\sin[(n+\frac{1}{2})u]}{2\sin(\frac{u}{2})}\, du,
\end{aligned}$$

since the integrand is even.

We rewrite

$$\begin{aligned}
s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\Big[\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\cos kt\cdot\cos kx\,dt + \sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\sin kt\cdot\sin kx\,dt\Big] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt\,\cos kx + \sin kt\,\sin kx)\Big]\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\Big]\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin(\frac{t-x}{2})}\,dt \\
&= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(u+x)\cdot\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \qquad \text{Let } u=t-x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\cdot\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du
\end{aligned}$$

since both $f(x+u)$ and $\dfrac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}$ have period 2π. □

Theorem 5.4. *Let f be absolutely integrable function with period 2π. Then at any continuity point where the right-hand and the left-hand derivatives exist, the Fourier series converges to $f(x)$. (In particular, the result holds if f has a derivative at x)*

Proof. Let x be a continuity point of f and

$$\begin{aligned}
f'_+(x) &= \lim_{\substack{u\to 0\\ u>0}} \frac{f(x+u)-f(x+0)}{u} \\
f'_-(x) &= \lim_{\substack{u\to 0\\ u<0}} \frac{f(x+u)-f(x-0)}{u}
\end{aligned} \tag{5.1}$$

exist. We want to show that

$$\begin{aligned}
f(x) &= \lim_{n\to\infty} s_n(x) \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}du.
\end{aligned}$$

That is,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \Big[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\ \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du - f(x)\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du\Big] \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+u)-f(x)]\ \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\ du.
\end{aligned}
$$

Let

$$
\phi(u) = \frac{f(x+u)-f(x)}{2\sin\frac{u}{2}} = \frac{f(x+u)-f(x)}{u}\cdot\frac{u}{2\sin\frac{u}{2}} := \phi_1(u)\cdot\phi_2(u),
$$

where

$$
\phi_1(u) =: \frac{f(x+u)-f(x)}{u} \quad \text{and} \quad \phi_2(u) := \frac{u}{2\sin\frac{u}{2}}.
$$

We claim that ϕ is absolutely integrable over $[-\pi,\,\pi]$. That is,

$$
\int_{-\pi}^{\pi}|\phi(u)|\,du < +\infty.
$$

It is sufficient to show that $\phi_1(u)$ is absolutely integrable, since $\phi_2(u)$ is a bounded continuous function (being continuous) on $[-\pi,\,\pi]$. But then ϕ is a product of an absolutely integrable function and a bounded function, so it is also absolutely integrable. Now we need to verify that $\phi_1(u)$ is absolutely integrable. But it follows from (5.1) that both the left- and right-handed derivatives exist, so there is some $\delta > 0$ such that

$$
|\phi_1(u)| = \Big|\frac{f(x+u)-f(x)}{u}\Big| \le M
$$

for some $M > 0$, on $[-\delta,\,\delta]$.

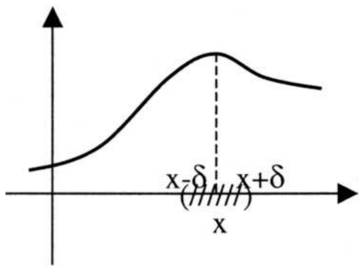

FIGURE 2

Since $f(x)$ and hence $f(x+u)$ are absolutely integrable, so we deduce that $\phi_1(u)$ (x is fixed) can have at most a finite number of discontinuities when $u \neq 0$. We conclude that it is absolutely integrable (with respect to u) on the interval $[-\delta,\,\delta]$. When u lies outside $[-\delta,\,\delta]$, we notice that

$$
|\frac{1}{u}| \le \frac{1}{\delta}
$$

and that $f(x+u)-f(x)$ is clearly absolutely integrable (with respect to u). So $\phi(u)$ is a product of an absolutely integrable function and a bounded function (with respect to u) outside $[-\delta,\,\delta]$. So the $\phi_1(u)$ is absolutely integrable on $[-\pi,\,\pi]$. Hence $\phi(u)$ is absolutely integrable over $[-\pi,\,\pi]$. We deduce from the Riemann-Lebesgue Lemma (Theorem 5.2) that

$$
\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\ \sin\big(n+\frac{1}{2}\big)u\ du = 0.
$$

