MATH4822E FOURIER ANALYSIS AND APPLICATIONS
CHAPTER 4 ORTHOGONAL FAMILIES

4. Orthogonal Families

We shall adopt the following understanding in this course unless we stated otherwise. We always assume that our f is a piece-wise continuous function defined on $[a, b]$ (i.e., continuous except for at most a finite number of discontinuities). When we say the integral exists, then we understand it exists in the Riemann sense.

An infinite sequence of real functions $\{\phi_j\}_{j=0}^{\infty}$ defined on $[a, b]$ is said to be *orthogonal on* $[a, b]$ if

$$\int_a^b \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \neq 0 & \text{if } m = n \end{cases}$$

for all m, n, where m, $n = 0, 1, 2, \ldots$

In fact, we have encountered many such examples in previous chapters already.

Example. The family $\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \ldots, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, \ldots\}$ is orthogonal on $[-l, l]$.

Example. The family $\{1, \cos x, \ldots, \cos nx, \ldots\}$ is orthogonal on $[0, \pi]$.

Example. The family $\{\sin x, \ldots, \sin nx, \ldots\}$ is orthogonal on $[0, \pi]$.

Example. The family $\{\sin x, \sin 3x, \ldots, \cos(2n+1)x, \ldots\}$ is orthogonal on $[0, \pi/2]$.

If our orthogonal family $\{\phi_j\}_{j=0}^{\infty}$ satisfies

$$\int_a^b \phi_n(x)^2\,dx = 1$$

for $n = 0, 1, 2, \ldots$, then $\{\phi_j\}$ is said to be *orthonormal* with respect to $[a, b]$. Any orthogonal family can be made into an orthonormal family by dividing $\int_a^b \phi_n^2\,dx$:

Let

$$\|\phi_n\| = \Big(\int_a^b \phi_n^2(x)\,dx\Big)^{\frac{1}{2}}$$

to be the *norm* of ϕ_n, then the family

$$\Big\{\frac{\phi_0}{\|\phi_0\|}, \frac{\phi_1}{\|\phi_1\|}, \cdots, \frac{\phi_n}{\|\phi_n\|}, \cdots\Big\}$$

is clearly orthonormal.

Example. Consider $\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \dots, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, \dots\}$ on $[-\pi, \pi]$. Then

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi, \qquad \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$$

$\|\cos nx\| = \|\sin nx\| = \sqrt{\pi}$.

$$\int_{-\pi}^{\pi} 1^2 \cdot\, dx = 2\pi, \qquad \|1\| = \sqrt{2\pi}.$$

So

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \cdots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \cdots\right\}$$

is orthogonal with respect to $[-\pi,\, \pi]$.

Example. Consider $\{\sin x, \dots, \sin nx, \dots\}$ is orthogonal on $[0,\, \pi]$. Then

$$\int_0^{\pi} (\sin nx)^2\, dx = \frac{\pi}{2}, \qquad \int_0^{\pi} \sin nx \sin mx\, dx = 0 \qquad \text{for } n \neq m.$$

So $\|\sin nx\| = \sqrt{\frac{2}{\pi}}$. Therefore

$$\left\{\sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\sin 2x, \cdots, \sqrt{\frac{2}{\pi}}\sin nx, \cdots\right\}$$

is orthonormal.

Fourier Series with respect to an Orthogonal Family

Let f be defined on $[a,\, b]$. Suppose that $\{\phi_j\}_{j=0}^{\infty}$ which is orthogonal over $[a,\, b]$. Then we say that the series is $\sum\limits_{j=0}^{\infty} c_j\phi_j$ is the Fourier series of f, written

$$f \sim c_0\phi_0 + c_1\phi_1 + \cdots + c_n\phi_n + \dots,$$

if

$$c_j = \frac{1}{\|\phi_j\|^2}\int_a^b f(x)\, \phi_j(x)\, dx, \qquad j = 0,\, 1,\, 2,\, \dots$$

The coefficients $\{c_j\}$ are called *Fourier coefficients* of f with respect to $\{\phi_j\}$. The above definition does not require the Fourier series to converge to f.

