MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

6. Further properties of Fourier series

6.1. Fourier Series with decreasing coefficients.

Lemma 6.1 (Abel's Lemma). *Let $\{u_n\}$ be a sequence of numbers and σ_n to be the $n-th$ partial sum of*

$$\sigma = u_0 + u_1 + u_2 + \dots$$

such that $|\sigma_n| \leq M$ for some $M > 0$, and for $n = 0, 1, 2, \dots$ Suppose α_0, α_1, α_2, α_3, $\dots$ are positive, monotone decreasing such that α_n decreases to 0. Then the series

$$\alpha_0 u_0 + \alpha_1 u_1 + \dots + \alpha_n u_n + \dots$$

converges to s, and $|s| \leq M\alpha_0$.

Proof. Let

$$\begin{aligned}
s_n &= \alpha_0 u_0 + \alpha_1 u_1 + \dots + \alpha_n u_n \\
&= \alpha_0 u_0 + \sum_{j=0}^{n-1} \alpha_{j+1}(\sigma_{j+1} - \sigma_j) \\
&= \alpha_0 u_0 + \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_{j+1} - \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_j \\
&= \alpha_0 u_0 + \sum_{j=1}^{n} \alpha_j\sigma_j - \sum_{j=0}^{n-1} \alpha_{j+1}\sigma_j \\
&= \alpha_0 u_0 + \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j + \alpha_n\sigma_n - \alpha_1\sigma_0
\end{aligned}$$

$$\begin{aligned}
|s_n - \alpha_n\sigma_n| &= \Big|\sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j\Big| \\
&\leq M \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1}) \\
&= M(\alpha_0 - \alpha_n)
\end{aligned}$$

Notice that the series

$$M \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1}) = M(\alpha_0 - \alpha_n)$$

is convergent to $M\alpha_0$. This implies the series

$$\sum_{j=1}^{n-1} (\alpha_j - \alpha_{j+1})\sigma_j$$

is absolutely convergent. This implies that $s_n - \alpha_n\sigma_n$ also converges

$$\lim_{n\to\infty}(s_n - \alpha_n\sigma_n) = \lim_{n\to\infty} s_n = s$$

since σ_n is bounded and $\alpha_n \to 0$ as $n \to \infty$. Moreover,

$$\lim_{n\to\infty} s_n = s \le M\alpha_0.$$

$\square$

6.2. **Some trigonometric identities and inequalities.** Recall that

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}. \tag{6.1}$$

We deduce that when $x \neq 2k\pi$ $(k = 0,\ \pm1,\ \pm2, \cdots)$

$$\Big|\frac{1}{2} + \sum_{k=1}^{n}\cos kx\Big| \le \frac{1}{2\sin(\frac{x}{2})}$$

(the sum equals $k+\frac{1}{2}$ when $x = 2k\pi,\ k = 0,\ \pm1,\ \pm2, \ldots$ and so unbounded). Now we consider

$$s = \sum_{k=1}^{n}\sin kx = \sin x + \sin 2x + \cdots + \sin nx.$$

We recall the identity:

$$2\sin A\sin B = \cos(A-B) - \cos(A+B).$$

Then

$$\begin{aligned}
2s\ \sin\frac{x}{2} &= \sum_{k=1}^{n} 2\ \sin(\frac{x}{2})\ \sin kx \\
&= \sum_{k=1}^{n}\Big[\cos(\frac{x}{2} - kx) - \cos(\frac{x}{2} + kx)\Big] \\
&= \cos\frac{x}{2} - \cos(n+\frac{1}{2})x.
\end{aligned}$$

Thus,

$$\Big|\sum_{k=1}^{n}\sin kx\Big| \le \frac{|\cos\frac{x}{2}| + |\cos(n+\frac{1}{2})x|}{2|\sin\frac{x}{2}|} \le \frac{1}{|\sin\frac{x}{2}|}.$$

provided $x \neq 2k\pi,\ \ k = 0, \pm1, \pm2, \ldots.$ In the case when $x = 2k\pi,\ \ k = 0,\ \pm1,\ \pm2,\ \ldots,$ then both sides of the sum vanish.

Theorem 6.2. *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad \sum_{k=1}^{\infty} b_n \sin nx$$

be two trigonometric series. If the coefficients a_n, b_n are positive and decrease monotonically to 0 *as $n \to \infty$, then both series converge uniformly on any interval $[a, b]$ that does not contain the points $x = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.*

Proof. Since the two series are very similar, we only consider the cosine series.