Rearranging the terms in the above equation yields the desired result. □

Theorem 5.5. *Let f be absolutely integrable function of period 2π. Then, at every point of discontinuity where f has a right-hand and left-hand derivatives, the Fourier series converges to*

$$\frac{f(x+0)+f(x-0)}{2}.$$

Proof. We aim to prove

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du = \frac{f(x+0)+f(x-0)}{2}.$$

It suffices to prove that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{0}^{\pi} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du = \frac{f(x+0)}{2} \tag{5.2}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{0} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du = \frac{f(x-0)}{2} \tag{5.3}$$

hold. We first note that

$$\frac{f(x+0)}{2} = \frac{1}{\pi}\int_{0}^{\pi} f(x+0)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du$$

Thus the (5.2) is equivalent to

$$\lim_{n\to\pi}\frac{1}{\pi}\int_{0}^{\pi}\left[f(x+u)-\frac{f(x+0)}{2}\right]\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du = 0 \tag{5.4}$$

Again denote

$$\phi(u) = \frac{f(x+u)-f(x)}{2\sin\frac{u}{2}} = \frac{f(x+u)-f(x)}{u}\cdot\frac{u}{2\sin\frac{u}{2}}.$$

Since $f'_+(x)$ exists, then we argue as in the proof of Theorem 5.4 that the ratio $\dfrac{f(x+u)-f(x+0)}{u}$ is (bounded and) absolutely integrable when $u\to 0$ and $u>0$. Similarly, the ratio is also absolutely integrable on $[-\pi,\,\pi]$. Finally, we again argue as in Theorem 5.4 that $\phi(u)$ is a product of the absolutely integrable function above and the bounded function $\dfrac{u}{2\sin(\frac{u}{2})}$ on $[-\pi,\,\pi]$. Hence $\phi(u)$ is absolutely integrable on $[-\pi,\,\pi]$. We then use the Riemann-Lebesgue Lemma to show that (5.4) holds. A similar argument gives (5.3). We omit the details here. □

Theorem 5.6. *Let f be an absolutely integrable function of period 2π which is piecewise smooth on $[a,b]$. Then for all $x\in(a,b)$, the Fourier series converges to $f(x)$ at points of continuity, and to*

$$\frac{f(x+0)+f(x-0)}{2}$$

at points of discontinuity.

Proof. Since f is piecewise smooth on $[a,b]$, so the left-hand and right-hand derivatives exist and equal for each $x\in(a,b)$ except at at most a finite number of points, where either f has a "corner" or is discontinuous. If f has a "corner" at x, then it is easy to see that it has both left-hand and right-hand derivatives at x. In fact, one has, via the *mean value theorem* that

$$\lim_{\substack{u\to 0\\ u>0}} \frac{f(x+u)-f(x)}{u} = \lim_{\substack{u\to 0\\ u>0}} f'(\xi) = f'(x+0),$$

for $x < \xi < x+u$. The left-hand limit case can be dealt with similarly. Hence one can apply the Theorem 5.4 to arrive at the desired conclusion.

Suppose that f has a discontinuity at x. Then the mean value theorem again implies

$$\lim_{\substack{u\to 0\\ u>0}} \frac{f(x+u)-f(x+0)}{u} = \lim_{\substack{u\to 0\\ u>0}} f'(\zeta) = f'(x+0),$$

for $x < \zeta < x+u$. The discontinuity case follows from Theorem 5.5. □

Remark. The above argument does not apply to the end points of the interval since only one-sided limits exist. However, one can extend f periodically beyond the 2π period so that the above theorem applies at these end points.

Example. Let us revisit the example $f(x) = -\ln|2\sin(\frac{x}{2})|$.

