MATH4822E FOURIER ANALYSIS AND APPLICATIONS CHAPTER 2 POWER SERIES AND CONVERGENCE

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2. Introduction

Theorem 2.1. *Suppose that the series* $\sum a_n x_0^n$ *converges for some* $x_0 > 0$*. Then the two series* $\sum_{n=0}^{\infty} a_n z^n$ *and* $\sum_{n=1}^{\infty} n a_n z^{n-1}$ *both converge absolutely for* $|x| < x_0$*.*

Proof. Since $a_n x_0^n \to 0$ as $n \to \infty$, we can find a $M > 0$ such that

$$a_n x_0^n \le M \qquad \text{for all } n.$$

For $|x| < x_0$,

$$|a_n x^n| = \left| a_n x_0^n \Big(\frac{x}{x_0}\Big)^n \right| \le M \left| \frac{x}{x_0} \right|^n \qquad \text{for all } n.$$

Hence $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely by comparison test. We leave the case $\sum_{n=1}^{\infty} n a_n z^{n-1}$ as an exercise. □

Theorem 2.2. *A power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

either

(i) *converges for all* x *(absolutely); or*

(ii) *there exists a* $R > 0$ *such that* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $|x| < R$ *and diverges for* $|x| > R$*; or*

(iii) *it converges only for* $x = 0$*.*

Proof. Let $S = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n z^n \text{ converges}\}$ for $|z| = x$. Since $0 \in S$, so S is non-empty. Suppose S is unbounded, then given any x_0, we have

$$|x_0| < x, \qquad \text{where } x \in S,$$

hence $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $z = x_0$. Suppose S is bounded above. Then $R = \sup S$ exists. If $R = 0$, then we proves (iii), so suppose $R > 0$. For any $|x_0| < R$ there exists $x \in S$

such that $|x_0| < x$. Then the Theorem 2.1 implies that $\sum_{n=0}^{\infty} a_n x_0^n$ converges absolutely. Moreover, if $|x_0| > R$, then $x_0 \notin S$ and hence $\sum_{n=0}^{\infty} a_n x_0^n$ diverges. □

Definition. The number R is called the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 2.3. *If the power series*

$$\sum_{n=0}^{\infty} a_n x^n = f(x)$$

has radius of convergence R. Then

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

also has the same radius of convergence, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof. From Theorem 2.1, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges for $|x| < R$. For $|x| > R$, $\sum_{n=1}^{\infty} a_n x^n$ diverges, so the terms $\{a_n x^n\}$ are note bounded (Hint: Theorem 2.1). Hence $n a_n x^{n-1}$ are surely unbounded, so $\sum_{n=1}^{\infty} n a_n x^{n-1}$ does not converge. Hence $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.

To prove $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, it is sufficient to make

$$\Big|\frac{f(x+h) - f(x)}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1}\Big|$$

small when h is small. Let $\epsilon > 0$ be given. We write

$$\begin{aligned}
\Big|\frac{f(x+h) - f(x)}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1}\Big| &= \Big|\sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1}\Big| \\
&\le \Big|\sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n=1}^{N} a_n \frac{(x+h)^n - x^n}{h}\Big| \qquad (1)\\
&+ \Big|\sum_{n=1}^{N} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n=1}^{N} n a_n x^{n-1}\Big| \qquad (2)\\
&+ \Big|\sum_{n=1}^{N} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1}\Big| \qquad (3)
\end{aligned}$$

We show that for a given $\epsilon > 0$, each of the three terms can be made less than $\frac{\epsilon}{3}$ by choosing N sufficiently large and h sufficiently small. To begin with, we choose x_0, $|x_0| < R$ and h such that $|x| < |x_0| < R$ and $|x+h| < |x_0| < R$. Notice that

$$\frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}.$$

by the identity $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots b^{n-1})$. Hence

$$\Big|\frac{(x+h)^n - x^n}{h}\Big| = \Big|\frac{(x+h)^n - x^n}{(x+h) - x}\Big| = \Big|\underbrace{(x+h)^{n-1} + x_0(x+h)^{n-2} + \cdots + x^{n-1}}_{\text{n terms}}\Big|$$
$$\leq n|x_0|^{n-1}.$$

Hence for (1), we have

$$\Big|\sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n=1}^{N} a_n \frac{(x+h)^n - x^n}{h}\Big| = \Big|\sum_{n=N+1}^{\infty} a_n \frac{(x+h)^n - x^n}{h}\Big| \leq \sum_{n=N+1}^{\infty} |a_n| n |x_0|^{n-1}.$$

But

$$\sum_{n=1}^{\infty} n|a_n||x_0|^{n-1}$$

is convergent, so we may choose N sufficiently large such that

$$\sum_{n=N+1}^{\infty} n|a_n||x_0|^{n-1} < \frac{\epsilon}{3}.$$

We may use a similar method to prove (3).

For (2), the polynomial

$$g(x) = \sum_{n=0}^{N} a_n x^n$$

is certainly differentiable. Therefore we may choose h so small such that

$$\Big|\sum_{n=1}^{N} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n=1}^{N} n a_n x^{n-1}\Big| < \frac{\epsilon}{3}$$

since

$$\lim_{n\to 0} \sum_{n=1}^{N} a_n \frac{(x+h)^n - x^n}{h} = \sum_{n=1}^{N} n a_n x^n.$$

This proves the Theorem. □

Example. From the series definition of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

We obtain, by Theorem 2.3,

$$\frac{d}{dx} e^x = e^x,$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\sin x = \cos x.$$

Theorem 2.4. *(Some properties of* $\sin x$ *and* $\cos x$*)*

Let sine and cosine be defined by the above series expansion. Then

(i) $\sin 0 = 0, \quad \cos 0 = 1$

(ii) $\sin(x+y) = \sin x \cos y + \sin y \cos x$

(iii) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

(iv) $\sin^2 x + \cos^2 x = 1.$

Proof. (i) From the series definition.

(ii) Let $f(t) = \sin(a+t)\cos(b-t) + \cos(a+t)\sin(b-t)$. Then

$$\begin{aligned} f'(t) &= \cos(a+t)\cos(b-t) + \sin(a+t)\sin(b-t) \\ &\quad + -\sin(a+t)\sin(b-t) - \cos(a+t)\cos(b-t) = 0 \end{aligned}$$

Hence

$$f(t) = \text{ constant } = f(0) = f(b).$$

But

$$f(0) = \sin a \cos b + \cos a \sin b$$
$$f(b) = \sin(a+b).$$

Hence we obtain (ii).

(iii) To prove (iii): as above or differentiate result result of (ii) with respect to x.

(iv)

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = 2\sin x \cos x - 2\cos x \sin x = 0.$$

Hence

$$\sin^x + \cos^2 x = \text{ constant } = \sin^2 0 + \cos^2 0 = 1.$$

□

Theorem 2.5. *There exists a unique number* α *with* $\sqrt{2} < \alpha < \sqrt{3}$ *such that* $\cos\alpha = 0$. *Also if* $0 \le x < \alpha$, *then* $\cos x > 0$.