Let $s_n(x)$ be the partial sum of the cosine series, and let $\sigma_n(x)$ be the $n-$th partial sum of (6.1):

$$\frac{1}{2} + \cos x + \cos 2x + \ldots$$

Let $\epsilon > 0$ be given, $0 < a \leq x \leq b < 2\pi$, and

$$\tau_m(x) = \sigma_{n+m}(x) - \sigma_n(x) = \sum_{k=n+1}^{n+m} \cos kx.$$

We deduce

$$|\tau_m(x)| \leq |\sigma_{n+m}(x)| + |\sigma_n(x)| \leq 2 \cdot \frac{1}{2|\sin\frac{x}{2}|} = \frac{1}{|\sin\frac{x}{2}|}.$$

Let

$$\sin\frac{x}{2} \geq \mu = \min\left\{\sin\frac{a}{2},\, \sin\frac{b}{2}\right\} > 0$$

and we set

$$|\tau_m(x)| \leq \frac{1}{\mu} := M \qquad \text{for all } x \in [a,\, b].$$

Let

$$s_n(x) = \sum_{k=1}^{n} a_k \cos kx$$

and we write formally

$$s = \sum_{k=1}^{\infty} a_k \cos kx.$$

Then,

$$s - s_n(x) = a_{n+1}\cos(n+1)x + a_{n+2}\cos(n+2)x + \ldots$$

Since $|\tau_m(x)| \leq M$ for $m \geq 1$, and $a_k > 0$ with a_k decreases to 0. So by Abel's lemma,

$$|s(x) - s_n(x)| \leq a_{n+1} M < \epsilon \qquad \text{for } x \in [a,\, b],$$

provided that n is chosen to be sufficiently large. This also implies that the convergence of s_n to s is uniform on $[a,\, b]$. □

6.3. **Complex Function Method on Fourier Series.** Let $F(z)$ be analytic on $\{z : |z| \leq 1\}$. Then

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \qquad |z| \leq 1.$$

If c_j, $j = 0, 1, 2, \dots$ are all real, then

$$\begin{aligned} F(e^{i\theta}) &= \sum_{j=0}^{\infty} c_j e^{ji\theta} = \sum_{j=0}^{\infty} c_j (\cos j\theta + i \sin j\theta) \\ &= \sum_{j=0}^{\infty} c_j \cos(j\theta) + i \sum_{j=0}^{\infty} c_j \sin(j\theta). \end{aligned}$$

If $F(e^{i\theta}) = f(\theta) + ig(\theta)$, where f, g are real functions of x, then

$$f(\theta) = \sum_{j=0}^{\infty} c_j \cos(j\theta), \qquad g(\theta) = \sum_{j=0}^{\infty} c_j \sin(j\theta).$$

Example. $F(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$.

Then

$$F(e^{i\theta}) = e^{e^{i\theta}} = \Big(1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots\Big) + i\Big(\sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \cdots\Big)$$

But

$$e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\Big(\cos(\sin\theta) + i\sin(\sin\theta)\Big).$$

So

$$\begin{aligned} e^{\cos\theta}\cos(\sin\theta) &= 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots \\ e^{\cos\theta}\sin(\sin\theta) &= \sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \cdots \end{aligned}$$

Example. Find the sums of the series

$$1 + \frac{\cos x}{p} + \frac{\cos 2x}{p^2} + \cdots + \frac{\cos nx}{p^n} + \cdots$$

and

$$\frac{\sin x}{p} + \frac{\sin 2x}{p^2} + \cdots + \frac{\sin nx}{p^n} + \cdots$$

where p is a real constant such that $|p| > 1$. Thus the two series converge for all x.