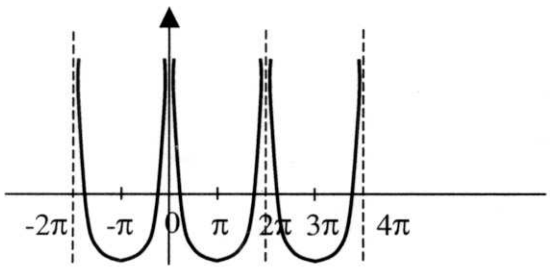

FIGURE 3

We have shown that f is absolutely integrable over $[-\pi,\pi]$. Thus,
Theorem 5.5 asserts that the Fourier series of f converges to f at every continuity point and to

$$\frac{1}{2}[f(x+0)+f(x-0)]$$

at every discontinuity point that has a left-hand and right-hand derivatives. It is clear that this f do not have points of this second category. Thus the above results do not apply to points at $2k\pi$, $k = 0, \pm1, \pm2, \dots$ However, since the Fourier series is given by (check!), we have

$$-\ln\left|2\sin\frac{x}{2}\right| = \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots, \qquad x \neq 2k\pi.$$

However, we see that both sides become infinite at $x = 2k\pi$, $k = 0, \pm1, \pm2, \dots$. So we may still regard that the Fourier series is value for all x.

Putting $x = \pi$ in the Fourier series above, we obtain

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which is the *harmonic sum.*

Theorem 5.7. *The Fourier series of a continuous, piecewise smooth function of f of period 2π converges to f absolutely and uniformly.*

Proof. Since f is continuous, piecewise smooth function of period 2π, so f' exists everywhere except at the "corner" of f, and is bounded on $[a, b]$. So the products $f'(x)\sin nx$ and $f'(x)\cos nx$ are absolutely integrable. Thus

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx &= \frac{1}{n\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx \\ &= 0 - \frac{1}{n}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx \\ &= -\frac{1}{n}\, b_n' \end{aligned}$$

where b_n' is the Fourier coefficient of $\sin nx$ of $f'(x)$. That is,

$$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\ \sin nx\ dx.$$

Similarly,

$$b_n = \frac{1}{n}\, a_n' = \frac{1}{n}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx.$$

Since $f'(x)$ is bounded (except at a finite number of points) and so it is square integrable on $[-\pi,\, \pi]$. By the Bessel inequality,

$$\infty > \int_a^b f'(x)^2\,dx \geq \sum_{n=1}^{\infty} |a_n'|^2 + |b_n'|^2.$$

So

$$\sum_{n=1}^{\infty} {a_n'}^2 + {b_n'}^2$$

also converges. Notice that

$$0 \leq (|a_k'| - \frac{1}{k})^2 = |a_k'|^2 - \frac{2}{k}|a_k'| + \frac{1}{k^2}$$

and

$$0 \leq (|b_k'| - \frac{1}{k})^2 = |b_k'|^2 - \frac{2}{k}|b_k'| + \frac{1}{k^2}.$$

Hence,

$$\frac{|a_k'|}{k} + \frac{|b_k'|}{k} \leq \frac{1}{2}({a_k'}^2 + {b_k'}^2) + \frac{1}{k^2},$$

so

$$\sum_{k=1}^{\infty} |a_k| + |b_k| = \sum_{k=1}^{\infty} \frac{|a_k'|}{k} + \frac{|b_k'|}{k}$$

converges absolutely. We deduce

$$\Big|\sum_{k=1}^{\infty} a_k\cos kx + b_k\sin kx\Big| < \sum_{k=1}^{\infty} |a_k| + |b_k|.$$

By Weierstrass $M-$test, the Fourier series of f converges absolutely and uniformly to $f(x)$ for all x. □

Lemma 5.8. *Let f be a continuous function of period 2π, which has an absolutely integrable derivative on $[a, b]$, and let $\omega(u)$ be a continuously differentiable function on $[a, b]$. Then given any $\epsilon > 0$, there is N such that*

$$\Big| \int_a^b f(x+u)\,\omega(u)\,\sin(mu)\,du \Big| < \epsilon$$

whenever $m \geq N$ for all x.

