MATH4822E FOURIER ANALYSIS AND APPLICATIONS
CHAPTER 3 TRIGONOMETRIC FOURIER SERIES

3. Trigonometric Fourier Series

Let us have a closer look at trigonometric Fourier series.

The sine function $y = A\sin(\omega x + \phi)$ is called a *harmonic*, where $|A|$ is the *amplitude*, ω is the *frequency*, and ϕ is the *initial phase*. The *period* of the above harmonic is $T = \dfrac{2\pi}{\omega}$. A trigonometric polynomial of period $2l$ ($l > 0$) is given by

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \Big(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l} \Big) \tag{3.1}$$

where a_k and b_k are some constants.

It is easy to see that the above polynomial is periodic, but it may be difficult to see its shape. An infinite trigonometric series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l} \Big) \tag{3.2}$$

If we make a change of variable in (3.2), we have

$$\phi(t) = f(\frac{tl}{\pi}) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{3.3}$$

which has period 2π. So we shall only consider sums of period 2π henceforth.

For each $n \neq 0$,

$$\int_{-\pi}^{\pi} \cos nx\, dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0, \qquad \int_{-\pi}^{\pi} \sin nx\, dx = -\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} = 0.$$

On the other hand,

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$$

so that

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \begin{cases} \pi, & n = n \\ 0, & m \neq n \end{cases} = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx$$

and

$$\int_{-\pi}^{\pi} \sin nx\, \cos mx\, dx = 0$$

for any integers m and n. We multiply $\cos nx$ on both sides of (3.1) and integrate the resulting identity from $-\pi$ to π.

$$\begin{aligned}\int_{-\pi}^{\pi} s_n(x)\cos nx\,dx &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx + \sum_{k=1}^{n}\Big(a_k\int_{-\pi}^{\pi}\cos kx\cos nx\,dx + b_k\int_{-\pi}^{\pi}\sin kx\cos nx\,dx\Big)\\ &= \pi a_n.\end{aligned}$$

So

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} s_n(x)\cos n\,dx,$$

Similarly, we can multiple $\sin nx$ on both sides of (3.1) and integrate from $-\pi$ to π to obtain

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} s_n(x)\sin nx\,dx.$$

Finally, we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s_n(x)\,dx$$

if we just integrate on both sides of (3.1) directly.

Definition. Let $f(x)$ be a periodic function of period 2π, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\Big(a_k\cos kx + b_k\sin kx\Big) \tag{3.4}$$

if

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \qquad k=0,1,2,\ldots \qquad \text{and}\\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \qquad k=0,1,2,\ldots \end{aligned} \tag{3.5}$$

The series (3.4) is called *the Fourier series of* f and the a_k, b_k (3.5) are called *the Fourier coefficients of* f.

Remark. The series (3.4) may *not* converge to $f(x)$.

Theorem 3.1. *Suppose that a 2π periodic function f on $[-\pi,\,\pi]$ can be expanded in a trigonometric series which converges uniformly on the whole real axis, then this is the Fourier series (period 2π).*

Proof. We have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\Big(a_k\cos kx + b_k\sin kx\Big) \tag{3.6}$$

which converges uniformly on $[-\pi,\pi]$. Multiply $\cos nx$ on both sides yields

$$f(x)\cos nx = \frac{a_0}{2}\cos nx + \frac{a_0}{2} + \sum_{k=1}^{n}\Big(a_k\cos kx\cos nx + b_k\sin kx\cos nx\Big) \tag{3.7}$$

But (3.6) converges uniformly. That is, given $\epsilon > 0$, there is N such that

$$\Big|f(x) - \Big[\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)\Big]\Big| < \epsilon,$$

where $n > N$ and $x \in [-\pi, \pi]$. Hence given $\epsilon > 0$, we can use the same N to shows that the (3.7) to have

$$\begin{aligned}
&\Big|f(x)\cos nx - \Big[\frac{a_0}{2}\cos nx + \sum_{k=1}^{n}(a_k \cos kx \cos nx + b_k \sin kx \cos nx)\Big]\Big| \\
&= \Big|f(x) - \Big[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big]\Big| |\cos nx| \\
&\le \Big|f(x) - \Big[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big]\Big| < \epsilon
\end{aligned}$$

where $n > N$ and $x \in [-\pi, \pi]$. Thus the convergence in (3.7) is also uniform on $[-\pi, \pi]$. So according to Theorem 2.10 that we can integrate (3.7) term by term to get (as in the case of $s_n(x)$)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx.$$