It is easy to see that

$$\begin{aligned}
&\Big(1+\frac{\cos x}{p}+\frac{\cos 2x}{p^2}+\cdots+\frac{\cos nx}{p^n}+\cdots\Big)+i\Big(\frac{\sin x}{p}+\frac{\sin 2x}{p^2}+\cdots+\frac{\sin nx}{p^n}+\cdots\Big)\\
&=1+\frac{e^{ix}}{p}+\frac{e^{2ix}}{p^2}+\cdots+\frac{e^{nix}}{p^n}+\cdots\\
&=1+\frac{e^{ix}}{p}+\Big(\frac{e^{ix}}{p}\Big)^2+\cdots+\Big(\frac{e^{ix}}{p}\Big)^n+\cdots\\
&=\frac{1}{1-(\frac{e^{ix}}{p})}=\frac{p}{p-e^{ix}}=\frac{p}{p-\cos x-i\sin x}\\
&=\frac{p(p-\cos x+i\sin x)}{(p-\cos x)^2+\sin x^2}=\frac{p(p-\cos x)}{(p-\cos x)^2+\sin^2 x}+i\frac{p\sin x}{(p-\cos x)^2+\sin^2 x}
\end{aligned}$$

That is,

$$\frac{p(p-\cos x)}{p^2-2p\cos x+1}=1+\frac{\cos x}{p}+\frac{\cos 2x}{p^2}+\cdots$$

and

$$\frac{p\sin x}{p^2-2p\cos x+1}=\frac{\sin x}{p}+\frac{\sin 2x}{p^2}+\cdots$$

Exercise

Q1 Let $a_0+a_1+a_2+\cdots$ be a convergent series. Use Abel's convergence test to determine the convergence of the following power series. Please discuss the region of convergence.

(1) $\sum_{k=0}^{\infty} a_k x^k$.

(2) $\sum_{k=0}^{\infty} a_k k^{-x}$.

Q2 For which values of x do the following series converge:

(1) $\sum_{k=1}^{\infty} \frac{\cos kx}{\sqrt{k}}$,

(2) $\sum_{k=1}^{\infty} \frac{\sin kx}{\sqrt{k}}$,

(3) $\sum_{k=1}^{\infty} \frac{\cos kx+\sin kx}{\sqrt{k}}$.

For which values of x are the sums of the series continuous. Which of the series are square integrable functions?

Q3 Find the sums of the series

(1) $\cos x-\frac{\cos 3x}{3!}+\frac{\cos 5x}{5!}-\cdots$. $(F(x)=\sin x)$.

(2) $\frac{\sin 2x}{2!}-\frac{\sin 4x}{4!}+\frac{\sin 6x}{6!}-\cdots$. $(F(x)=\cos x)$.

You may assume $\sin(\alpha+i\beta)=\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta$, and $\cos(\alpha+i\beta)=\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta$.

6.4. **Addition and Subtraction on Fourier series.**

Suppose

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$
$$F(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx.$$

Then

$$\begin{aligned}\alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x) \pm F(x)\big) \cos nx \ dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \ dx \pm \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos nx \ dx \\ &= a_n \pm A_n,\end{aligned}$$

and similarly,

$$\beta_n = b_n \pm B_n,$$

even though the Fourier series of f and F might not necessary be convergent at x (so is $f \pm F$).

6.5. **Integration of Fourier series.**

Theorem 6.3. *Let $f(x)$be an absolutely integrable function of period 2π. Then the Fourier series can be integrated term by term, whether or not the series converges, that is,*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$

then

$$\begin{aligned}\int_a^b f(x) \ dx &= \int_a^b \frac{a_0}{2} \ dx + \sum_{k=1}^{\infty} \int_a^b (a_k \cos kx + b_k \sin kx) \ dx \\ &= \frac{a_0}{2}(b-a) + \sum_{k=1}^{\infty} \frac{a_k(\sin kb - \sin ka) - b_k(\cos kb - \cos ka)}{k}\end{aligned}$$

Remark. Since $\{\cos kx, \sin kx\}$ is a complete orthogonal system, and if f is square integrable, then the above theorem holds automatically from the Theorem 4.9.

Proof. Let

$$F(x) = \int_0^x \big(f(x) - \frac{a_0}{2}\big) \ dx$$

F is clearly continuous and has an absolutely integrable derivative (so it may not exist for a finite number of points). Observe that

$$\begin{aligned} F(x+2\pi) &= \int_0^x \big(f(x) - \frac{a_0}{2}\big)\, dx + \int_x^{x+2\pi} \big(f(x) - \frac{a_0}{2}\big)\, dx \\ &= F(x) + \int_{-\pi}^{\pi} \big(f(x) - \frac{a_0}{2}\big)\, dx \\ &= F(x) + \int_{-\pi}^{\pi} f(x)\, dx - \pi a_0 \\ &= F(x) + 0 = F(x). \end{aligned}$$

