MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

6. Further properties of Fourier series

6.1. Fourier Series with decreasing coefficients.

Lemma 6.1 (Abel's Lemma). *Let $\{u_n\}$ be a sequence of numbers and σ_n to be the $n-th$ partial sum of*

$$\sigma = u_0 + u_1 + u_2 + \dots$$

such that $|\sigma_n| \leq M$ for some $M > 0$, and for $n = 0, 1, 2, \dots$ Suppose α_0, α_1, α_2, α_3, $\dots$ are positive, monotone decreasing such that α_n decreases to 0. Then the series

$$\alpha_0 u_0 + \alpha_1 u_1 + \dots + \alpha_n u_n + \dots$$

converges to s, and $|s| \leq M\alpha_0$.

Proof. Let

$$\begin{aligned}
s_n &= \alpha_0 u_0 + \alpha_1 u_1 + \dots + \alpha_n u_n \\
&= \alpha_0 u_0 + \sum_{j=0}^{n-1} \alpha_{j+1}(\sigma_{j+1} - \sigma_j) \\
&= \alpha_0 u_0 + \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_{j+1} - \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_j \\
&= \alpha_0 u_0 + \sum_{j=1}^{n} \alpha_j\sigma_j - \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_j \\
&= \alpha_0 u_0 + \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j + \alpha_n\sigma_n - \alpha_1\sigma_0
\end{aligned}$$

$$\begin{aligned}
|s_n - \alpha_n\sigma_n| &= \Big|\sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j\Big| \\
&\leq M \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1}) \\
&= M(\alpha_0 - \alpha_n)
\end{aligned}$$

Notice that the series

$$M \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1}) = M(\alpha_0 - \alpha_n)$$

is convergent to $M\alpha_0$. This implies the series

$$\sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j$$

is absolutely convergent. This implies that $s_n - \alpha_n\sigma_n$ also converges

$$\lim_{n\to\infty}(s_n - \alpha_n\sigma_n) = \lim_{n\to\infty} s_n = s$$

since σ_n is bounded and $\alpha_n \to 0$ as $n \to \infty$. Moreover,

$$\lim_{n\to\infty} s_n = s \le M\alpha_0.$$

□

6.2. **Some trigonometric identities and inequalities.** Recall that

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}. \tag{6.1}$$

We deduce that when $x \neq 2k\pi$ $(k = 0, \pm 1, \pm 2, \cdots)$

$$\Big|\frac{1}{2} + \sum_{k=1}^{n}\cos kx\Big| \le \frac{1}{2\sin(\frac{x}{2})}$$

(the sum equals $k + \frac{1}{2}$ when $x = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ and so unbounded). Now we consider

$$s = \sum_{k=1}^{n}\sin kx = \sin x + \sin 2x + \cdots + \sin nx.$$

We recall the identity:

$$2\sin A \sin B = \cos(A - B) - \cos(A + B).$$

Then

$$\begin{aligned} 2s\,\sin\frac{x}{2} &= \sum_{k=1}^{n} 2\,\sin(\frac{x}{2})\,\sin kx \\ &= \sum_{k=1}^{n}\Big[\cos(\frac{x}{2} - kx) - \cos(\frac{x}{2} + kx)\Big] \\ &= \cos\frac{x}{2} - \cos(n + \frac{1}{2})x. \end{aligned}$$

Thus,

$$\Big|\sum_{k=1}^{n}\sin kx\Big| \le \frac{|\cos\frac{x}{2}| + |\cos(n+\frac{1}{2})x|}{2|\sin\frac{x}{2}|} \le \frac{1}{|\sin\frac{x}{2}|}.$$

provided $x \neq 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. In the case when $x = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, then both sides of the sum vanish.

Theorem 6.2. *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad \sum_{k=1}^{\infty} b_n \sin nx$$

be two trigonometric series. If the coefficients a_n, b_n are positive and decrease monotonically to 0 *as $n \to \infty$, then both series converge uniformly on any interval $[a,\, b]$ that does not contain the points $x = 2k\pi$, $k = 0,\, \pm 1,\, \pm 2, \ldots$.*

Proof. Since the two series are very similar, we only consider the cosine series.

Let $s_n(x)$ be the partial sum of the cosine series, and let $\sigma_n(x)$ be the $n-$th partial sum of (6.1):

$$\frac{1}{2} + \cos x + \cos 2x + \ldots$$

Let $\epsilon > 0$ be given, $0 < a \le x \le b < 2\pi$, and

$$\tau_m(x) = \sigma_{n+m}(x) - \sigma_n(x) = \sum_{k=n+1}^{n+m} \cos kx.$$

We deduce

$$|\tau_m(x)| \le |\sigma_{n+m}(x)| + |\sigma_n(x)| \le 2 \cdot \frac{1}{2|\sin \frac{x}{2}|} = \frac{1}{|\sin \frac{x}{2}|}.$$

