MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

7. Summability methods

7.1. Cesàro summability.

Arithmetic means. The following idea is due to the Italian geometer Ernesto Cesàro (1859 - 1906). He shows that even if the Fourier series of a given continuous function does not converge, the infinite sum produced by his average summation method always converge.

Let us consider the infinite series

$$u_0 + u_1 + u_2 + u_3 + \dots,$$

and let

$$s_n = u_0 + u_1 + \dots + u_n.$$

Definition. We say the above series is *Cesàro summable to the limit* σ (or summable by the method of arithmetic means to σ) if

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{s_0 + s_1 + \dots + s_{n-1}}{n} = \sigma.$$

We note that a series is *Cesàro* summable does not necessary imply that it is summable in the ordinary sense. For example, the very simple divergent series given by

$$1 - 1 + 1 - 1 + \dots$$

gives

$$s_0 = 1, \qquad s_1 = 0, \qquad s_2 = 1, \dots$$

Thus,

$$\frac{s_0 + s_1 + \dots + s_{n-1}}{n} = \frac{n/2}{n} \to \frac{1}{2}, \qquad \text{as } n \longrightarrow \infty.$$

Hence although the divergent series is *Cesàro* summable.

Conversely, we have

Theorem 7.1. *If a series is convergent and has a limit* σ*, then it is Cesàro summable to the same limit* σ*.*

Proof. Suppose we are given that

$$\lim_{n\to\infty} s_n = \sigma.$$

For any given $\epsilon > 0$, there is $m > 0$ such that

$$|s_n - \sigma| < \frac{\epsilon}{2},$$

for all $n \geq m$. Consider the difference

$$\begin{aligned}\sigma_n - \sigma &= \frac{(s_0 - \sigma) + (s_1 - \sigma) + \cdots + (s_{n-1} - \sigma)}{n} = \frac{1}{n}\sum_{j=0}^{n-1}(s_j - \sigma) \\ &= \frac{1}{n}\sum_{j=0}^{m-1}(s_j - \sigma) + \frac{1}{n}\sum_{j=m}^{n-1}(s_j - \sigma),\end{aligned}$$

where we have assumed that $n > m$. Since m is fixed, so we may choose n so large such that

$$\frac{1}{n}\sum_{j=0}^{m-1}|s_j - \sigma| < \frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned}|\sigma_n - \sigma| &\leq \frac{1}{n}\sum_{j=0}^{m-1}|s_j - \sigma| + \frac{1}{n}\sum_{j=m}^{n-1}|s_j - \sigma| \\ &< \frac{\epsilon}{2} + \frac{n-1-m}{n}\cdot\frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

for $n > m$, that is, when n is sufficiently large. This proves the theorem. □

Cesàro summable of Fourier series. If

$$\begin{aligned}f(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx,\end{aligned}$$

we define

$$\begin{aligned}\sigma_n(x) &= \frac{s_0(x) + s_1(x) + \ldots s_{n-1}(x)}{n} = \frac{1}{n}\sum_{j=0}^{n-1} s_j(x) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n-1}\Big(\frac{n-k}{n}\Big)(a_k \cos kx + b_k \sin kx).\end{aligned}$$

Thus,

$$
\begin{aligned}
\sigma_n(x) &= \frac{1}{n}s_0(x) + \frac{1}{n}s_1(x) + \cdots + \frac{1}{n}s_{n-1}(x) \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(0+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&\qquad + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(1+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&\qquad \vdots \\
&\qquad + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(n-1+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} \frac{f(x+u)}{2\sin\frac{u}{2}}\Big(\sum_{k=0}^{n-1}\sin\big(k+\frac{1}{2}\big)u\Big)\,du.
\end{aligned}
$$

But

$$
2\sin\big(\frac{u}{2}\big)\,\sin\big(k+\frac{1}{2}\big)u = \cos ku - \cos(k+1)u.
$$