Hence $F(x)$ is a continuous periodic function with period 2π, which F has an absolutely integrable derivative. Theorem 5.10 asserts that F can be expanded by its Fourier series which converges uniformly. Let

$$F(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos kx + B_k \sin kx,$$

say. Integrating-by-parts yields

$$\begin{aligned} A_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx\, dx \\ &= \frac{1}{\pi} \Big[\frac{F(x)\sin kx}{k}\Big]_{-\pi}^{\pi} - \frac{1}{k\pi} \int_{-\pi}^{\pi} \big(f(x) - \frac{a_0}{2}\big) \sin kx\, dx \\ &= 0 + \Big(-\frac{b_k}{k}\Big) = -\frac{b_k}{k}. \end{aligned}$$

Similarly, we obtain

$$B_n = \frac{a_n}{n}.$$

Thus,

$$F(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{k}.$$

Hence

$$\int_0^x f(x)\, dx = \frac{a_0 x}{2} + \frac{A_0}{2} + \sum_{k=1}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{k} \tag{6.2}$$

and we obtain the desired series by considering

$$\Big(\int_0^b - \int_0^a\Big) f(x)\, dx.$$

□

Theorem 6.4. *Let f be an absolutely integrable function, with*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$

whether or not the series converges. Then we can integrate the series term by term, that is,

$$\int_0^x f(x)\ dx = \sum_{k=1}^{\infty} \frac{b_k}{k} + \sum_{k=1}^{\infty} \frac{-b_k \cos kx + \big(a_k + (-1)^{k+1} a_0\big) \sin kx}{k}, \tag{6.3}$$

$x \in (-\pi, \pi)$. *In particular, if* $a_0 = 0$*, then*

$$\int_0^x f(x)\ dx = \sum_{k=1}^{\infty} \frac{b_k}{k} + \sum_{k=1}^{\infty} \frac{-b_k \cos kx + a_k \sin kx}{k} \tag{6.4}$$

for all x*.*

Proof. Setting $x = 0$ in the formula (6.2) gives

$$\frac{A_0}{2} = \sum_{k=1}^{\infty} \frac{b_k}{k}.$$

On the other hand, recall that the Fourier series of an absolutely integrable function of x on $(-\pi, \pi)$ is given by

$$\frac{x}{2} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k},$$

thus the (6.2) becomes

$$\begin{aligned} \int_0^x f(x)\ dx = \sum_{k=1}^{\infty} \frac{b_k}{k} &+ a_0 \Big(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k} \Big) \\ &+ \sum_{k=1}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{k} \end{aligned}$$

This gives the (6.3). It remains to check that (6.4) holds for all x. This follows from the fact that $\int_0^x f\, dx$ is of period 2π:

$$\begin{aligned} \int_0^{x+2\pi} f(x)\ dx &= \int_0^x f(x)\ dx + \int_x^{x+2\pi} f(x)\ dx \\ &= \int_0^x f(x)\ dx + \frac{a_0}{2}(x + 2\pi - x) \\ &= \int_0^x f(x)\ dx. \end{aligned}$$

□

Remark. As a by-product, we have established that for an absolutely integrable function f, its Fourier series satisfies

$$\sum_{k=1}^{\infty} \frac{b_k}{k}$$

being convergent. Thus although the series $\sum\limits_{k=1}^{\infty} \dfrac{\sin kx}{\ln k}$ is convergent, $\sum\limits_{k=1}^{\infty} \dfrac{1}{k \ln k}$ is divergent, so it is not the Fourier series of an absolutely integrable function.

Example. Since the Fourier series of an absolutely integrable function of x on $(-\pi, \pi)$ is given by

$$\frac{x}{2} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k},$$

Theorem 6.4 gives

$$\begin{aligned}
\frac{x^2}{4} = \int_0^x \frac{x}{2}\, dx &= \sum_{k=1}^{\infty} \frac{b_k}{k} + \sum_{k=1}^{\infty} \frac{-b_k \cos kx}{k} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos kx \\
&= K - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos kx.
\end{aligned}$$

So it remains to determine the constant K. Since the right-hand side converges uniformly, we may integrate term-by-term over $(-\pi, \pi)$ to obtain

$$\begin{aligned}
\frac{\pi^3}{6} = \int_{-\pi}^{\pi} \frac{x^2}{4}\, dx &= 2\pi K - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \int_{-\pi}^{\pi} \cos kx dx \\
&= 2\pi K - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \frac{\sin kx}{k} \Big|_{-\pi}^{\pi} \\
&= 2\pi K + 0.
\end{aligned}$$

Hence $K = \dfrac{\pi^2}{12}$. Thus,

$$-\frac{x^2}{4} + \frac{\pi^3}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos kx,$$

a formula that we have obtained in Chapter 2. Integrating this formula on both sides again from 0 to x yields

$$-\frac{x^3}{12} + \frac{\pi^2 x}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sin kx.$$

This is another Fourier series.