Theorem 4.1. *Let $\{\phi_n\}_0^{\infty}$ be an orthogonal family of continuous functions defined on $[a,\, b]$. If $f(x) = \sum_{j=0}^{\infty} c_j\phi_j$ converges uniformly on $[a,\, b]$, then the above sum is the Fourier series of f with respect to $\{\phi_n\}$ and*

$$c_k = \frac{1}{\|\phi_k\|^2}\int_a^b f(x)\, \phi_k(x)\, dx, \qquad k = 0, 1, 2, \dots$$

The above theorem asserts that if $f(x) = \sum_{k=0}^{\infty} c_k\phi_k$ and the convergence is uniform over $[a, b]$, then the series $\sum\limits_{k=0}^{\infty} c_k\phi_k$ *must be* the Fourier series of f with respect to the system $\{\phi_j\}_0^{\infty}$.

Proof. We multiply the series $\sum_{k=0}^{\infty} c_k\phi_k$ on both sides by ϕ_k before integrate the resulted equation over $[a,\, b]$. Since each ϕ_k is continuous on $[a,\, b]$ and the convergence of $\sum_{k=0}^{\infty} c_k\phi_k$ is uniform, so one can one can mimic the proof of Theorem 3.1 to interchange the summation and integration arrive at the conclusion. □

Theorem 4.2 (Bessel's Inequality)**.** *Let f be a square integrable function on $[a,\, b]$ and $\sum\limits_{k=0}^{\infty} c_k\phi_k$ be the Fourier series of f. Then the inequality*

$$\int_a^b f(x)^2\, dx \ge \sum_{k=0}^{n} c_k^2\, \|\phi_k\|^2$$

holds for each n, and in particular, we have

$$\int_a^b f(x)^2\, dx \ge \sum_{k=0}^{\infty} c_k^2\, \|\phi_k\|^2.$$

holds.

Remark. For the Bessel inequality, we have not assumed that the series necessarily converges to f.

Proof. Let $\sigma_n(x) = \sum\limits_{k=0}^{n} \gamma_k\, \phi_k(x)$ be an arbitrary sum of $\{\phi_0,\, \phi_1,\, \dots, \phi_n, \dots\}$ (that is, $\gamma_0,\, \gamma_1,\, \dots,\, \gamma_n$ are arbitrary constants). Consider the error term

$$\delta_n = \int_a^b \big(f(x) - \sigma_n(x)\big)^2\, dx.$$

$$\begin{aligned}
\delta_n &= \int_a^b [f(x)^2 - 2f(x)\,\sigma_n(x) + \sigma_n^2(x)]\, dx \\
&= \int_a^b f(x)^2\, dx - 2\int_a^b f(x)\sigma_n(x)\, dx + \int_a^b \sigma_n^2(x)\, dx.
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_a^b f(x)\,\sigma_n(x)\, dx &= \sum_{k=0}^{n} \int_a^b f(x)\,\gamma_k\phi_k(x)\, dx \\
&= \sum_{k=0}^{n} \gamma_k \int_a^b f(x)\,\phi_k(x)\, dx \\
&= \sum_{k=0}^{n} \gamma_k c_k \|\phi_k\|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_a^b \sigma_n^2(x)\,dx &= \int_a^b \Big(\sum_{k=0}^n \gamma_k \phi_k(x)\Big)^2\,dx \\
&= \sum_{k=0}^n \gamma_k^2 \int_a^b \phi_k^2(x)\,dx + \sum_{\substack{k=1 \\ k=p+q \\ p\neq q}}^n \gamma_p\gamma_q \int_a^b \phi_p(x)\phi_q(x)\,dx \\
&= \sum_{k=0}^n \gamma_k^2\,\|\phi_k\|^2 + 0 \\
&= \sum_{k=0}^n \gamma_k^2\,\|\phi_k\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta_n &= \int_a^b f(x)^2\,dx - 2\sum_{k=0}^n \gamma_k c_k \|\phi_k\|^2 + \sum_{k=0}^n \gamma_k^2\,\|\phi_k\|^2 \\
&= \int_a^b f(x)^2\,dx + \sum_{k=0}^n (c_k-\gamma_k)^2\|\phi_k\|^2 - \sum_{k=0}^n c_k^2\,\|\phi_k\|^2
\end{aligned}$$

