MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

7. Summability methods

7.1. Cesàro summability.

Arithmetic means. The following idea is due to the Italian geometer Ernesto Cesàro (1859 - 1906). He shows that even if the Fourier series of a given continuous function does not converge, the infinite sum produced by his average summation method always converge.

Let us consider the infinite series

$$u_0 + u_1 + u_2 + u_3 + \dots,$$

and let

$$s_n = u_0 + u_1 + \cdots + u_n.$$

Definition. We say the above series is *Cesàro summable to the limit* σ (or summable by the method of arithmetic means to σ) if

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \sigma.$$

We note that a series is *Cesàro* summable does not necessary imply that it is summable in the ordinary sense. For example, the very simple divergent series given by

$$1 - 1 + 1 - 1 + \dots$$

gives

$$s_0 = 1, \qquad s_1 = 0, \qquad s_2 = 1, \dots$$

Thus,

$$\frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{n/2}{n} \to \frac{1}{2}, \qquad \text{as } n \longrightarrow \infty.$$

Hence although the divergent series is *Cesàro* summable.

Conversely, we have

Theorem 7.1. *If a series is convergent and has a limit σ, then it is Cesàro summable to the same limit σ.*

Proof. Suppose we are given that

$$\lim_{n\to\infty} s_n = \sigma.$$

For any given $\epsilon > 0$, there is $m > 0$ such that

$$|s_n - \sigma| < \frac{\epsilon}{2},$$

for all $n \geq m$. Consider the difference

$$\begin{aligned}\sigma_n - \sigma &= \frac{(s_0 - \sigma) + (s_1 - \sigma) + \cdots + (s_{n-1} - \sigma)}{n} = \frac{1}{n}\sum_{j=0}^{n-1}(s_j - \sigma) \\ &= \frac{1}{n}\sum_{j=0}^{m-1}(s_j - \sigma) + \frac{1}{n}\sum_{j=m}^{n-1}(s_j - \sigma),\end{aligned}$$

where we have assumed that $n > m$. Since m is fixed, so we may choose n so large such that

$$\frac{1}{n}\sum_{j=0}^{m-1}|s_j - \sigma| < \frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned}|\sigma_n - \sigma| &\leq \frac{1}{n}\sum_{j=0}^{m-1}|s_j - \sigma| + \frac{1}{n}\sum_{j=m}^{n-1}|s_j - \sigma| \\ &< \frac{\epsilon}{2} + \frac{n-1-m}{n}\cdot\frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

for $n > m$, that is, when n is sufficiently large. This proves the theorem. □

Cesàro summable of Fourier series. If

$$\begin{aligned}f(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx,\end{aligned}$$

we define

$$\begin{aligned}\sigma_n(x) &= \frac{s_0(x) + s_1(x) + \ldots s_{n-1}(x)}{n} = \frac{1}{n}\sum_{j=0}^{n-1} s_j(x) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n-1}\Big(\frac{n-k}{n}\Big)(a_k \cos kx + b_k \sin kx).\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma_n(x) &= \frac{1}{n}s_0(x) + \frac{1}{n}s_1(x) + \cdots + \frac{1}{n}s_{n-1}(x) \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(0+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&\quad + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(1+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&\quad \vdots \\
&\quad + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(n-1+\frac{1}{2})u}{2\sin(\frac{u}{2})}\,du \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} \frac{f(x+u)}{2\sin\frac{u}{2}}\Big(\sum_{k=0}^{n-1}\sin\big(k+\frac{1}{2}\big)u\Big)\,du.
\end{aligned}$$

But

$$2\sin\big(\frac{u}{2}\big)\,\sin\big(k+\frac{1}{2}\big)u = \cos ku - \cos(k+1)u.$$

So

$$\begin{aligned}
\sum_{k=0}^{n-1}\sin\big(k+\frac{1}{2}\big)\,u &= \frac{1}{2\sin(\frac{u}{2})}\sum_{k=1}^{n-1}\big(\cos ku - \cos(k+1)u\big) \\
&= \frac{1}{2\sin(\frac{u}{2})}(1-\cos nu) = \frac{2\sin^2(\frac{nu}{2})}{2\sin(\frac{u}{2})} \\
&= \frac{\sin^2(\frac{nu}{2})}{\sin(\frac{u}{2})}.
\end{aligned}$$