Proof. Integration-by-parts yields

$$\int_a^b f(x+u)\,\omega(u)\,\sin mu\,du = \frac{1}{m}\Big(-f(x+u)\omega(u)\cos mu\Big|_a^b\Big) + \frac{1}{m}\int_a^b [f(x+u)\omega(u)]'\,\cos mu\,du,$$

where

$$[f(x+u)\omega(u)]' = f'(x+u)\,\omega(u) + f(x+u)\,\omega'(u).$$

Since $\omega(u)$ and $f(x+u)\omega'(u)$ are bounded, that is,

$$|\omega(u)| \leq M \qquad \text{and} \qquad |f(x+u)\,\omega'(u)| \leq M,$$

for some $M > 0$. Hence

$$\begin{aligned}\Big| \int_a^b [f(x+u)\,\omega(u)]'\cos mu\,du\Big| &\leq M\int_a^b |f'(x+u)|\,du + M(b-a)\\ &\leq M\int_{-\pi}^{\pi} |f'(u)|\,du + M(b-a) < \infty.\end{aligned}$$

Assume that $b - a \leq 2\pi$. Therefore

$$\begin{aligned}\Big| \int_a^b f(x+u)\,\omega(u)\,\sin mu\,du\Big| &\leq \Big|\frac{1}{m}\Big(-f(x+u)\omega(u)\cos mu\Big|_a^b\Big)\Big|\\ &\quad + \frac{1}{m}\Big|\int_a^b [f(x+u)\omega(u)]'\,\cos mu\,du\Big|\\ &\to 0\end{aligned}$$

since ω' is continuous. □

Lemma 5.9. *The integral*

$$I = \int_0^u \frac{\sin(mt)}{2\sin(\frac{t}{2})}\,dt$$

is bounded for $-\pi \leq u \leq \pi$ for any m.

Proof. We note that

$$
\begin{aligned}
I & \int_0^u \frac{\sin(mt)}{2\sin(\frac{t}{2})}\,dt \\
& \int_0^u \frac{\sin mt}{t} - \frac{\sin mt}{t} + \frac{\sin(mt)}{2\sin(\frac{t}{2})}\,dt \\
&= \int_0^u \frac{\sin mt}{t} + \Big(\frac{1}{2\sin t/2} - \frac{1}{t}\Big)\,\sin mt\;dt \\
&= \int_0^u \frac{\sin mt}{t}\,dt + \int_0^u \omega(t)\,\sin mt\;dt
\end{aligned}
$$

where

$$\omega(t) = \frac{1}{2\sin t/2} - \frac{1}{t}.$$

Since we have $\frac{\sin(\alpha t)}{t} \to \alpha$ as $t \to 0$, so

$$\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t} \to \frac{1}{2(t/2)} - \frac{1}{t},$$

as $t \to 0$. So the second integral above is bounded. On the other hand, we can rewrite the first integral as

$$\int_0^u \frac{\sin mt}{t}\,dt = \int_0^{mu} \frac{\sin x}{x}\,dx$$

after the substitution $x = mt$. It is left as an exercise to show that this integral is also bounded. □

Theorem 5.10. *The Fourier series of a continuous function f of period 2π with an absolutely integrable derivative converges uniformly to $f(x)$ for all x.*

Proof. We apply the partial sum formula from Theorem 5.3 to consider the difference

$$
\begin{aligned}
s_n(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x+u) - f(x)\big)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du \\
&= \frac{1}{\pi}\Big(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\Big)\big(f(x+u) - f(x)\big)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du \\
&:= \frac{1}{\pi}\,(I_1 + I_2 + I_3),
\end{aligned}
$$

where $0 < \delta < \pi$. Let ϵ be given. Integration-by-parts yields

$$
\begin{aligned}
I_2 &= \int_{-\delta}^{\delta} \big(f(x+u) - f(x)\big)\,\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du \\
&= [f(x+u) - f(x)]\Big(\int_0^u \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\Big)\Big|_{-\delta}^{\delta} - \int_{-\delta}^{\delta} f'(x+u)\cdot\Big(\int_0^u \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\Big)\,du.
\end{aligned}
$$