Theorem 6.5. *Let f be a continuous function of period 2π, which has $m-1$ continuous derivatives and the $m-$th derivative is absolutely integrable (That is, $f^{(m)}(x)$ may not exist for a finite number of points). Then, the Fourier series of all m derivatives can be obtained by term-by-term differentiation of the Fourier series of $f(x)$, where all the series, except possibly the last one, converge to the corresponding derivatives. Moreover, the Fourier series coefficients of $f(x)$ satisfy*

$$\lim_{n\to\infty} n^m a_n = 0 = \lim_{n\to\infty} n^m b_n.$$

Proof. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

and suppose a_n', b_n' are the Fourier coefficients of $f'(x)$. Then

$$\begin{aligned} a_n' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx \; dx = \frac{1}{\pi} f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx \\ &= 0 + nb_n. \end{aligned}$$

Similarly, $b_n' = -na_n$. Thus,

$$f'(x) \sim \sum_{k=1}^{\infty} k(b_k \cos kx - a_k \sin kx).$$

Since f has continuous derivatives, Theorem 5.10 gives

$$f'(x) = \sum_{k=1}^{\infty} k\,(b_k \cos kx - a_k \sin kx).$$

We may repeat the above argument to get

$$\begin{aligned} a_n &= -\frac{b_n'}{n} = -\frac{a_n''}{n^2} = \frac{b_n'''}{n^3} = \cdots = \frac{\alpha_n}{n^m}, \\ b_n &= \frac{a_n'}{n} = -\frac{b_n''}{n^2} = -\frac{a_n'''}{n^3} = \cdots = \frac{\beta_n}{n^m}, \end{aligned}$$

where a_n', a_n'', $\ldots, b_n'$, b_n'', $\ldots$ are the Fourier coefficients of $f'(x)$, $f''(x)$, $\ldots$, and α_n and β_n are the Fourier coefficients of $f^{(m)}(x)$. Since $f^{(m)}$ is absolutely integrable, the Riemann-Lebesgue Lemma implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} n^m b_n, \\ 0 &= \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} n^m a_n. \end{aligned}$$

□

Theorem 6.6. *Suppose that the coefficients of*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{6.5}$$

satisfy

$$|n^m a_n| \leq M, \qquad |n^m b_n| \leq M \qquad (M \geq 2) \tag{6.6}$$

for some positive constant M, then the series (6.5) has $(m-2)$ continuous derivatives, which can be obtained by term-by-term differentiation of the series (6.5).

Proof. Let us write

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k k^m}{k^m} \cos kx + \frac{b_k k^m}{k^m} \sin kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{\alpha_k}{k^m} \cos kx + \frac{\beta_k}{k^m} \sin kx \qquad (6.7) \end{aligned}$$

where the hypothesis (6.6) indicates that

$$|\alpha_k| = |k^m a_k| \leq M \qquad \text{and} \qquad |\beta_k| = |k^m b_k| \leq M$$

for all k. We now differentiate the series (6.7) n times ($n \leq m-2$). It is clear that the absolute values of the coefficients of the differentiated series is bounded above by

$$\frac{M}{k^{m-n}}.$$

It follows that the sum of the absolute values of these coefficients converge for each $n = 1, \ldots, m-2$. A simple application of the Weierstrass M-test (Theorem 2.11) shows that (6.7) converges uniformly for each $n = 1, 2, \ldots, m-2$. Theorem 2.10 implies that term-by-term differentiation of the series for $n = 1, 2, \ldots, m-2$ is valid. □

Theorem 6.7. *Let f be a continuous function defined on $[-\pi, \pi]$ with an absolutely integrable derivative (so it may not exist at certain points). Then*

$$f'(x) \sim \frac{c}{2} + \sum_{k=1}^{\infty} \Big[(kb_k + (-1)^k c)\cos kx - ka_k \sin kx\Big],$$

where a_k and b_k are the Fourier coefficients of $f(x)$ and c is given by

$$c = \frac{1}{\pi}\Big[f(\pi) - f(-\pi)\Big].$$

Proof. Set

$$f'(x) \sim \frac{a_0'}{2} + \sum_{k=1}^{\infty} \big(a_k' \cos kx + b_\prime' \sin kx\big).$$