which can be minimized to be

$$\Delta_n = \delta_n = \int_a^b f(x)^2\,dx - \sum_{k=0}^n c_k^2\|\phi_k\|^2$$

if we choose $\gamma_k = c_k$, $k = 0, 1, 2, \ldots, n$. ($\delta_n = \delta_n(\gamma_0,\ldots,\gamma_n) = \delta_n(c_0,\ldots,c_n) = \Delta_n$). Thus $\Delta_n \geq 0$ if and only if

$$\int_a^b f(x)^2\,dx \geq \sum_{k=0}^n c_k^2\,\|\phi_k\|^2.$$

□

Remark. The quantity

$$\delta_n = \int_a^b \big(f(x) - \sigma_n(x)\big)^2 dx$$

is called the *mean square error of* f being approximated by the system $\{\phi_0,\ldots,\phi_n\}$.

Corollary 4.3. *Let* $\sum c_k\phi_k$ *be the Fourier series of a square integrable function* f *on* $[a, b]$. *Then*

$$\lim_{k\to\infty} c_k\|\phi_k\| = 0.$$

Moreover, if $\{\phi_k\}$ *are orthogonal, then*

$$\lim_{k\to\infty} c_k = 0.$$

Definition. The orthogonal system $\{\phi_k\}_0^\infty$ is said to be *complete* if

$$\int_a^b f(x)^2\,dx = \sum_{k=0}^\infty c_k^2\|\phi_k\|^2$$

for every square integrable function f ($\Delta_n \to 0$ as $n \to \infty$).

Definition. The Fourier series $\sum_{k=0}^{n} c_k\phi_k$ of f is said to *converge to f in the mean* or (*mean square convergence*) if

$$\lim_{n\to\infty}\int_a^b \Big(f(x)-\sum_{k=0}^{\infty} c_k\phi_k\Big)^2\,dx = 0.$$

We clearly can deduce

Corollary 4.4. *A necessary and sufficient condition for an orthogonal family to be complete is that the Fourier series of f converges to f in the mean for each square integrable function f.*

Remark. Convergence in the mean is thought of as a generalised convergence.

Theorem 4.5. *Suppose the orthogonal family $\{\phi_k\}_0^\infty$ converges to f in the mean, that is,*

$$\lim_{n\to\infty}\int_a^b \Big(f(x)-\sum_{k=0}^{n} c_k\phi_k\Big)^2\,dx = 0,$$

where $\{c_k\}$ are the Fourier series of f. If

$$\lim_{n\to\infty}\int_a^b \Big(F(x)-\sum_{k=0}^{n} c_k\phi_k\Big)^2\,dx = 0,$$

then $F(x)=f(x)$ except at at most a finite number of points in $[a,\, b]$.

Proof.

$$\begin{aligned}
0 &\le \int_a^b \Big(F(x)-f(x)\Big)^2\,dx\\
&= \int_a^b \Big(F(x)-\sum_{k=0}^{n} c_k\phi_k+\sum_{k=0}^{n} c_k\phi_k - f(x)\Big)^2\,dx\\
&\le 2\int_a^b \Big(F(x)-\sum_{k=0}^{n} c_k\phi_k\Big)^2 dx + 2\int_a^b \Big(f(x)-\sum_{k=0}^{n} c_k\phi_k\Big)^2 dx \qquad \Big((a+b)^2\le 2a^2+2b^2\Big)\\
&\to 0.
\end{aligned}$$

as $n\to\infty$. □

Theorem 4.6. *If the orthogonal family $\{\phi_k\}$ is complete, then every square integrable function is completely determined by its Fourier series (except perhaps at a finite number of points), whether or not the series converges.*