We can also integrate (3.6) directly to get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,$$

since (3.6) converges uniformly. Hence (3.6) is the Fourier series of f. □

We recall that the *left-hand* and *right-hand* limits of a function f at x_0 are defined, respectively, by

$$f(x_0 - 0) = \lim_{\substack{x\to x_0\\ x<x_0}} f(x), \qquad f(x_0 + 0) = \lim_{\substack{x\to x_0\\ x>x_0}} f(x).$$

Then f has a limit at x_0 if an only if $f(x_0 - 0) = f(x_0 + 0)$. If the two one-sided limits are not equal, we can measure their difference by

$$\delta = f(x_0 + 0) - f(x_0 - 0).$$

We say that f has a *jump discontinuity* at x_0.

Theorem 3.2. *Let f be a piecewise continuous periodic function of x. Then the Fourier series of f converges for all x, and the sum equals to f if f is continuous there, and the sum equals*

$$\frac{f(x_0 + 0) + f(x_0 - 0)}{2},$$

if f has a jump discontinuity at x_0. If f is continuous everywhere, then the Fourier series of f converges to f uniformly and absolutely.

The proof will be given in due course.

Example. Expand $f(x) = x^2$, $x \in [-\pi, \pi]$ in Fourier series.

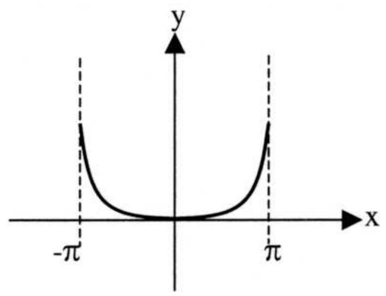

FIGURE 1

Since f is continuous over $[-\pi,\ \pi]$ so the Theorem 3.2 asserts that

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos kx + b_k \sin kx \Big)$$

for each $x \in [-\pi, \pi]$. Moreover, the infinite sum converges uniformly and absolutely. In this case,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx d\,x = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx = (-1)^n \frac{4}{n^2}. \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0 \text{ (because } x^2 \sin nx \text{ is odd with respect to the } y-\text{axis.)} \\
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}.
\end{aligned}$$

So

$$\begin{aligned}
x^2 &= \frac{\pi^2}{3} - 4\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots \\
&= \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \cdots
\end{aligned}$$

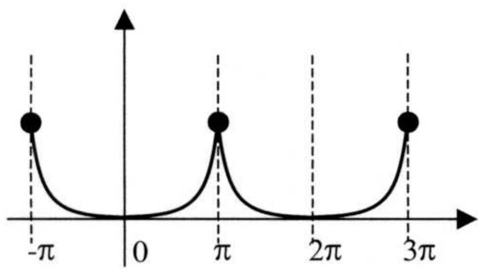

FIGURE 2. Extension of x^2 beyond $[-\pi, \pi]$.

In fact, the Fourier series of f converges absolutely and uniformly to the periodic extension of f for all x.

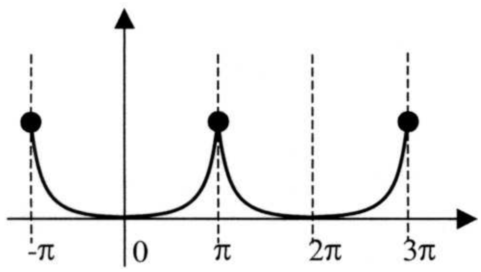

FIGURE 3. Extension of x^2 beyond $[-\pi, \pi]$.

Let us put $x = \pi$ on both sides of the above series. Then we obtain

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2},$$

from which one deduces

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Example. Expand $f(x) = x,\ (-\pi < x < \pi)$.

The Theorem 3.2 implies that we have equality

$$x = f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos kx + b_k \sin kx\Big)$$

on $-\pi < x < \pi$.