We obtain the new representation that

$$\sigma_n(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin^2(\frac{nu}{2})}{2\sin^2\frac{u}{2}}\,du.$$

We note in the above consideration when $f(x)=1$ and hence $\sigma_n(x)=1$ give

$$1 = \frac{1}{\pi n}\int_{-\pi}^{\pi}\frac{\sin^2(\frac{nu}{2})}{2\sin^2(\frac{u}{2})}\,du = \frac{2}{\pi n}\int_0^{\pi}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du \tag{7.1}$$

for each $n = 1, 2, 3, \ldots$.

Definition. Let f be defined on $[a, b]$ with at most a finite number of discontinuities. Then f is said to be *absolutely integrable* if $|f(x)|$ is absolutely integrable over $[a, b]$. That is,

$$\int_a^b |f(x)|\,dx$$

exists.

Theorem 7.2. *The Fourier series of an absolutely integrable function $f(x)$ of period 2π is Cesàro summable to the limit $f(x)$ at every point of continuity and to the limit*

$$\frac{f(x+0)+f(x-0)}{2}$$

at every point of jump discontinuity.

Proof. It will be sufficient to prove

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_0^{\pi} f(x+u)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{f(x+0)}{2} \tag{7.2}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_{-\pi}^{0} f(x+u)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{f(x-0)}{2} \tag{7.3}$$

since they imply

$$\lim_{n\to\infty}\sigma_n(x)=\frac{f(x+0)+f(x-0)}{2}$$

which gives $\lim\limits_{n\to\infty}\sigma_n(x)=f(x)$ at a continuity point a x.

We will only prove the limit (7.2) above. This is equivalent to the statement

$$\lim_{n\to\infty}\frac{1}{\pi n}\int_0^{\pi}\Big(f(x+u)-f(x+0)\Big)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=0.$$

Given $\epsilon>0$, there is $\delta>0$ such that for $0<u\le\delta$,

$$|f(x+u)-f(x+0)|<\epsilon.$$

Writing the above integral in the form of the sum of two integrals $\int_0^{\delta}\cdots$ and $\int_{\delta}^{\pi}\cdots$ which we denote by I_1 and I_2 respectively.

Thus,

$$\begin{aligned}|I_1|&=\Big|\frac{1}{\pi n}\int_0^{\delta}\big(f(x+u)-f(x+0)\big)\,\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\Big|\\&\le\frac{\epsilon}{\pi n}\int_0^{\delta}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\\&<\frac{\epsilon}{\pi n}\int_0^{\pi}\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du=\frac{\epsilon}{2}.\end{aligned}$$

Also,

$$\begin{aligned}|I_2|&=\Big|\frac{1}{\pi n}\int_{\delta}^{\pi}\big(f(x+u)-f(x+0)\big)\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\,du\Big|\\&\le\frac{1}{2\pi n}\frac{1}{\sin^2\frac{\delta}{2}}\int_{\delta}^{\pi}\big|f(x+u)-f(x+0)\big|\,du,\end{aligned}$$

since $\sin\frac{\delta}{2} \le \sin\frac{u}{2}$ for $u \in [\delta, \pi]$. It follows that $|I_2| < \frac{\epsilon}{2}$ when we choose n to be sufficiently large (f is absolutely integrable). This proves that $|I_1| + |I_2|$ can be made arbitrary small when n tends to infinity. This establishes (7.2). The remaining integral from $-\pi$ to 0 follows similarly. This completes the proof of the Theorem. □

Remark. We note that the estimation of I_2 depends on x. So the convergence of $\sigma_n(x)$ is pointwise.

Definition. The series $\sigma_n(x) = u_0(x) + u_1(x) + \dots$ is said to be *uniformly (Cesàro) summable* on $[a, b]$ to f, if given $\epsilon > 0$, there is N such that

$$\Big|\sum_{k=0}^{\infty} \sigma_n(x) - f(x)\Big| < \epsilon$$

for $n > N$ and for all $x \in [a, b]$.