So

$$
\begin{aligned}
\sum_{k=0}^{n-1}\sin\big(k+\frac{1}{2}\big)\,u &= \frac{1}{2\sin(\frac{u}{2})}\sum_{k=1}^{n-1}\big(\cos ku - \cos(k+1)u\big) \\
&= \frac{1}{2\sin(\frac{u}{2})}(1-\cos nu) = \frac{2\sin^2(\frac{nu}{2})}{2\sin(\frac{u}{2})} \\
&= \frac{\sin^2(\frac{nu}{2})}{\sin(\frac{u}{2})}.
\end{aligned}
$$

We obtain the new representation that

$$
\sigma_n(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin^2(\frac{nu}{2})}{2\sin^2\frac{u}{2}}\,du.
$$

We note in the above consideration when $f(x)=1$ and hence $\sigma_n(x)=1$ give

$$
1 = \frac{1}{\pi n}\int_{-\pi}^{\pi}\frac{\sin^2(\frac{nu}{2})}{2\sin^2(\frac{u}{2})}\,du = \frac{2}{\pi n}\int_0^{\pi}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du \tag{7.1}
$$

for each $n = 1, 2, 3, \ldots$.

Definition. Let f be defined on $[a, b]$ with at most a finite number of discontinuities. Then f is said to be *absolutely integrable* if $|f(x)|$ is absolutely integrable over $[a, b]$. That is,

$$
\int_a^b |f(x)|\,dx
$$

exists.

Theorem 7.2. *The Fourier series of an absolutely integrable function $f(x)$ of period 2π is Cesàro summable to the limit $f(x)$ at every point of continuity and to the limit*

$$\frac{f(x+0)+f(x-0)}{2}$$

at every point of jump discontinuity.

Proof. It will be sufficient to prove

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_0^{\pi} f(x+u)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{f(x+0)}{2} \tag{7.2}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_{-\infty}^{0} f(x+u)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{f(x-0)}{2} \tag{7.3}$$

since they imply

$$\lim_{n\to\infty}\sigma_n(x)=\frac{f(x+0)+f(x-0)}{2}$$

which gives $\lim\limits_{n\to\infty}\sigma_n(x)=f(x)$ at a continuity point a x.

We will only prove the limit (7.2) above. This is equivalent to the statement

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_0^{\pi}\Big(f(x+u)-f(x+0)\Big)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=0.$$

Given $\epsilon>0$, there is $\delta>0$ such that for $0<u\le\delta$,

$$|f(x+u)-f(x+0)|<\epsilon.$$

Writing the above integral in the form of the sum of two integrals $\int_0^{\delta}\cdots$ and $\int_{\delta}^{\pi}\cdots$ which we denote by I_1 and I_2 respectively.

Thus,

$$\begin{aligned}|I_1|&=\Big|\frac{1}{\pi n}\int_0^{\delta}\big(f(x+u)-f(x+0)\big)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\Big|\\&\le\frac{\epsilon}{\pi n}\int_0^{\delta}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\\&<\frac{\epsilon}{\pi n}\int_0^{\pi}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{\epsilon}{2}.\end{aligned}$$

Also,

$$\begin{aligned}|I_2|&=\Big|\frac{1}{\pi n}\int_{\delta}^{\pi}\big(f(x+u)-f(x+0)\big)\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\Big|\\&\le\frac{1}{2\pi n}\frac{1}{\sin^2\frac{\delta}{2}}\int_{\delta}^{\pi}\big|f(x+u)-f(x+0)\big|\,du,\end{aligned}$$

since $\sin\dfrac{\delta}{2} \le \sin\dfrac{u}{2}$ for $u \in [\delta, \pi]$. It follows that $|I_2| < \dfrac{\epsilon}{2}$ when we choose n to be sufficiently large (f is absolutely integrable). This proves that $|I_1| + |I_2|$ can be made arbitrary small when n tends to infinity. This establishes (7.2). The remaining integral from $-\pi$ to 0 follows similarly. This completes the proof of the Theorem. □

Remark. We note that the estimation of I_2 depends on x. So the convergence of $\sigma_n(x)$ is pointwise.