Proof. Consider

$$\cos x = \Big(1 - \frac{x^2}{2!}\Big) + \Big(\frac{x^4}{4!} - \frac{x^6}{6!}\Big) + \cdots \tag{2.1}$$

Note that

$$\frac{x^{n+2}}{(n+2)!} < \frac{x^n}{n!} \qquad \text{if } x^2 < (n+1)(n+2).$$

Thus if $0 \le x \le \sqrt{2}$ the first bracketed pair of (2.1) above is 0 and the others are positive. Hence if s_n denotes the partial sum of the series (2.1), then

$$a_1 - a_2 + a_3 - a_4 + \ldots$$

and

$$a_1 - a_2 < s_n < a_1 = 1.$$

By Leibniz's theorem, if $0 < x < \sqrt{2}$, we have

$$1 - \frac{x^2}{2} < \cos x < 1,$$

so that

$$0 < \cos x < 1.$$

Leibniz's Theorem

Let $\{a_n\}$ be a sequence of positive numbers such that $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Then the infinite sum

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

has a limit. Moreover,

$$a_1 - a_2 < \sum_{n=1}^{N} (-1)^{n+1} a_n < a_1$$

for any integer N.

Now

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \Big(\frac{x^6}{6!} - \frac{x^8}{8!} - \cdots\Big)$$

As before,

$$\frac{x^{n+2}}{(n+2)!} > \frac{x^n}{n!} \qquad \text{if } x^2 < (n+1)(n+2).$$

and this certainly holds if $x = \sqrt{3}$ and $n \geq 6$. Also when $x = \sqrt{3}$,

$$\cos\sqrt{3} = 1 - \frac{\sqrt{3}^2}{2!} + \frac{\sqrt{3}^4}{4!} + \cdots < 1 - \frac{3}{2} + \frac{9}{24} = \frac{-1}{8} < 0.$$

Hence $\cos x = 0$ *for some* x *between* $\sqrt{2}$ *and* $\sqrt{3}$.

Recall that

$$\frac{d}{dx}(\cos x) = -\sin x,$$

and

$$\sin x = \Big(x = \frac{x^3}{3!}\Big) + \Big(\frac{x^5}{5!} - \frac{x^7}{7!}\Big) + \frac{x^9}{9!} - \cdots$$

Again

$$\frac{x^{n+2}}{(n+2)!} < \frac{x^n}{n!} \qquad \text{if } x^2 < (n+1)(n+2).$$

if $0 < x \leq 2$ and in particular for $\sqrt{2} < x < \sqrt{3}$.

Thus $\cos x$ is strictly decreasing on $\sqrt{2} \leq x \leq \sqrt{3}$. Hence $\cos x = 0$ for just one x with $\sqrt{2} \leq x \leq \sqrt{3}$. Hence result follows. □

Definition. The number α in the last theorem is denoted by $\frac{\pi}{2}$.

Uniform Convergence

From the previous arguments in this chapter we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

can be thought as

$$e^x = f_1(x) + f_2(x) + f_3(x) + \ldots$$

where

$$f_n(x) = \frac{x^{n-1}}{(n-1)!}.$$

Hence we are interested in

$$s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x).$$

And in the case of e^x,

$$\lim_{n\to\infty} s_n(x) = e^x.$$

More generally, we consider

$$f(x) = \lim_{n\to\infty} f_n(x)$$

instead of special case of $s_n(x)$.

Our basic questions are
(1) if f_n are continuous, is $f = \lim_{n\to\infty} f_n$ continuous?
(2) if f_n are differentiable, is $f = \lim_{n\to\infty} f_n$ differentiable?

Let us have some historical background of the problem first.

Cauchy and Continuity

This section is largely an extraction from Bressoud.

On page 120 of his *Cours d'analyse*, Cauchy proves his first theorem about infinite series. Let S be an infinite series of continuous functions,

$$S(x) = f_1(x) + f_2(x) + f_3(x) + \dots.$$

Let S_n be the partial sum of the first n terms,

$$S_n(x) = f_1(x) + f_2(x) + f_3(x) + \dots + f_n(x),$$

and let $R_n(x)$ be the remainder,

$$S(x) - S_n(x) = f_{n+1}(x) + f_{n+2}(x) + \dots$$

Cauchy remarks that S_n, a finite sum of continuous functions, must be continuous, and then goes on to state:

> Let us consider the changes in these three functions when we increase x by an infinitely small value α. For all possible values of n, the change in $S_n(x)$ will be infinitely small; the change in $R_n(x)$ will be as insignificant as the size of $R_n(x)$ when n is made very large. It follows that the change in the function $S(x)$ can only be an infinitely small quantity. From this remark, we immediately deduce the following proposition:
>
> **Theorem.** *When the terms of a series are functions of a single variable x and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the neighborhood of this particular value, a continuous function of x.*

Cauchy has proven that any infinite series of continuous functions is continuous.

There is only one problem with this theorem. It is wrong. The Fourier series

$$\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots$$

is an infinite series of continuous functions. We will see in Chapter 5 that it is not continuous at $x = 1$. No one seems to have noticed this contradiction until 1826 when Niel Abel pointed it out in a footnote to his paper on infinite series.

Even though Dirichlet definitively established the validity of Fourier series in 1829, it was 1847 before anyone was able to make progress on resolving the contradiction between Cauchy's theorem and the properties of Fourier series. The first light was shed by George Stokes (1819-1903). A year later, Dirichlet's student Phillip Seidel (1821-1896) went a long way toward clarifying Cauchy's error. Cauchy corrected his error in 1853, but the conditions required the continuity of an infinite series were not generally recognized until the 1860s when Weierstrass began to emphasize their importance.

Cauchy's Proof

Before we search for the flaw in Cauchy's argument, we need to restate it more carefully using our definitions of continuity and convergence. The simple act of putting it into precise language may reveal the problem.

To prove the continuity of $S(x)$ at $x = a$, we must show that for any given $\epsilon > 0$, there is δ such that as long as x stays within δ of a, $S(x)$ will be within ϵ of $S(a)$:

$$|x - a| < \delta \text{ implies that } |S(x) - S(a)| < \epsilon.$$

Cauchy's analysis begins with the observation that

$$\begin{aligned} |S(x) - S(a)| &= |S_n(x) + R_n(x) - S_n(a) - R_n(a)| \\ &\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|. \end{aligned} \tag{1}$$

We can divide the allowable error three ways, giving $\frac{\epsilon}{3}$ to each of the terms in the last line. The continuity of $S_n(x)$ guarantees that we can make

$$|S_n(x) - S_n(a)| < \frac{\epsilon}{3}.$$

The convergence of $S(x)$ at $x = a$ and at all points close to a tells us that the remainders can each be made arbitrarily small:

$$|R_n(x)| < \frac{\epsilon}{3} \text{ and } |R_n(a)| < \frac{\epsilon}{3}.$$

If you still do not see what is wrong with this proof, you should not be discouraged. It took mathematicians over a quarter of a century to find the error.