Corollary 4.7. *If the orthogonal family $\{\phi_j\}$ is complete, then any square integrable function f which is orthogonal to each ϕ_j (for $j=0,1,2,\dots$) must satisfy $f(x)=0$ except perhaps at a finite number of points.*

Proof. Since $\{\phi_j\}$ is complete, that is,

$$\lim_{n\to\infty}\int_a^b \Big(f(x)-\sum_{k=0}^{n} c_k\phi_k\Big)^2 dx = 0, \tag{4.1}$$

$$c_k = \frac{1}{\|\phi_k\|^2}\int_a^b f(x)\phi_k(x)\,dx = 0$$

for $k = 0,\ 1,\ 2, \ldots$. Then (4.1) becomes

$$\int_a^b f(x)^2\,dx = 0.$$

We deduce $f(x) = 0$ except at a finite number of points (if any). □

Theorem 4.8. *If the system $\{\phi_j\}$ is complete and if all the members of $\{\phi_j\}$ are continuous, and if the Fourier series of the continuous f converges uniformly, then the sum of the Fourier series of f at x equals $f(x)$.*

Proof. Suppose

$$f(x) \sim \sum_{k=0}^{\infty} c_k\phi_k(x)$$

is the Fourier series of f, and that

$$s(x) = \sum_{k=0}^{\infty} c_k\phi_k(x)$$

converges uniformly. Since ϕ_k, $k = 0, 1, 2, \ldots$ are continuous, so the $s(x)$ is continuous. Thus the Theorem 4.1 asserts that $\sum_{k=0}^{\infty} c_k\phi_k$ is the Fourier series of $s(x)$. Thus

$$f(x) = s(x) = \sum_{k=0}^{\infty} c_k\phi_k(x)$$

for all x (since f is continuous) by Theorem 4.6. □

Theorem 4.9. *If the orthogonal system $\{\phi_j\}$ on $[a,\, b]$ is complete, then the Fourier series of a square integrable function $f(x)$ can be integrated term by term, whether or not the series converges. Moreover,*

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} c_k \int_a^b \phi_k(x)\,dx.$$

Proof. Since $\{\phi_j\}$ is complete, then Cauchy-Schwarz's inequality gives

$$\begin{aligned}
\Big|\int_a^b f(x)\,dx - \sum_{k=0}^{n} c_k\int_a^b \phi_k(x)\,dx\Big| &\le \int_a^b \Big|f(x)-\sum_{k=0}^{n} c_k\phi_k(x)\Big|\,dx\\
&\le \Big[\int_a^b \Big(f(x)-\sum_{k=0}^{n} c_k\phi_k(x)dx\Big)^2\cdot\int_a^b 1^2 dx\Big]^{\frac12}\\
&\Big(\Big(\int fg\Big)^2 \le \int f^2\cdot\int g^2\Big)\\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence,

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} c_k \int_a^b \phi_k(x)\,dx.$$

□

A criterion for the Completeness of an Orthogonal Family $\{\phi_j\}$.

Theorem 4.10. *Let $\{\phi_j\}$ be a family of functions on $[a, b]$. If for every continuous function $F(x)$ on $[a, b]$ and any $\epsilon > 0$, there exists a linear combination*

$$\sigma_n(x) = \gamma_0\phi_0(x) + \gamma_1\phi_1(x) + \cdots + \gamma_n\phi_n(x)$$

for which

$$\int_a^b [F(x) - \sigma_n(x)]^2\,dx \le \epsilon,$$

then the system $\{\phi_j\}$ is complete. (Note that we assume $\{\phi_j\}$ is orthogonal in this theorem)