We consider the first term above:

$$
\begin{aligned}
&[f(x+u)-f(x)]\Big(\int_0^u \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\Big)\Big|_{-\delta}^{\delta}\\
&=[f(x+\delta)-f(x)]\int_0^{\delta}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt-[f(x-\delta)-f(x)]\int_0^{-\delta}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\\
&=[f(x+\delta)-f(x)]\int_0^{\delta}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt+[f(x-\delta)-f(x)]\int_0^{\delta}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\\
&=[f(x+\delta)+f(x-\delta)-2f(x)]\int_0^{\delta}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}dt<\frac{\epsilon}{2}
\end{aligned}
$$

when $\delta>0$ is chosen to be sufficiently small since $f(x)$ is continuous by our assumption and the definite integral is bounded by the Lemma 5.9. We apply the Lemma 5.9 to the second integral above to deduce

$$\Big|\int_{-\delta}^{\delta} f'(x+u)\Big(\int_0^u \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\Big)\,du\Big|\le M\int_{-\delta}^{\delta}|f'(x+u)|\,du$$

for a suitable $M>0$. This integral is clearly bounded since f' is absolutely integrable, also

$$M\int_{-\delta}^{\delta}|f'(x+u)|\,du<\frac{\epsilon}{2}$$

if $\delta>0$ is sufficiently small. Therefore, $I_2<\epsilon$. Next the Lemma 5.8 implies that

$$
\begin{aligned}
|I_3|&=\Big|\int_{\delta}^{\pi}\big(f(x+u)-f(x)\big)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}\,du\Big|\\
&\le\Big|\int_{\delta}^{\pi} f(x+u)\,\omega(u)\,\sin(n+\frac{1}{2})u\,du\Big|+\Big|\int_{\delta}^{\pi} f(x)\,\omega(u)\,\sin(n+\frac{1}{2})u\,du\Big|\\
&<\epsilon,
\end{aligned}
$$

where $\omega(u)=\frac{1}{2\sin(\frac{u}{2})}$ (which is clearly continuously differentiable over $[\delta,\,\pi]$) provided that n is chosen sufficiently large. We skip the derivation for I_1 which is similar. Hence, given $\epsilon>0$ we can find a N such that

$$|s_n(x)-f(x)|\le\frac{|I_1|+|I_2|+|I_3|}{\pi}\le\frac{3\epsilon}{\pi}<\epsilon$$

for all x provided $n>N$. □

Theorem 5.11. *Let f be an absolutely integrable function of period 2π, which is continuous and has an absolutely integrable derivative on $[a,\,b]$. Then the Fourier series of f converges uniformly to $f(x)$ on each interval $[a+\delta,\,b-\delta]$, where $\delta>0$.*

Proof. If the length of $[a,b]$ is not less than 2π, it is clear that $f(x)$ is continuous for all x and has an absolutely integrable derivative, so by Theorem 4.10, the convergence of its Fourier series is uniform on the whole $x-$axis. Assume $b-a<2\pi$, let $F(x)$ be a continuous function of period 2π, which equals $f(x)$ for $a\le x\le b$, equals $f(a)$ for $x=a+2\pi$, and is linear on the interval $[b,a+2\pi]$. Outside $[a,a+2\pi]$, the values of $F(x)$ are obtained by periodic extension. $F(x)$ has an absolutely integrable derivative. Let $\phi(x)=f(x)-F(x)$, where $\phi(x)$ is absolutely integrable and $\phi(x)=0$ for $a\le x\le b$.

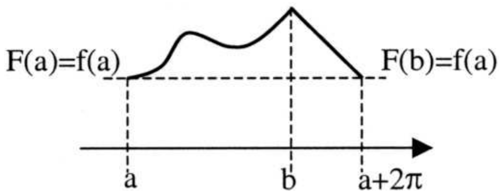

FIGURE 4

Then $f(x) = \phi(x) + F(x)$, and

$$\begin{aligned} s_n(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+u) - F(x)]\frac{\sin mu}{2\sin(\frac{u}{2})}\ du + \frac{1}{\pi}\int_{-\pi}^{\pi}[\phi(x+u) - \phi(x)]\frac{\sin mu}{2\sin(\frac{u}{2})}\ du \\ &= I_1 + I_2, \end{aligned}$$

where we have set $m = n + \dfrac{1}{2}$.