An Example

It is easiest to see where Cauchy went wrong by analyzing an example of an infinite series of continuous functions that is itself discontinuous. Fourier series are rather complicated. We shall use a simpler example:

$$S(x) = \sum_{k=1}^{\infty} \frac{x^2}{(1 + kx^2)(1 + (k-1)x^2)}. \tag{2}$$

Each of the summands is a continuous function of x. The partial sums are particularly easy to work with. We observe that

$$\frac{x^2}{(1 + kx^2)(1 + (k-1)x^2)} = \frac{1}{1 + (k-1)x^2} - \frac{1}{1 + kx^2},$$

and therefore

$$\begin{aligned} S_n(x) &= \Big(1 - \frac{1}{1 + x^2}\Big) + \Big(\frac{1}{1 + x^2} - \frac{1}{1 + 2x^2}\Big) \\ &\quad + \Big(\frac{1}{1 + 2x^2} - \frac{1}{1 + 3x^2}\Big) + \cdots + \Big(\frac{1}{1 + (n-1)x^2} - \frac{1}{1 + nx^2}\Big) \\ &= 1 - \frac{1}{1 + nx^2} \\ &= \frac{nx^2}{1 + nx^2}. \end{aligned} \tag{3}$$

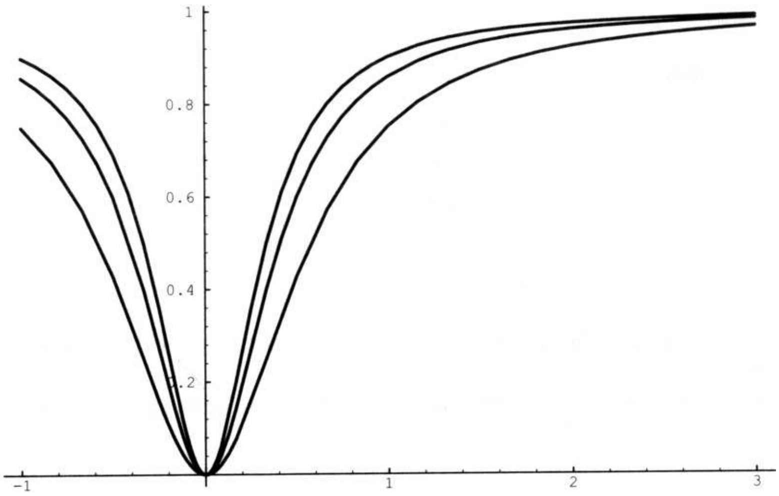

FIGURE 1. Graphs of $S_3(x), S_6(x)$ and $S_9(x)$.

We see that $S_n(0) = 0$ for all values of n, and so $S(0) = 0$. If x is not zero, then

$$S_n(x) = \frac{x^2}{n^{-1} + x^2}$$

which approaches 1 as n gets large,

$$S(x) = 1, \qquad x \neq 0.$$

The series is definitely discontinuous at $x = 0$.

We can see what is happening if we look at the graphs of the partial sums (Figure 1). As n increases, the graphs become steeper near $x = 0$. In the limit, we get a vertical jump.

Where is the Mistake?

Cauchy must be making some unwarranted assumption in his proof. To see what it might be, we return to his proof and use our specific example:

$$S(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0, \end{cases} \tag{4}$$

$$S_n(x) = \frac{nx^2}{1 + nx^2}, \tag{5}$$

$$R_n(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{1}{1 + nx^2}, & \text{if } x \neq 0, \end{cases} \tag{6}$$

The critical point at which we want to investigate continuity is $a = 0$. If x is close to but not equal to 0, then inequality (1) becomes

$$
\begin{aligned}
|S(x) - S(0)| &\le |S_n(x) - S_n(0)| + |R_n(x)| + |R_n(0)| \\
&= \left|\frac{nx^2}{1+nx^2} - 0\right| + \left|\frac{1}{1+nx^2}\right| + |0| && (7)\\
&= \frac{nx^2}{1+nx^2} + \frac{1}{1+nx^2} \\
&= 1. && (8)
\end{aligned}
$$

Something is wrong with the assertion that we can make each of the terms in (7) arbitrarily small.

We make the first piece small by taking x close to 0. How close does it have to be? We want

$$\frac{nx^2}{1+nx^2} < \frac{\epsilon}{3}. \tag{9}$$

Multiplying through $1 + nx^2$ and then solving for x^2, we see that

$$
\begin{aligned}
nx^2 &< (\frac{\epsilon}{3})(1+nx^2), \\
x^2(n - n\frac{\epsilon}{3}) &< \frac{\epsilon}{3}, \\
x^2 &< \frac{\frac{\epsilon}{3}}{n - n\frac{\epsilon}{3}} = \frac{\epsilon}{n(3-\epsilon)}, \\
|x| &< \sqrt{\frac{\epsilon}{3n - \epsilon n}}. && (10)
\end{aligned}
$$

The size of our response δ depends on n. As n gets larger, δ must be smaller. This makes sense if we think of the graph in Figure xx. If $\epsilon = 0.1$ so that we want $S_n(x) < 0.1$, we need to take a much tighter interval when $n = 9$ than we do when $n = 3$.

To make the second piece small,

$$\frac{1}{1+nx^2} < \frac{\epsilon}{3}, \tag{11}$$

we have to take a large value of n. If we solve this inequality for n, we see that we need

$$
\begin{aligned}
1 &< (\frac{\epsilon}{3})(1+nx^2), \\
\frac{3}{\epsilon} - 1 &< nx^2 \\
\frac{3-\epsilon}{\epsilon x^2} &< n. && (12)
\end{aligned}
$$

The size of n depends of x As $|x|$ gets smaller, n must be taken larger. This also makes sense when we look at the graph. If we take an x that is very close to 0, then we need a very large value of n before we are near $S(x) = 1$.

Here is our difficulty. The size of x depends on n, and the size of n depends on x. We can make the first piece small by making x small, but that increases the size of the second piece.

If we increase n to make the second piece small, the first piece increases. We are in a vicious cycle. *We cannot make both pieces small simultaneously.*

Example.