Proof. Let $\epsilon > 0$ be given. We first show that for any square integrable function $f(x)$, there is continuous function $F(x)$ on $[a, b]$ such that

$$\int_a^b \Big(f(x) - F(x)\Big)^2\,dx < \epsilon.$$

Since f is square integrable, so it may have at most a finite number of discontinuities. We cover the discontinuities by some open intervals (a_i, b_i), $i = 1, 2, \ldots, m$, say, such that

$$\sum_{i=1}^{m} \int_{a_i}^{b_i} f(x)^2\,dx < \frac{\epsilon}{4}.$$

Let

$$\Phi(x) = \begin{cases} f(x), & x \notin (a_i, b_i) \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\int_a^b \big(f(x) - \Phi(x)\big)^2\,dx &= \int_{(a,b)\setminus\cup_i(a_i,b_i)} \big(f(x) - \Phi(x)\big)^2\,dx + \int_{\cup_i(a_i,b_i)} \big(f(x) - \Phi(x)\big)^2\,dx \\ &\le 0 + \epsilon/4.\end{aligned}$$

Let $M = \max_{a\le x\le b} |\Phi(x))|$. We choose (a_i, b_i) so small such that

$$\sum_{i=1}^{m} (b_i - a_i) < \frac{\epsilon}{4M^2}.$$

We now construct a continuous function $F(x)$ by joining the left and right pieces of Φ at each discontinuity within each (a_i, b_i). The value of F being the same as f (also the Φ) outside $\cup_i(a_i, b_i)$. Thus

$$\begin{aligned}\int_a^b \big(f(x)-F(x)\big)^2\,dx &\le 2\int_a^b \big(f(x)-\Phi(x)\big)^2\,dx + 2\int_a^b \big(F(x)-\Phi(x)\big)^2\,dx\\ &\le 2(\frac{\epsilon}{4}) + 2M^2\sum_{i=1}^{m}(b_i-a_i)\\ &< \frac{\epsilon}{2} + 2M^2\cdot\frac{\epsilon}{4M^2} = \epsilon,\end{aligned}$$

It follows from this inequality and the assumption that the existence of the linear combination σ_n that approximates F:

$$\begin{aligned}\int_a^b \big(f(x)-\sigma_n(x)\big)^2 dx &\le 2\int_a^b \big(f(x)-F(x)\big)^2 dx + 2\int_a^b \big(F(x)-\sigma_n(x)\big)^2 dx\\ &\le 2\epsilon + 2\epsilon = 4\epsilon.\end{aligned}$$

But we know that the Fourier coefficient of f gives the minimum mean square error. So

$$\int_a^b \big(f(x)-\sum_{k=0}^{n} c_k\phi_k\big)^2\,dx \le \int_a^b \big(f(x)-\sigma_n(x)\big)^2\,dx < 4\epsilon$$

and

$$0 \le \int_a^b f(x)^2\,dx - \sum_{k=0}^{n} c_k^2\,\|\phi_k\|^2 < 4\epsilon$$

for each n.

Thus,

$$0 \le \int_a^b f(x)^2\,dx - \sum_{k=0}^{\infty} c_k^2\|\phi_k\|^2 < 4\epsilon$$

and

$$\int_a^b f(x)^2\,dx = \sum_{k=0}^{\infty} c_k^2\|\phi_k\|^2,$$

since $\epsilon > 0$ is arbitrary. This proves that $\{\phi_k\}$ is complete. □

Remark. (1) A real-valued function f is called *smooth* on $[a, b]$ if it has a continuous derivative on $[a, b]$.

(2) A real-valued function f is called a *piecewise continuous* (or *piecewise smooth*) on $[a, b]$ if either f and f' are both continuous on $[a, b]$ or they have only a finite number of jump discontinuities on $[a, b]$.
(3) The family of "piecewise continuous" functions is not a good model to study space of functions when regarded as an infinite dimensional vector space. This is because the limit of an infinite sequence of piecewise smooth functions is not piecewise continuous.
(4) In modern literature, a square integrable real-valued functions on $[a, b]$ have a special name denoted by $L^2(a, b)$. That is,

$$L^2(a, b) = \Big\{f : \int_a^b |f(x)|^2\,dx < \infty\Big\}.$$

Note that we replace $f(x)^2$ by $|f(x)|$ because the notation can be flexible enough to include complex valued functions in other context. The $L^2(a,\, b)$ certainly has a vector space structure.