Let $\epsilon > 0$ be arbitrary. By Theorem 5.10, the Fourier series of $F(x)$ converges uniformly to $F(x)$, so that

$$|I_1| \leq \frac{\epsilon}{2}$$

for all x, provided that n is sufficiently large.

Now let $a + \delta \leq x \leq b - \delta$. Then $\phi(x) = 0$ and therefore

$$I_2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x+u)\frac{\sin mu}{2\sin(\frac{u}{2})}\ du.$$

If $-\delta \leq u \leq \delta$, we have

$$a \leq x + u \leq b$$

and hence $\phi(x+u) = 0$. Therefore,

$$I_2 = \frac{1}{\pi}\int_{-\pi}^{-\delta}\phi(x+u)\frac{\sin mu}{2\sin(\frac{u}{2})}\ du + \frac{1}{\pi}\int_{\delta}^{\pi}\phi(x+u)\frac{\sin mu}{2\sin(\frac{u}{2})}\ du.$$

By Lemma 5.8, $|I_2| < \frac{\epsilon}{2}$ for $a + \delta \leq x \leq b - \delta$ and n is sufficiently large.

Hence,

$$|s_n(x) - f(x)| \leq |I_1| + |I_2| < \epsilon$$

for all x in the interval $[a + \delta,\, b - \delta]$, and n is sufficiently large. □

EXERCISES

Q1 Show that the Theorem 5.4 can be proved if we assume $f(x)$ is an absolutely integrable function of period 2π and if f is continuous at x_0 and that there exist positive constants c and α such that

$$|f(x) - f(x_0)| \leq c|x - x_0|^{\alpha}$$

for all x in some neighbourhood of x_0, then the Fourier series of $f(x)$ converges to $f(x_0)$ at $x = x_0$.

Q2 Let $f(x)$ and $g(x)$ be absolutely integrable functions with period 2π, and whose Fourier series are

$$f(x) \sim \sum_{n=1}^{\infty} a_n e^{inx}, \qquad g(x) \sim \sum_{n=1}^{\infty} b_n e^{inx}.$$

Let

$$h(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)\, g(t)\, dt,$$

and

$$h(x) \sim \sum_{n=1}^{\infty} c_n e^{inx}.$$

Show that

$$\frac{1}{2\pi} \int_0^{2\pi} |h(x)|\, dx \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|\, dx \right) \left(\frac{1}{2\pi} \int_0^{2\pi} |g(x)|\, dx \right),$$

and that $c_n = a_n b_n$. In particular, if f and g are square integrable, show that

$$\sum_{n=1}^{\infty} |c_n| < +\infty.$$

Q3 Assuming the following Fourier series expansion

$$f(x) = \frac{\pi - x}{2} = \sum_{k=1}^{\infty} \frac{\sin kx}{k}.$$

Let $s_n(x)$ be the $n-$th partial sum of the series:

$$s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

Let

$$D_n(x) \frac{\sin(n + 1/2)x}{2\sin(x/2)}.$$

Show that

- (a)

$$\frac{x}{2} + s_n(x) = \int_0^x D_n(t)\, dt,$$

- (b)

$$\int_0^x D_n(t)\, dt = \int_0^x \frac{\sin nt}{t}\, dt + \omega_n(x) = \int_0^{nx} \frac{\sin t}{t}\, dt + \omega_n(x),$$

where $\omega_n(x) \to 0$ as $n \to \infty$;

- (c)

$$\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Q4 Using the notation in Q3 to show that

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{n}\right) = \int_0^{\pi} \frac{\sin t}{t}\, dt > \int_0^{\infty} \frac{\sin t}{t}\, dt.$$