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1 \\ 1 & x \ge 1. \end{cases} \quad \xrightarrow{n\to\infty} \quad f(x) = \begin{cases} 0 & 0 \le x \le 1 \\ 1 & x \ge 1. \end{cases}$$

continuous discontinuous

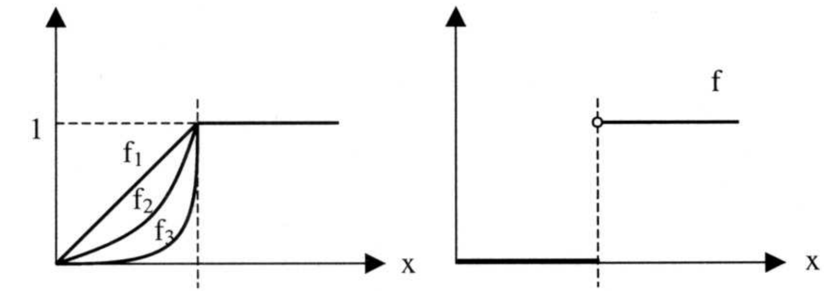

FIGURE 2

Example.

$$f_n(x) = \begin{cases} -1 & x \le -\frac{1}{n} \\ nx & -\frac{1}{n} \le x \le \frac{1}{n} \\ 1 & x \ge \frac{1}{n} \end{cases} \quad \xrightarrow{n\to\infty} \quad f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

continuous discontinuous

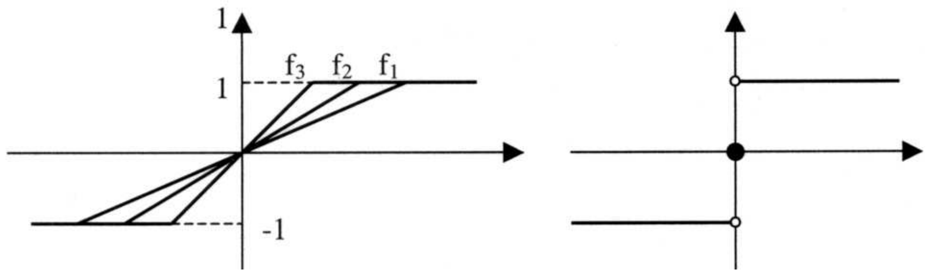

FIGURE 3

Example.

$$f_n(x) = \begin{cases} -1 & x \le -\frac{1}{n} \\ \sin\frac{n\pi x}{2} & -\frac{1}{n} \le x \le \frac{1}{n} \\ 1 & x \ge \frac{1}{n} \end{cases} \quad \xrightarrow{n\to\infty} \quad f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

differentiable non-differentiable

Example.

$$f_n(x) = \begin{cases} 2n^2, x & 0 \le x \le \frac{1}{2n}, \\ 2n - 2n^2 x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1. \end{cases}$$

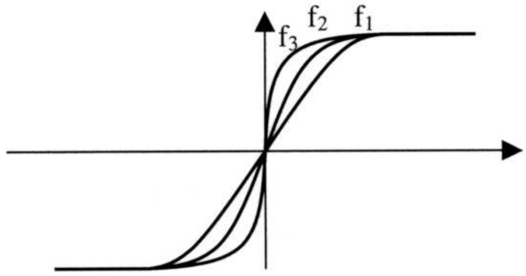

FIGURE 4

Hence $\lim_{n\to\infty} f_n(x) = 0$ for all x.

Since for any given $x > 0$ there is N such that $f_n(x) = 0$ for $n > N$ and so $f(0) = 0$.

However,

$$\int_0^1 f_n(x)dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

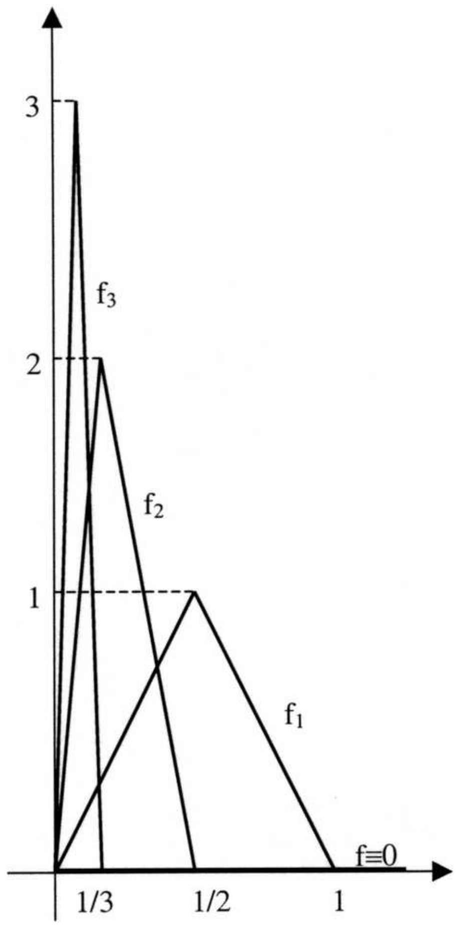

FIGURE 5

From the above pictures, f_n do not "get close" to f. For each of the above examples, take an $x \in A$ $(x \neq 0)$. Then for any $\epsilon > 0$,

$$|f_n(x) - f(x)| \geq 2\epsilon, \qquad n > N,$$

for some N sufficiently large.

Fixing the Problem up with Uniform Convergence

Part of the reason that Cauchy made his mistake is that many infinite series of continuous functions *are* continuous. Having found what is wrong with Cauchy's proof, we can attempt to find criteria that will identify infinite series that are continuous. If we are going to be able to break our cycle, then either the size of the first piece does not depend on n or the size of the second piece does not depend on x.

The usual solution is the second: that is size of $|R_n(x)|$ does not depend on x. When this happens, we say that the series is **uniformly convergent**. Specifically, we have the following definition:

Definition. Let $\{f_n\}_1^\infty$ be a sequence of functions defined on $[a, b]$. Then f is called the uniform limit of $\{f_n\}$ on $[a, b]$ if given $\epsilon > 0$ there is some N such that for all $x \in [a, b]$,

$$\Big| f(x) - f_n(x) \Big| < \epsilon \qquad \text{whenever } n > N.$$

We also say that $\{f_n\}$ *converges uniformly* to f on $[a, b]$.

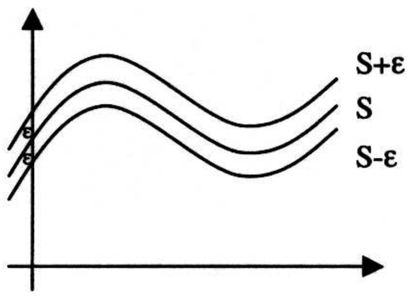

Figure 6. The ϵ around S.

Graphically, this implies that if we put an *envelop* extending distance ϵ above and below S (Figure 6), then there is a response N such that $n \geq N$ implies that the graph of S_n lies entirely inside this envelop. Using the example from equation (2) (Figure 7), we see that when ϵ is small (less than $\frac{1}{2}$), none of the partial sums stay inside the ϵ envelop. This example was not uniformly convergent.

Example. $f_n(x) = x(1-x)^n$, $0 \leq x \leq 1$, $f(x) = 0$ on $0 \leq x \leq 1$. Then $f_n \to f$ uniformly on $[0, 1]$.