Definition. The series $\sigma_n(x) = u_0(x)+u_1(x)+\dots$ is said to be *uniformly (Cesàro) summable* on $[a, b]$ to f, if given $\epsilon > 0$, there is N such that

$$\Big|\sum_{k=0}^{\infty}\sigma_n(x) - f(x)\Big| < \epsilon$$

for $n > N$ and for all $x \in [a, b]$.

We can prove

Theorem 7.3. *The Fourier series of an absolutely integrable function f of period 2π is uniformly Cesàro summable to f on every $[\alpha, \beta] \subset [a, b]$ where f is continuous.*

Proof. Let $x \in [\alpha, \beta]$. We choose $\delta > 0$ so small such that $x+u$ lying within $[\alpha, \beta]$. We apply the (7.1) and to write

$$\sigma_n(x) - f(x) = \frac{1}{\pi n}\int_0^{\pi}\big(f(x+u) - f(x)\big)\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du = J_1 + J_2.$$

where J_1 and J_2 stand for the integrals $\int_{-\pi}^{0}\cdots$ and $\int_0^{\pi}\cdots$ respectively. We split the J_1 into the integrals as the sum of $\int_0^{\delta}\cdots$ and $\int_{\delta}^{\pi}\cdots$ as it the proof of the last theorem.

Since f is continuous over $[\alpha, \beta]$, so given $\epsilon > 0$, we choose $\delta > 0$ such that

$$|f(x+u) - f(x)| < \epsilon/2.$$

holds for all $x \in [\alpha, \beta]$ provided $|u| < \delta$. Moreover, we can define $M = \max_{\alpha\le x\le\beta}|f(x)|$. It is easy to see that

$$|I_1| < \frac{\epsilon}{2}, \qquad x \in [\alpha, \beta],$$

On the other hand,

$$\begin{aligned}
|I_2| &= \Big|\frac{1}{\pi n}\int_{\delta}^{\pi}\big(f(x+u) - f(x)\big)\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\Big| \\
&\le \frac{1}{2\pi n}\frac{1}{\sin^2\delta/2}\int_{\delta}^{\pi}\big|f(x+u) - f(x)\big|\,du \\
&\le \frac{1}{2\pi n}\frac{1}{\sin^2\delta/2}\int_{-\pi}^{\pi}\big|f(x+u)\big|\,du + M\pi \\
&\le \frac{1}{2\pi n}\frac{M'}{\sin^2\delta/2}
\end{aligned}$$

for some constant M' since f is absolutely integrable. Hence we can choose $N > 0$ large enough so that

$$|I_2| < \frac{\epsilon}{2}, \qquad x \in [\alpha, \beta].$$

It follows that $|J_1| < \epsilon$. Similarly, we can show $|J_2| < \epsilon$, so that $|J_1| + |J_2| < 2\epsilon$ for all $x \in [\alpha, \beta]$. □

Previous consideration about convergence of Fourier series requires the f to be piecewise continuous and absolutely integrable.

If, however, f is merely continuous, the Fourier series may diverge at certain points. But we obtain

Theorem 7.4. *The Fourier series of a continuous function $f(x)$ of period 2π is uniformly Cesàro summable to $f(x)$.*

Thus, *Cesàro* summable is both superior and surprising.

It follows from our study of Cesàro summability that

Theorem 7.5. *If the Fourier series of an absolutely integrable function f converges at a point x of continuity (respectively a jump discontinuity), then the Fourier series must converge to $f(x)$ (respectively $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$).*

We recall from an earlier theorem that a square integrable function f is completely defined by its trigonometric Fourier series. Theorem 7.1 implies that

Theorem 7.6. *Any absolutely integrable function is completely determined (except for its values at a finite number of points) by its trigonometric Fourier series, whether or not the series converges.*

To be continued ...