This gives

$$a_0' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x)\ dx = \frac{1}{\pi}\Big(f(\pi) - f(-\pi)\Big).$$

It is not difficult (why?) to see that

$$f'(x) - \frac{a_0}{2} \sim \sum_{k=1}^{\infty} \big(a_k \cos kx + b_k \sin kx\big). \tag{6.8}$$

The Fourier series of the function (according to Theorem 6.4)

$$F(x) = \int_0^x \big(f'(x) - \frac{a_0'}{2}\big)\ dx = f(x) - f(0) - \frac{a_0'}{2} x \qquad (F(0) = 0;\ f \text{ continuous}) \tag{6.9}$$

can be obtained from (6.8) by term-by-term integration. This implies, therefore that the Fourier series of (6.8) can be obtained from (6.9) by term-by-term differentiation. On the other hand, since f has an absolutely integrable derivative, so we know from the Theorem 5.10 that that the Fourier series of f converges uniformly:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

and

$$\frac{x}{2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx.$$

We deduce from the Theorem 6.4 that

$$f(x) - f(0) - \frac{a_0' x}{2} = \frac{a_0}{2} - f(0) + \sum_{k=1}^{\infty} a_k \cos kx + \Big(b_k + \frac{(-1)^k a_0'}{k}\Big) \sin kx$$

and hence

$$f'(x) - \frac{a_0'}{2} \sim \sum_{k=1}^{\infty} \Big(- k a_k \sin kx + \big(k b_k + (-1)^k a_0'\big) \cos kx\Big),$$

which is the desired result after setting $a_0' = c$. □

The following theorem is important to know the analytic nature of $f(x)$ if we are only given the Fourier series of f.

Corollary 6.8. *If $f(\pi) = f(-\pi)$, that is, $c = 0$, then the Fourier series of f' can be obtained directly from the Theorem 6.7 via term-by-term differentiation*

$$f'(x) \sim \sum_{k=1}^{\infty} k\,(b_k \cos kx - a_k \sin kx).$$

Remark. We note that this also follows from the Theorem 6.5 since we can extend f to the $x-$axis with period 2π.

Improving the Convergence of Fourier Series.

Example. Consider

$$f(x) = \sum_{k=2}^{\infty} (-1)^k \frac{k^3}{k^4-1} \sin kx.$$

Since $\dfrac{k^3}{k^4-1} \sim \dfrac{1}{k}$ as $k \to \infty$, we consider

$$\frac{k^3}{k^4-1} - \frac{1}{k} = \frac{1}{k^5-k} \sim \frac{1}{k^5}.$$

Recall that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k} = \frac{x}{2}$$

on $(-\pi, \pi)$. Hence

$$\begin{aligned} f(x) &= \sum_{k=2}^{\infty} \frac{(-1)^k k^3}{k^4-1} \sin kx + \sin x - \sin x \\ &= \sum_{k=2}^{\infty} (-1)^k \left[\frac{k^3}{k^4-1} - \frac{1}{k}\right] \sin kx + (-1) \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx + \sin x - \sin x \\ &= -\frac{x}{2} + \sin x + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^5-k} \sin kx, \qquad x \in (-\pi, \pi). \end{aligned}$$

This improves the convergence from $\dfrac{1}{k}$ to $\dfrac{1}{k^5}$.

Example. Consider

$$f(x) = \sum_{k=1}^{\infty} \frac{k^4-k^2+1}{k^2(k^4+1)} \cos nx.$$

Notice that

$$\frac{k^4-k^2+1}{k^2(k^4+1)} \sim \frac{1}{k^2}.$$

We consider

$$\frac{k^4-k^2+1}{k^2(k^4+1)} - \frac{1}{k^2} = \frac{-1}{k^4+1}.$$

Recall that (see Chapter 2)

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{3x^2 - 6x\pi + 2\pi^2}{12}, \qquad x \in [0, 2\pi].$$

Hence

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{1}{k^2} \cos kx - \sum_{k=1}^{\infty} \frac{1}{k^4+1} \cos kx \\ &= \frac{2x^2 - 6x\pi + 2\pi^2}{12} - \sum_{k=1}^{\infty} \frac{\cos kx}{k^4+1}, \qquad x \in [0,\, 2\pi]. \end{aligned}$$

This improves the convergence from $\dfrac{1}{k^2}$ to $\dfrac{1}{k^4}$.