Proof. Given $\epsilon > 0$, choose N such that $(1-\epsilon)^N < \epsilon$. Then for $n \leq N$ we have

$$|f_n(x) - f(x)| = x(1-x)^n \leq x < \epsilon \qquad \text{if } 0 \leq x < \epsilon,$$

and

$$|f_n(x) - f(x)| = x(1-x)^n < \epsilon \qquad \text{if } \epsilon \leq x \leq 1.$$

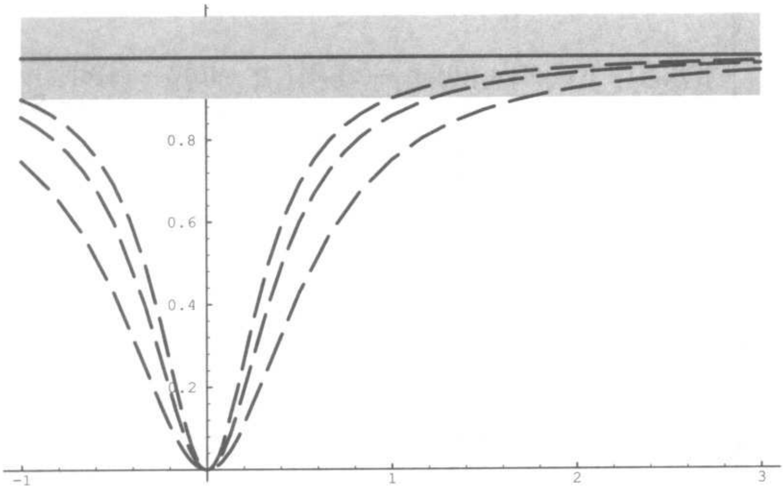

FIGURE 7. Figure 1 with ϵ envelop.

So for $n > N$,

$$|f_n(x) - f(x)| = x(1-x)^n < \epsilon$$

for all $x \in [0,1]$. □

Example.

$$f_n(x) = (1-x)^n, \qquad 0 \le x \le 1$$

$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & x = 0. \end{cases}$$

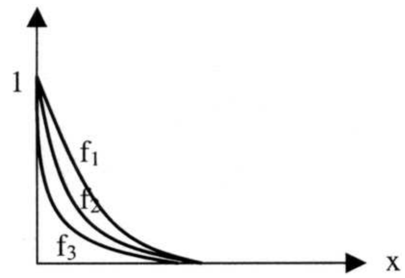

FIGURE 8

Choose $\epsilon = \dfrac{1}{2e}$, we choose $x_n \to 0$ say $\dfrac{1}{n}$.

$$|f_n\Big(\frac{1}{n}\Big) - f\Big(\frac{1}{n}\Big)| = \Big|\Big(1-\frac{1}{n}\Big)^n - 0\Big| \to \frac{1}{e} > \frac{1}{2e}.$$

So

$$|f_n\Big(\frac{1}{n}\Big) - f\Big(\frac{1}{n}\Big)| > \frac{1}{2e}$$

for n sufficiently large.

Hence we cannot choose N such that

$$|f_n\Big(\frac{1}{n}\Big) - f\Big(\frac{1}{n}\Big)| < \frac{1}{2e}$$

for $n > N$ and $x \in [0, 1]$.

Definition. $\{f_n\}$ is said to converge to f pointwise on $[a, b]$ if

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each x in $[a, b]$.

Except $f_n(x) = x(1-x)^n$, all the above examples have pointwise limits only.

Theorem 2.6. *Suppose that $\{f_n\}$ is a sequence of functions which are integrable on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f which is integrable on $[a, b]$. Then*

$$\int_a^b \lim_{n\to\infty} f_n\,dx = \int_a^b f dx = \lim_{n\to\infty}\int_a^b f_n\,dx.$$

Proof. Given $\epsilon > 0$, there is $N > 0$ such that

$$|f_n(x) - f(x)| < \epsilon$$

if $n > N$ and for all $x \in [a, b]$. Then if $n > N$ we have

$$\begin{aligned}\Big|\int_a^b f(x)dx - \int_a^b f_n(x)dx\Big| &= \Big|\int_a^b \Big(f(x)dx - f_n(x)\Big)dx\Big| \\ &\le \int_a^b \Big|f(x)dx - f_n(x)\Big|dx < \epsilon\int_a^b 1dx = (b-a)\epsilon.\end{aligned}$$

Since this is true for all $\epsilon > 0$, hence

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

∎

Theorem 2.7. *Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to f. Then f is also continuous on $[a, b]$.*

Proof. By the "$\frac{\epsilon}{3}$" method. For each $x \in [a, b]$, we deal only with $x \in (a, b)$ with usual justification, for $x = a$ or b.

Given $\epsilon > 0$, we may choose N so large such that

$$\Big|f(y) - f_n(y)\Big| < \frac{\epsilon}{3}$$

whenever $n > N$ and all $y \in [a, b]$. Let $x \in (a,\, b)$ and h so small such that $x + h \in [a, b]$. Since f_n is continuous on $[a, b]$, there is $\delta > 0$ such that

$$\Big|f_n(x+h) - f_n(x)\Big| < \frac{\epsilon}{3} \qquad \text{if } |h| < \delta.$$

Hence

$$\begin{aligned}\Big|f(x+h)-f(x)\Big| &= \Big|f(x+h)-f_n(x+h)+f_n(x+h)-f_n(x)+f_n(x)-f(x)\Big| \\ &\le \Big|f(x+h)-f_n(x+h)\Big| + \Big|f_n(x+h)-f_n(x)\Big| + \Big|f_n(x)-f(x)\Big| \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Hence f is continuous on $[a, b]$. □

Remark. In this way, we can fix Cauchy's mistake up with uniform convergence. We repeat Cauchy's proof, being careful to choose n first. We choose any $a \in (\alpha, \beta)$ and use inequality (1):

$$|S(x) - S(a)| \le |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|.$$

As before, we assign a third of our error bound to each of these terms. Using the uniform convergence, we can find an n for which both $|R_n(x)|$ and $|R_n(a)|$ are less than $\dfrac{\epsilon}{3}$, regardless of our choice of x. Once n is chosen, we turn to the first piece and use the continuity of $S_n(x)$ to find a δ for which

$$|x-a| < \delta \text{ implies that } |S(x) - S(a)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

Theorem 2.8. *Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ and converges uniformly to f. Then*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f dx.$$

Proof. trivial. □

Question. Is it true that if $f_n \to f$ uniformly and f_n differentiable on $[a, b]$, then f is also differentiable on $[a, b]$?

Answer. No.