Clearly,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \cos nx\, dx = 0,$$

since $x \cos nx$ is an odd function. Besides,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\, dx = \frac{2}{n\pi}\int_{0}^{\pi} -x\, d(\cos nx) \\ &= \frac{-2}{n\pi}\, x \cos nx\Big|_0^{\pi} + \int_0^{\pi} \frac{2}{n\pi} \cos nx\, dx \\ &= \frac{-2}{n}(-1)^n + \frac{2}{n^2\pi} \sin nx\Big|_0^{\pi} \\ &= (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

So

$$\begin{aligned} x &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\sin nx}{n} \\ &= \frac{2\sin x}{1} - \frac{2\sin 2x}{2} + \frac{2\sin 3x}{3} - \frac{2\sin 4x}{4} + \cdots \\ &= 2\Big[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\Big] \end{aligned}$$

for x in $(-\pi,\, \pi)$. However, we note that the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos kx + b_k \sin kx\Big)$$

is invariant under a shift of $2j\pi$ where j is an integer. That is, the series converges uniformly and absolutely on intervals $\big((2j-1)\pi,\, (2j+1)\pi\big)$ despite that $f(x) = x$ is defined only on $(-\pi,\, \pi)$. Thus we can extend f periodically as $f(x + 2j\pi) = f(x)$ for all integers j, for x in $(-\pi < x < \pi)$. This extended function has discontinuities at $x = \pm\pi,\, \pm 2\pi, \cdots$. Now the Theorem 3.2 guarantees that

$$s_n(\pi) \to \frac{f(\pi - 0) + f(\pi + 0)}{2} = \frac{\pi + (-\pi)}{2} = 0,$$

as $n \to \infty$, since f has a discontinuity at $x = \pi$, and similar limit holds at other discontinuities of f.

Example. Expand $f(x) = x^2$ on $0 < x < 2\pi$ in Fourier series.

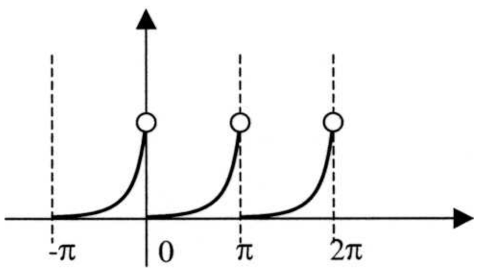

FIGURE 4

Clearly x^2 is continuous over $0 < x < 2\pi$ so its Fourier series converges absolutely and uniformly over $(0,\, 2\pi)$ (Theorem 3.2). It is routine to check that

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\, dx = \frac{1}{\pi}\frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^2}{3}$$

and

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx = \frac{1}{n\pi}x^2 \sin nx\Big|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} 2x \sin nx\, dx \\
&= 0 + \frac{2}{n^2\pi}\, x\cos nx\,\Big|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \cos nx\, dx \\
&= \frac{2}{n^2\pi}\cdot 2\pi - \frac{2}{n^3\pi}\sin nx\Big|_0^{2\pi} \\
&= \frac{4}{n^2}, \qquad n \geq 1.
\end{aligned}
$$

Similarly, we have

$$
b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx = -\frac{4\pi}{n},
$$

for $n = 1,\, 2,\, \cdots$.

The Fourier series of $f(x) = x^2$ is therefore given by

$$
\begin{aligned}
x^2 &= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\Big(a_n \cos nx + b_n \sin nx\Big) \\
&= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\Big(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\Big) \\
&= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx
\end{aligned}
$$

where we have re-arranged the terms in the infinite sums above because that both

$$
\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n}\sin nx
$$

converge absolutely and uniformly.

We extend f periodically to other interviews of 2π as shown in the figure. Then Theorem 3.2 again implies that the partial sum of the Fourier series

$$
s_m(2\pi) \to \frac{f(2\pi+0) + f(2\pi-0)}{2} = \frac{0^2 + (2\pi)^2}{2} = 2\pi^2.
$$

$$
\begin{aligned}
s_m(2\pi) &= \frac{4\pi^2}{3} + 4\sum_{n=1}^{m}\frac{1}{n^2}\cos 2n\pi - 4\pi\sum_{n=1}^{m}\frac{2n\pi}{n} \\
&= \frac{4\pi^2}{3} + 4\sum_{n=1}^{m}\frac{1}{n^2}.
\end{aligned}
$$

$$
s_m(2\pi) \to \frac{4\pi^2}{3} + 4(\frac{\pi^2}{6}) = \frac{6\pi^2}{3} = 2\pi^2
$$

as $m \to \infty$.