Example. Consider

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k+a}, \qquad 0 < a < 1.$$

We write

$$\begin{aligned}
\frac{1}{k+a} &= \frac{1}{k}\frac{1}{1+\frac{a}{k}} = \frac{1}{k}\Big(1 - \frac{a}{k} + \frac{a^2}{k^2} - \frac{a^3}{k^4} + \cdots\Big) \\
&= \frac{1}{k}\Big(1 - \frac{a}{k} + \frac{a^2}{k^2} - \frac{a^2}{k^2}\Big[\frac{a}{k} - \frac{a^2}{k^2} + \cdots\Big]\Big) \\
&= \frac{1}{k}\Big[1 - \frac{a}{k} + \frac{a^2}{k^2} - \frac{a^2}{k^2}\Big(\frac{\frac{a}{k}}{1+\frac{a}{k}}\Big)\Big] \\
&= \frac{1}{k}\Big[1 - \frac{a}{k} + \frac{a^2}{k^2} - \frac{a^3}{k^3 + ak^2}\Big] \\
&= \frac{1}{k} - \frac{a}{k^2} + \frac{a^2}{k^3} - \frac{a^3}{k^3(k+a)}.
\end{aligned}$$

We again recall that

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad (0 < x < 2\pi)$$

and

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\ln(2\sin\frac{x}{2}), \qquad (0 < x < 2\pi);$$

and

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12} \qquad 0 \le x \le 2\pi.$$

Integrating the second and third series gives

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^2} = -\int_0^x \ln(2\sin\frac{x}{2})\ dx,$$

and

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \int_0^x \frac{3x^2 - 6\pi x + 2\pi^2}{12}\ dx = \frac{x^3 - 3\pi x^2 + 2\pi^2 x}{12}.$$

Thus, we obtain

$$\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k} - a\sum_{k=1}^{\infty} \frac{\sin kx}{k^2} + a^2\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} - a^3\sum_{k=1}^{\infty} \frac{\sin kx}{k^3(k+a)} \\
&= \frac{\pi - x}{2} + a\int_0^x \ln(2\sin\frac{x}{2})\ dx + a^2(\frac{x^3 - 3\pi x^2 + 2\pi^2 x}{12}) - a^3\sum_{k=1}^{\infty} \frac{\sin kx}{k^3(k+a)}.
\end{aligned}$$

This improves the convergence from $\dfrac{1}{k}$ to $\dfrac{1}{k^4}$.

Remark. We now offer a "theory" concerning the understanding behind the method that improves the convergence of the Fourier series. Suppose a function defined on $[-\pi, \pi]$ with $f(-\pi) \neq f(\pi)$ is to be extended onto the whole real axis. Since the the Fourier series of f converges to the value $\frac{f(\pi+0)+f(\pi-0)}{2}$, and hence the approximations to f cannot be too good, that is, one needs more terms to get a good approximation (small error). If, however, we adjust the function by adding a new function ϕ such that $f(\pi)+\phi(\pi) = f(-\pi)+\phi(-\pi)$, then the Fourier series of $f+\phi$ will converge better (smaller error term). This can usually be done by $\phi(x) = ax + b$ for some suitably chosen constants a and b.

Exercises

Q1 Using the list of identities in section 12, Chapter 5 of Tolstov's book and the Parseval theorem to calculate the sums

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$.

Q2 Using the list of identities in section 12, Chapter 5 of Tolstov's book and the Parseval theorem to calculate

$$\int_0^{\pi} \ln^2\left(2\sin\frac{x}{2}\right)\, dx.$$

Q3 Find the Fourier series of the derivative of

$$f(x) = \sum_{k=1}^{\infty}\left(\frac{\cos kx}{k^3} + (-1)^k \frac{\sin kx}{k+1}\right).$$

Q4 Improve the convergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+k+1}{k(k^3+1)} \sin kx$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{k+1}$

Q5 Show that if $f(x)$ is square integrable and if a_k, and b_k are the Fourier coefficients of f, then

$$\frac{1}{2h}\int_{x-h}^{x+h} f(u)\, du = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k \sin kx)\frac{\sin kh}{kh},$$

where $|h| \leq \pi$.