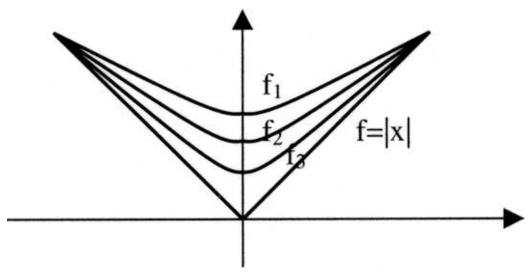

Figure 9

Even if f is differentiable, it may not be true that

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Example. Take $f_n(x) = \frac{1}{n}\sin(n^2x)$. $f_n(x) = \frac{1}{n}\sin(n^2x) \to f(x) = 0$ uniformly, as $n \to \infty$. But $f_n'(x) = n\cos(n^2x)$ and $\lim\limits_{n\to\infty} n\cos(n^2x)$ does not exist, say, at $x = 0$.

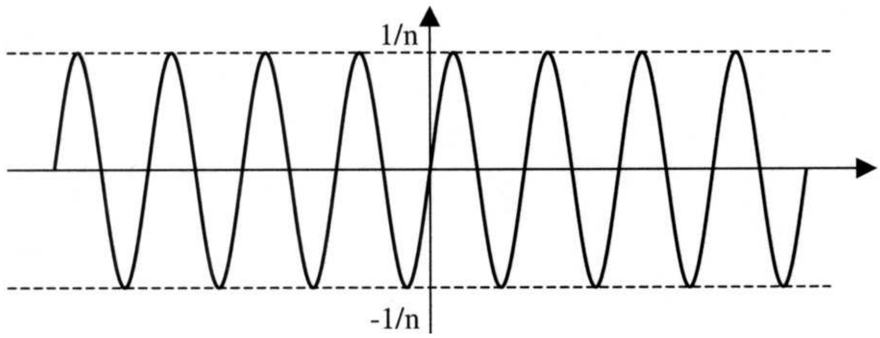

FIGURE 10

Theorem 2.9. *Suppose that $\{f_n\}$ is a sequence of functions which are differentiable on $[a, b]$, and that $\{f_n\}$ converges uniformly to f. Suppose, moreover, that $\{f_n'\}$ converges uniformly to some continuous function g. Then f is differentiable, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof. Applying Theorem 2.6 to $[a, x]$, we see that for each x we have

$$\begin{aligned}\int_a^x g dx &= \lim_{n\to\infty} \int_a^x f_n' dx \\ &= \lim_{n\to a} \Big(f_n(x) - f_n(a)\Big) \\ &= f(x) - f(a).\end{aligned}$$

Since g is continuous,

$$\frac{d}{dx}\int_a^x g dx = f'(x).$$

That is,

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x) \qquad \text{for all } x \in [a, b].$$

□

Definition. We say the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A if the sequence

$$f_1, \qquad f_1 + f_2, \qquad f_1 + f_2 + f_3, \qquad \ldots$$

converges uniformly to f on A.

Theorem 2.10. *Let $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$.*

(i) *If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.*

(ii) *If f and each f_n is integrable on $[a, b]$, then*

$$\int_a^b f dx = \sum_{n=1}^{\infty} \int_a^b f_n \, dx.$$

(iii) *If* $\sum_{n=1}^{\infty} f_n$ *converges (even pointwise) to* f *on* $[a,b]$*, and* $\sum_{n=1}^{\infty} f'_n$ *converges uniformly on* $[a,b]$ *to some continuous function, then*

$$f'(x) = \sum_{n=1}^{\infty} f'_n(x),$$

for all x *in* $[a,b]$.

Proof. (i) Since each f_n is continuous, then so is each

$$f_1 + \cdots + f_n \qquad n = 1, 2, 3, \ldots$$

But then f is the uniform limit of continuous functions (the partial sum) on $[a,b]$ and hence continuous on $[a,b]$.

(ii) Since $f_1,\ f_1+f_2,\ f_1+f_2+f_3, \ldots$ converges uniformly to f, hence

$$\begin{aligned}\int_a^b f\,dx &= \int_a^b \lim_{N\to\infty} \Big(N\text{th partial sum}\Big)dx \\ &= \lim_{N\to\infty}\int_a^b \Big(N\text{th partial sum}\Big)dx \qquad \text{(Theorem 2.6)} \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_a^b f_n dx = \sum_{n=1}^{\infty}\int_a^b f_n dx.\end{aligned}$$

(iii) $f'_1,\ f'_1+f'_2,\ f'_1+f'_2+f'_3, \ldots$ converges uniformly to some continuous function. Hence f is differentiable and

$$f' = \lim_{N\to\infty}\Big(\underbrace{f'_1+\cdots+f'_N}_{s'_N}\Big) = \sum_{n=1}^{\infty} f'_n \qquad \text{(Theorem 2.9)}$$

□

Theorem 2.11. *(Weierstrass* $M-$*test) Let* $\{f_n\}$ *be a sequence of functions defined on* A*, and suppose that* $\{M_n\}$ *is a sequence of numbers such that*

$$|f_n(x)| \le M_n$$

for all x *in* A*. Suppose moreover that* $\sum_{n=1}^{\infty} M_n$ *converges. Then for each* x *in* A *the series* $\sum_{n=1}^{\infty} f_n$ *converges (absolutely) and* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* A *to the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Proof. For each x in A the series $\sum_{n=1}^{\infty} |f_n|$ converges (absolutely) by the comparison test. Given $\epsilon > 0$ choose N so large such that $\sum_{N+1}^{\infty} M_n < \epsilon$ For all x in A

$$\Big|f(x) - (f_1 + f_2 + \cdots + f_N)\Big| = \Big|\sum_{N+1}^{\infty} f_n\Big| \leq \sum_{N+1}^{\infty} M_n < \epsilon$$

for all $x \in A$. □

Theorem 2.12. *Suppose that the series*

$$f(x_0) = \sum_{n=0}^{\infty} a_n x_0^n$$

converges, and let a be any number with $0 < a < |x_0|$. Then on $[-a, a]$ the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges uniformly. Moreover, the same is true for the series

$$g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Finally, f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

for all x with $|x| < |x_0|$.

Proof. (second proof) Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges and hence $\{a_n x_0^n\}$ is bounded. That is,

$$|a_n x_0^n| \leq M$$

for all n. If $x \in [-a, a]$, then $|x| \leq a$, so

$$\begin{aligned} |a_n x^n| = |a_n||x^n| &\leq |a_n||a^n| \\ &= |a_n x_0^n|\Big|\frac{a^n}{x_0^n}\Big| \\ &\leq M\Big|\frac{a^n}{x_0^n}\Big|. \end{aligned}$$

Hence $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $x \in [-a, a]$ by comparison test since $\Big|\frac{a}{x_0}\Big| < 1$.

Choose $M_n = M\Big|\frac{a}{x_0}\Big|^n$ in the $M-$test, it follows that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-a, a]$.