Let us substitute $x = \pi$ in the above Fourier series:

$$\begin{aligned}\pi^2 &= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi - 4\pi \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi \\ &= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 0\end{aligned}$$

That is,

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{1}{4}\,(\frac{4\pi^2}{3} - \pi^2) = \frac{\pi^2}{12}.$$

Example. Expand

$$f(x) = \begin{cases} \cos x, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

in Fourier cosine series.

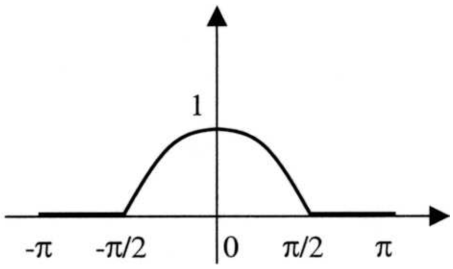

FIGURE 5

Since the extension makes the graph continuous everywhere, so its Fourier series converges uniformly everywhere. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{\pi}.$$

and

$$a_1 = \frac{2}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{2}{\pi}\int_{0}^{\pi/2} \cos 2x + 1)\,dx = \frac{1}{2}.$$

When $n \geq 2$, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,\cos nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos x\,\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} [\cos(n+1)x + \cos(n-1)x]\,dx \\
&= \frac{1}{\pi}\Big(\frac{1}{n+1}\sin(n+1)x + \frac{1}{n-1}\sin(n-1)x\Big)\Big|_0^{\frac{\pi}{2}} \qquad (n \geq 2) \\
&= \frac{1}{\pi}\Big[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\Big] \\
&= \begin{cases} 0, & n \text{ is odd} \\ (-1)\cdot\dfrac{(-1)^{\frac{n}{2}}\cdot 2}{(n^2-1)\pi,} & n \text{ is even} \end{cases} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\sin nx\,dx = 0 \qquad (n \geq 1)
\end{aligned}$$

because $\cos x \sin nx$ is an odd function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore,

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\cos x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 2}{(4n^2-1)\pi}\cos 2n\pi.$$

Even and odd extensions. In general, when we are given a function on $[0,\, l]$, we can extend it to $[-l,\, 0]$ by (i) an odd extension, or by (ii) an even extension.

(i) odd extension:

$$f(-x) = -f(x), \qquad x \in [-l,\, l].$$

Then

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\Big\{\int_0^l f(x)\cos\frac{n\pi x}{l}dx + \int_{-l}^{0} f(x)\cos\frac{n\pi x}{l}\,dx\Big\} \\
&= \frac{1}{l}\Big\{\int_0^l f(x)\cos\frac{n\pi x}{l}dx - \int_{l}^{0} f(-x)\cos\frac{-n\pi x}{l}\,dx\Big\} \\
&= \frac{1}{l}\Big\{\int_0^l f(x)\cos\frac{n\pi x}{l}dx - \int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\Big\} \\
&= 0, \qquad n \geq 0
\end{aligned}$$

(ii) even extension:

$$f(-x) = f(x), \qquad x \in [-l,\, l].$$

Then we can similarly show that

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = 0$$

for $n = 1, 2, 3, \ldots$ We obtain respectively

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin kx, \qquad \text{which is known as the sine series of } f, \text{ and}$$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \qquad \text{which is known as the cosine series of } f.$$

Remark. All previous results on Fourier series apply to the sine and cosine series.

Example. Expand $f(x) = x$ $(0 < x < l)$ in sine series.

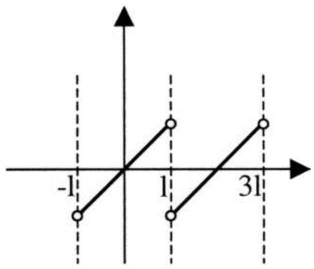

FIGURE 6

We need to construct an odd extension of f to $[-l, l]$ as shown in the figure. Thus all $a_n = 0$.

$$\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})\,dx \\
&= \frac{2}{l}\int_0^l x\sin(\frac{n\pi x}{l})\,dx \\
&= \frac{-2}{n\pi}\int_0^l xd\Big(\cos\frac{n\pi x}{l}\Big) \\
&= \frac{-2}{n\pi}x\cos(\frac{n\pi x}{l})\Big|_0^l + \frac{2}{n\pi}\int_0^l \cos(\frac{n\pi x}{l})\,dx \\
&= \frac{-2}{n\pi}[(-1)^n l] + \frac{2l}{n^2\pi^2}\sin(\frac{n\pi x}{l})\Big|_0^l \\
&= \frac{2l}{n\pi}(-1)^{n+1} + 0, \qquad n = 1, 2, 3\ldots
\end{aligned}$$

Hence,

$$x = \frac{2l}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(\frac{k\pi x}{l})$$

for $x \in (-\pi, \pi)$. One needs to check the convergence of the series at the end points.