Let

$$\sin \frac{x}{2} \ge \mu = \min \Big\{ \sin \frac{a}{2},\, \sin \frac{b}{2} \Big\} > 0$$

and we set

$$|\tau_m(x)| \le \frac{1}{\mu} := M \qquad \text{for all } x \in [a,\, b].$$

Let

$$s_n(x) = \sum_{k=1}^{n} a_k \cos kx$$

and we write formally

$$s = \sum_{k=1}^{\infty} a_k \cos kx.$$

Then,

$$s - s_n(x) = a_{n+1} \cos(n+1)x + a_{n+2} \cos(n+2)x + \ldots$$

Since $|\tau_m(x)| \le M$ for $m \ge 1$, and $a_k > 0$ with a_k decreases to 0. So by Abel's lemma,

$$|s(x) - s_n(x)| \le a_{n+1} M < \epsilon \qquad \text{for } x \in [a,\, b],$$

provided that n is chosen to be sufficiently large. This also implies that the convergence of s_n to s is uniform on $[a,\, b]$. □

6.3. **Complex Function Method on Fourier Series.** Let $F(z)$ be analytic on $\{z : |z| \le 1\}$. Then

$$f(x) = \sum_{j=1}^{\infty} c_j z^j, \qquad |z| \le 1.$$

If c_j, $j = 0, 1, 2, \ldots$ are all real, then

$$\begin{aligned} F(e^{i\theta}) &= \sum_{j=0}^{\infty} c_j e^{ji\theta} = \sum_{j=0}^{\infty} c_j(\cos j\theta + i \sin j\theta) \\ &= \sum_{j=0}^{\infty} c_j \cos(j\theta) + i \sum_{j=0}^{\infty} c_j \sin(j\theta). \end{aligned}$$

If $F(e^{i\theta}) = f(\theta) + ig(\theta)$, where f, g are real functions of x, then

$$f(\theta) = \sum_{j=0}^{\infty} c_j \cos(j\theta), \qquad g(\theta) = \sum_{j=0}^{\infty} c_j \sin(j\theta).$$

Example. $F(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$.

Then

$$F(e^{i\theta}) = e^{e^{i\theta}} = \Big(1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots\Big) + i\Big(\sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \cdots\Big)$$

But

$$e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\Big(\cos(\sin\theta) + i\sin(\sin\theta)\Big).$$

So

$$\begin{aligned} e^{\cos\theta}\cos(\sin\theta) &= 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots \\ e^{\cos\theta}\sin(\sin\theta) &= \sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \cdots \end{aligned}$$

Example. Find the sums of the series

$$1 + \frac{\cos x}{p} + \frac{\cos 2x}{p^2} + \cdots + \frac{\cos nx}{p^n} + \cdots$$

and

$$\frac{\sin x}{p} + \frac{\sin 2x}{p^2} + \cdots + \frac{\sin nx}{p^n} + \cdots$$

where p is a real constant such that $|p| > 1$. Thus the two series converge for all x.

It is easy to see that

$$\Big(1+\frac{\cos x}{p}+\frac{\cos 2x}{p^2}+\cdots+\frac{\cos nx}{p^n}+\cdots\Big)+i\Big(\frac{\sin x}{p}+\frac{\sin 2x}{p^2}+\cdots+\frac{\sin nx}{p^n}+\cdots\Big)$$

$$=1+\frac{e^{ix}}{p}+\frac{e^{2ix}}{p^2}+\cdots+\frac{e^{nix}}{p^n}+\cdots$$

$$=1+\frac{e^{ix}}{p}+\Big(\frac{e^{ix}}{p}\Big)^2+\cdots+\Big(\frac{e^{ix}}{p}\Big)^n+\cdots$$

$$=\frac{1}{1-(\frac{e^{ix}}{p})}=\frac{p}{p-e^{ix}}=\frac{p}{p-\cos x-i\sin x}$$

$$=\frac{p(p-\cos x+i\sin x)}{(p-\cos x)^2+\sin x^2}=\frac{p(p-\cos x)}{(p-\cos x)^2+\sin^2 x}+i\frac{p\sin x}{(p-\cos x)^2+\sin^2 x}$$

That is,

$$\frac{p(p-\cos x)}{p^2-2p\cos x+1}=1+\frac{\cos x}{p}+\frac{\cos 2x}{p^2}+\cdots$$

and

$$\frac{p\sin x}{p^2-2p\cos x+1}=\frac{\sin x}{p}+\frac{\sin 2x}{p^2}+\cdots$$

Exercise

Q1 Let $a_0+a_1+a_2+\cdots$ be a convergent series. Use Abel's convergence test to determine the convergence of the following power series. Please discuss the region of convergence.
(1) $\sum_{k=0}^{\infty} a_k x^k$.
(2) $\sum_{k=0}^{\infty} a_k k^{-x}$.

Q2 For which values of x do the following series converge:
(1) $\sum_{k=1}^{\infty}\frac{\cos kx}{\sqrt{k}}$,
(2) $\sum_{k=1}^{\infty}\frac{\sin kx}{\sqrt{k}}$,
(3) $\sum_{k=1}^{\infty}\frac{\cos kx+\sin kx}{\sqrt{k}}$.
For which values of x are the sums of the series continuous. Which of the series are square integrable functions?

Q3 Find the sums of the series
(1) $\cos x-\frac{\cos 3x}{3!}+\frac{\cos 5x}{5!}-\cdots$. $(F(x)=\sin x)$.
(2) $\frac{\sin 2x}{2!}-\frac{\sin 4x}{4!}+\frac{\sin 6x}{6!}-\cdots$. $(F(x)=\cos x)$.
You may assume $\sin(\alpha+i\beta)=\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta$, and $\cos(\alpha+i\beta)=\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta$.

To be continued ...