For $g(x) = \sum\limits_{n=1}^{\infty} na_n x^{n-1}$, notice that

$$\begin{aligned} |na_n x^{n-1}| &= na_n|x^{n-1}| \le n|a_n||a^{n-1}| \\ &= \Big|\frac{a_n}{a}\Big||x_0^n| \cdot n \cdot \Big|\frac{a}{x_0}\Big|^n \\ &\le \frac{M}{|a|} n \Big|\frac{a}{x_0}\Big|^n. \end{aligned}$$

since $\Big|\dfrac{a}{x_0}\Big| < 1$, the series

$$\sum_{n=1}^{\infty} \frac{M}{|a|} n \Big|\frac{a}{x_0}\Big|^n = \frac{M}{|a|} \sum_{n=1}^{\infty} n \Big|\frac{a}{x_0}\Big|^n$$

converges, applying the $M-$test again proves that $\sum\limits_{n=1}^{\infty} na_n x^{n-1}$ converges uniformly on $[-a, a]$. From Theorem 2.7, g is continuous and hence

$$f'(x) = g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

for $x \in [-a, a]$. (Theorem 2.9) □

Some examples:

(a) $s(x) = \sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2}$ is uniformly convergent on $[0, 1]$.

Proof. Given $\epsilon > 0$ choose N such that

$$\sum_{N+1}^{\infty} \frac{1}{n^2} < \epsilon.$$

Then

$$|s(x) - s_n(x)| = \sum_{r=n+1}^{\infty} \frac{x^r}{r^2} \le \sum_{r=n+1}^{\infty} \frac{1}{r^2} < \epsilon$$

for all $n \ge N$ and $x \in [a, b]$. So $s_n(x) \to s(x)$ uniformly on $[0, 1]$. That is, $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2}$ is uniformly convergent on $[0, 1]$.

□

(b) (Weierstrass $M-$test) We consider the above example again. Since

$$\Big|\frac{x^n}{n^2}\Big| \le \frac{1}{n^2} = M_n$$

if $0 \le x \le 1$, $n = 1, 2, \ldots$ and

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent. So by the $M-$test, the $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent on $[0,1]$.

(c) Show that

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x| < 1$.

Proof. Since

$$\begin{aligned} \tan^{-1} x &= \int_0^x \frac{1}{1+y^2} dy \\ &= \int_0^x \Big(\sum_{n=0}^{\infty} (-1)^n y^{2n} \Big) dy. \end{aligned}$$

Consider $\sum_{n=0}^{\infty} (-1)^n y^{2n}$ on $[-|x|, |x|]$. Since $|(-1)^n y^{2n}| \le |x|^{2n}$ and $\sum_{n=0}^{\infty} |x|^{2n}$ is convergent. Hence we may apply to $M-$test with $M_n = |x|^{2n}$.

So by Theorem 2.11, $\sum_{n=0}^{\infty} (-1)^n y^{2n}$ converges uniformly on $[-|x|, |x|]$. Clearly each $(-1)^n y^{2n}$ is integrable on $[0, x]$, Theorem 2.10 implies

$$\begin{aligned} \tan^{-1} x &= \int_0^x \sum_{n=0}^{\infty} (-1)^n y^{2n} dy \\ &= \sum_{n=0}^{\infty} \int_0^x (-1)^n y^{2n} dy \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \end{aligned}$$

□

Definition. We define $f(x) = \{x\}$ to be the distance from x to the nearest integer.

Example. $f(x) = \{x\}$ looks like

We consider

$$f_n(x) = \frac{1}{10^n} \{10^n x\}$$

then clearly

$$|f_n(x)| \le \frac{1}{10^n} \qquad \text{for all } x,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{10^n}$$

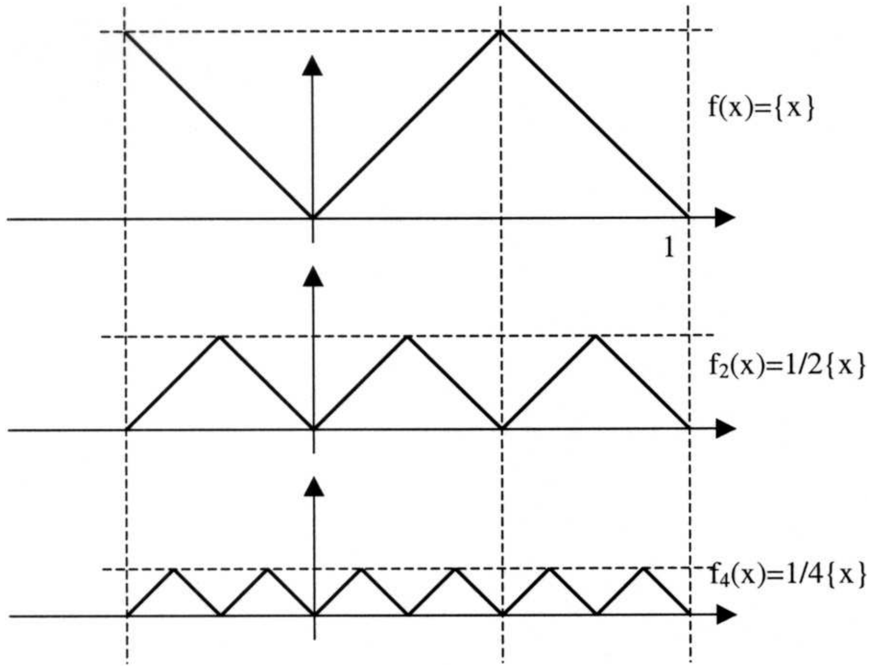

FIGURE 11

converges. Hence the function defined by

$$f(x)=\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty}\frac{1}{10^n}\{10^n x\}$$

is uniformly convergent series.

Theorem 2.13. *The function*

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{10^n}\{10^n x\}$$

is continuous everywhere and differentiable nowhere.

Proof. By the $M-$test and Theorem 1.7 that f is continuous. We will prove that f is not differentiable at a, for any a. It suffices to show that for a sequence $h_m\to 0$ the limit

$$\lim_{m\to\infty}\frac{f(a+h_m)-f(a)}{h_m}$$

does not exist. It obviously suffice to consider only a such that $0<a\le 1$.

Suppose a has the decimal representation

$$a=0.a_1a_2a_3\ldots$$

Let

$$h_m=\begin{cases}10^{-m} & \text{if } a_m\neq 4 \text{ or } 9\\ -10^{-m} & \text{if } a_m=4 \text{ or } 9\end{cases}$$

Then

$$\frac{f(a+h_m)-f(a)}{h_m} = \sum_{n=1}^{\infty} \frac{1}{10^n} \frac{\{10^n(a+h_m)\} - \{10^n a\}}{\pm 10^{-m}}$$
$$= \sum_{n=1}^{\infty} \pm 10^{m-n} \Big(\{10^n(a+h_m)\} - \{10^n a\}. \Big)$$

If $n \geq m$, then $10^n h_m \in \mathbb{Z}$ so

$$\{10^n(a+h_m)\} - \{10^n a\} = 0.$$

Hence it becomes a *finite* sum.