(5) One can define a *norm* on $L^2(a,\, b)$ by

$$\|f\|_2 = \Big(\int_a^b |f(x)|^2\, dx \Big)^{1/2}$$

which satisfies

(a) $\|x\|_2 = 0$ if and only if $x = 0$,

(b) $\|cx\|_2 = |c|\|x\|_2$ for any scalar c,

(c) $\|f + g\|_2 \le \|f\|_2 + \|g\|_2$.

Hence the $L^2(a,\, b)$ is a normed space.

(6) It is clear that the notion of "convergence in the mean" can be phrased as convergence in the $L^2(a,\, b)$ sense.

(7) When *null* (or zero) function f in $L^2(a,\, b)$ has

$$\|f\|_2 = \Big(\int_a^b |f(x)|^2\, dx \Big)^{1/2} = 0.$$

It is known that a null function does not imply $f(x) = 0$ in $[a,\, b]$. Instead one can show that given $\epsilon > 0$, there exists a subset E of $[a,\, b]$ which can be covered by a $\cup_{j=1}^{\infty} I_j$, where I_j is an interval with the *total length* less than ϵ. That is. $\sum_{j=1}^{\infty} |I_j| < \epsilon$. We say $f = 0$ a.e. which stands for *almost everywhere.* We also call E has *(Lebesgue) measure zero.* The $L^2(a,\, b)$ certainly contains Riemann integrable functions and piecewise continuous functions. But it is much bigger.

(8) When a space S has a norm $\|\cdot\|$ defined on it, one can define a *metric* d on S by $d(x,\, y) := \|x - y\|$.

(9) A metric space S is called *complete* if every Cauchy sequence converges to a member in S.

(10) A space S becomes a *Banach space* if which is a complete metric space with respect to the metric induced by its norm. The $L^2(a,\, b)$ is a Banach space.

(11) One can often define an *inner product* $(\cdot,\, \cdot)$ on a normed space S by, for example $L^2(a,\, b)$:

$$(f,\, g) := \int_a^b f(x)\, \overline{g(x)}\, dx$$

So $\|f\| = (f,\, f)^{1/2}$. The S is called an *Hilbert space* if S is an inner product space which is a complete metric space with respect to the metric induced by the inner product. The $L^2(a,\, b)$ is an Hilbert space. Here the completeness is equivalent to that mentioned in the definition 4.

Exercise

Q1 A system of $\{\psi_j\}_0^{\infty}$ defined on $[a,\, b]$ which is not necessary orthogonal is said to be *complete* if every square integrable function g can be approximated in the mean by a linear combination of the $\psi_j s$. This means that for any g on $[a,\, b]$, and any given $\varepsilon > 0$, we can find numbers $\gamma_0, \cdots, \gamma_n$ such that

$$\int_a^b (g(x) - (\gamma_0 \psi_0(x) + \cdots + \gamma_n \psi_n(x))^2\, dx < \varepsilon.$$

Prove that if the system $\{\psi_j\}_0^{\infty}$ is complete, then any continuous function which is orthogonal to all the functions of the system must be zero.

Q2 Explain why the system $\{\sin x,\ \sin 3x,\ \cdots, \sin(2n+1)x, \cdots\}$ can approximate any function on $[0,\ \pi/2]$. (Hint: there is no need to use completeness idea. Please read pp. 45-49 of Tolstov).

Q3 (optional) The *Legendre polynomial of degree* n is defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Show that

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{2}{2n+1} & \text{if } n = m. \end{cases}$$

Q4 (optional) Expand $f(x) = |x|$ in $(-1,\ 1)$ as a series in terms of the Legendre polynimials.