Example. Expand $f(x) = x$ $(0 < x < l)$ in cosine series.

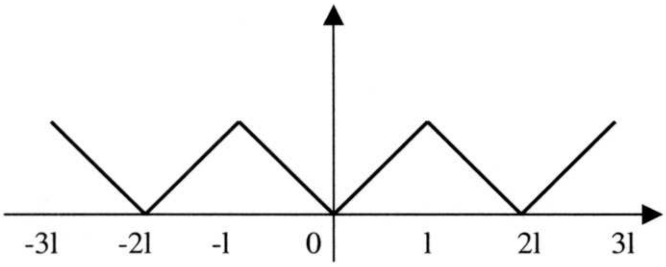

FIGURE 7

We know all $b_n = 0$. On the other hand,

$$a_0 = \frac{1}{l}\int_{-l}^{l} x\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{1}{l}x^2\Big|_0^l = l.$$

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} x\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2}{n\pi}x\sin\frac{n\pi x}{l}\Big|_0^l - \frac{2}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx \\
&= 0 + \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l}\Big|_0^l \\
&= \frac{2l}{n^2\pi^2}[(-1)^n - 1] \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4l}{n^2\pi^2}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore

$$x = \frac{l}{2} - \sum_{k=1}^{\infty}\frac{4l}{(2k-1)\pi^2}\cos\frac{\cos(2k-1)\pi x}{l}$$

for all x.

Complex Forms of Fourier Series

Suppose $f(x)$ is integrable on $[-\pi, \pi]$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\Big(a_k\cos kx + b_k\sin kx\Big)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx.$$

If we write $e^{i\theta} = \cos\theta + i\sin\theta$, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

So

$$\begin{aligned} f &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(\frac{a_k - ib_k}{2} e^{ik\theta} + \frac{a_k + ib_k}{2} e^{-ik\theta} \Big) \\ &\sim c_0 + \sum_{k=-\infty}^{\infty} c_k e^{ikx} \end{aligned}$$

where c_k are generally complex numbers.

Note that the above *complex Fourier series* is to be understood as the limit of the following partial sum:

$$s_n(x) = c_0 + \sum_{k=-n}^{n} c_k e^{ikx}.$$

Exercise

Q1 Expand the $f(x) = \cos ax$, where a is not an integer, by Fourier series on $-\pi \le x \le \pi$.

Q2 Expand the function

$$f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0, \\ x, & \text{for } 0 < x < \pi \end{cases}$$

in Fourier series.

Q3 Expand the function

$$f(x) = \begin{cases} 1 - \frac{x}{2h}, & \text{for } 0 \le x \le 2h, \\ 0, & \text{for } 2h \le x \le \pi \end{cases}$$

in (i) Fourier sine series (i.e., f has an odd extension) and (ii) in Fourier cosine series (i.e., f has an even extension).

Q4 Let

$$f(x) = \begin{cases} Ax + B, & \text{if } -\pi \le x < 0, \\ \cos x, & \text{if } 0 \le x \le \pi. \end{cases}$$

For what values of A and B does the Fourier series of f converge uniformly to f on $[-\pi, \pi]$?

Q5 Let f have period 2π and let

$$|f(x) - f(y)| \le c|x - y|^{\alpha}$$

for all x and y, for some positive constants c and α. That is, f satisfies the Hölder (also known as Lipschitz) condition of order α. Prove that

$$|a_n| \le \frac{c\pi^{\alpha}}{n^{\alpha}}, \qquad |b_n| \le \frac{c\pi^{\alpha}}{n^{\alpha}},$$

for each n, where a_n, b_n are the Fourier coefficients of f.

Q6 Expand $f(x) = x$ in $(-\pi, \pi)$ in complex Fourier series. Verify that the series is the same as the ordinary Fourier series worked out earlier in the lectures.