If however $n < m$ we can write

$$10^n a = \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3}\dots a_m \dots$$
$$\text{and} \qquad 10^n(a+h_m) = \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3}\dots(a_m \pm 1)\dots$$

(by the choice of h_m).

Now suppose

$$0.a_{n+1}a_{n+2}a_{n+3}\dots a_m \dots \leq \frac{1}{2},$$

then we also have

$$0.a_{n+1}a_{n+2}a_{n+3}\dots(a_m \pm 1)\dots \leq \frac{1}{2}$$

(by the choice of h_m, $h_m = 10^{-m}$ if $a_m = 4$).

This implies that

$$\Big(\{10^n(a+h_m)\} - \{10^n a\} \Big) = \pm 10^{n-m}$$

Exactly for the same reasoning for

$$0.a_{n+1}a_{n+2}a_{n+3}\dots > \frac{1}{2}.$$

Thus, for $n < m$,

$$10^{m-n} \Big(\{10^n(a+h_m)\} - \{10^n a\} \Big) = \pm 1.$$

That is, $\dfrac{f(a+h_m-f(a))}{h_m}$ is the sum of $m-1$ numbers, each of which is ± 1. Hence it equals an even integer if m is odd; and equals an odd integer if m is even. (Note that we always write 0.1240=0.123999...; another example will be 0.5=0.4999...). □

To show $\sum_{n=1}^{\infty} \dfrac{1}{2^n}\{2^n x\}$

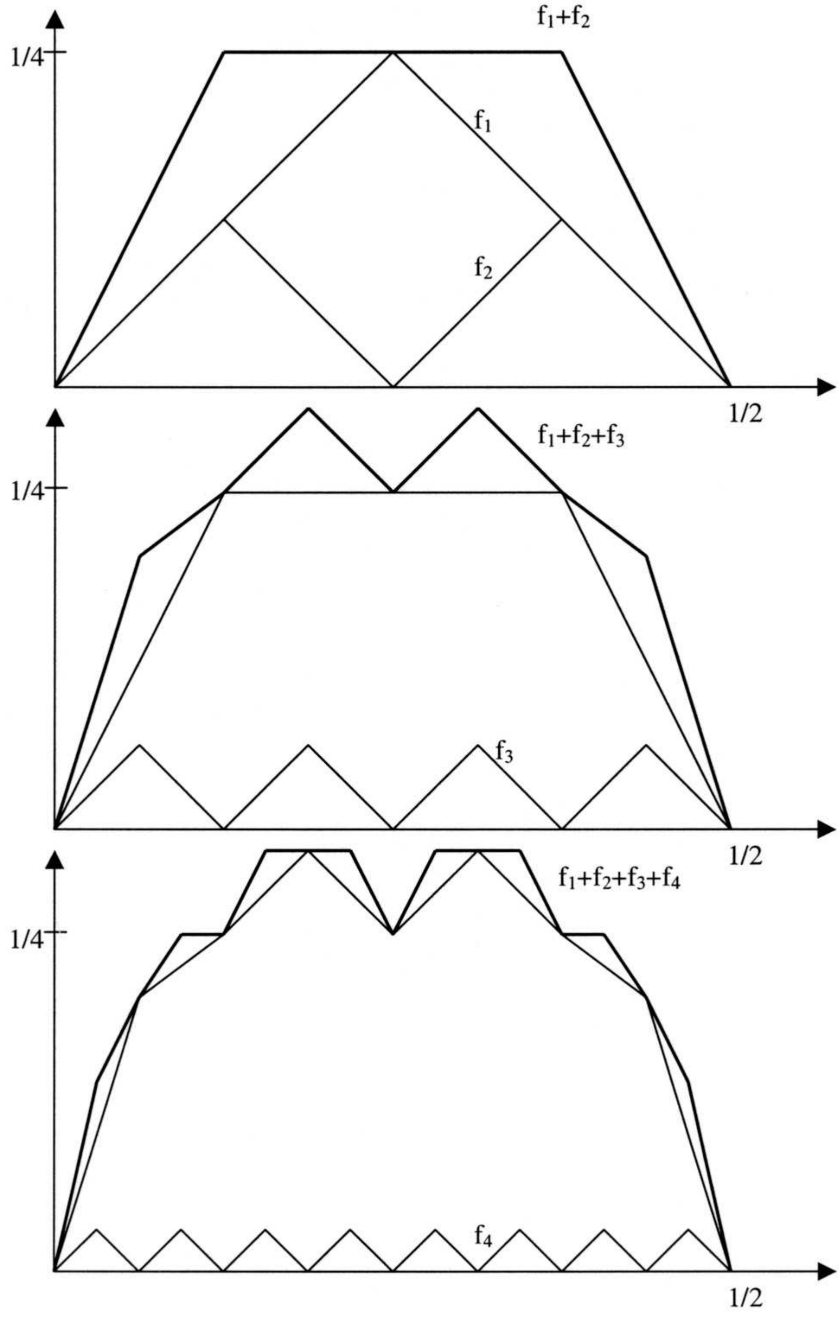

FIGURE 12

Exercise

Q1. Show that $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all $x \in \mathbb{R}$.

Q2 Show that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$ and $|\sin x - x| \leq |x|^3/6$ for all $x \in \mathbb{R}$.

Q3 Show that if $x > 0$ then

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Use this inequality to establish a lower bound for π.

Q4 Show that the convergence of $\lim_{n\to\infty} \frac{x^n}{1+x^n}$, $x \geq 0$ is uniform on $[0, b]$ whenever $b < 1$, but is not uniform on $[0, 1]$.

Q5 Show that the convergence of $\lim_{n\to\infty} \frac{\sin nx}{1+nx}$, $x \geq 0$ is uniform on $[a, +\infty)$ whenever $a > 0$, but is not uniform on $[0, +\infty)$.

Q6 Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, +\infty)$.

Q7 Let $f_n = nx/(1+nx^2)$ for $x \in A := [0, \infty)$. Show that each f_n is bounded on A, but the pointwise limit f of the sequence is not bounded on A. Does $\{f_n\}$ converge uniformly to f on A?

Q8 Let $f_n(x) = \frac{1}{(1+x)^n}$, $x \in [0, 1]$. Find the pointwise limit f of the sequence $\{f_n\}$ on $[0, 1]$. Does $\{f_n\}$ converge uniformly on $[0, 1]$ to f ?

Q9 Discuss the convergence and the uniform convergence of the series Σf_n, where f_n is given by (i)$1/(x^2 + n^2)$, (ii) $\sin(x/n^2)$, (iii) $x^n/(x^n + 1)$.

Q10 If Σa_n is absolutely convergent series, then the series $\Sigma a_n \sin(nx)$ is absolutely and uniformly convergent.