We can prove

Theorem 7.3. *The Fourier series of an absolutely integrable function f of period 2π is uniformly Cesàro summable to f on every $[\alpha, \beta] \subset [a, b]$ where f is continuous.*

Proof. Let $x \in [\alpha, \beta]$. We choose $\delta > 0$ so small such that $x + u$ lying within $[\alpha, \beta]$. We apply the (7.1) and to write

$$\sigma_n(x) - f(x) = \frac{1}{\pi n}\int_0^{\pi} \big(f(x+u) - f(x)\big)\, \frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\, du = J_1 + J_2.$$

where J_1 and J_2 stand for the integrals $\int_{-\pi}^{0}\cdots$ and $\int_0^{\pi}\cdots$ respectively. We split the J_1 into the integrals as the sum of $\int_0^{\delta}\cdots$ and $\int_{\delta}^{\pi}\cdots$ as it the proof of the last theorem.

Since f is continuous over $[\alpha, \beta]$, so given $\epsilon > 0$, we choose $\delta > 0$ such that

$$|f(x+u) - f(x)| < \epsilon/2.$$

holds for all $x \in [\alpha, \beta]$ provided $|u| < \delta$. Moreover, we can define $M = \max_{\alpha \le x \le \beta} |f(x)|$. It is easy to see that

$$|I_1| < \frac{\epsilon}{2}, \qquad x \in [\alpha, \beta],$$

On the other hand,

$$\begin{aligned}
|I_2| &= \Big|\frac{1}{\pi n}\int_{\delta}^{\pi} \big(f(x+u) - f(x)\big)\frac{\sin^2\frac{nu}{2}}{2\sin^2\frac{u}{2}}\, du\Big| \\
&\le \frac{1}{2\pi n}\frac{1}{\sin^2\delta/2}\int_{\delta}^{\pi} \big|f(x+u) - f(x)\big|\, du \\
&\le \frac{1}{2\pi n}\frac{1}{\sin^2\delta/2}\int_{-\pi}^{\pi} \big|f(x+u)\big|\, du + M\pi \\
&\le \frac{1}{2\pi n}\frac{M'}{\sin^2\delta/2}
\end{aligned}$$

for some constant M' since f is absolutely integrable. Hence we can choose $N > 0$ large enough so that

$$|I_2| < \frac{\epsilon}{2}, \qquad x \in [\alpha, \beta].$$

It follows that $|J_1| < \epsilon$. Similarly, we can show $|J_2| < \epsilon$, so that $|J_1| + |J_2| < 2\epsilon$ for all $x \in [\alpha, \beta]$. $\square$

Previous consideration about convergence of Fourier series requires the f to be piecewise continuous and absolutely integrable.

If, however, f is merely continuous, the Fourier series may diverge at certain points. But we obtain

Theorem 7.4. *The Fourier series of a continuous function $f(x)$ of period 2π is uniformly Cesàro summable to $f(x)$.*

Thus, *Cesàro* summable is both superior and surprising.

It follows from our study of Cesàro summability that

Theorem 7.5. *If the Fourier series of an absolutely integrable function f converges at a point x of continuity (respectively a jump discontinuity), then the Fourier series must converge to $f(x)$ (respectively $\frac{1}{2}\big(f(x+0) + f(x-0)\big)$).*

We recall from an earlier theorem that a square integrable function f is completely defined by its trigonometric Fourier series. Theorem 7.1 implies that

Theorem 7.6. *Any absolutely integrable function is completely determined (except for its values at a finite number of points) by its trigonometric Fourier series, whether or not the series converges.*

7.2. **Abel summability and Poisson kernel.**

Abel's summability. (N. H. Abel: Norwegian mathematician (1802-1829))
Consider the series

$$u_0 + u_1 + \cdots + u_n + \ldots, \tag{7.4}$$

and

$$\sigma(r) = u_0 + u_1 r + \cdots + u_n r^n + \ldots \tag{7.5}$$

We assume that the series (7.5) converges for $0 < r < 1$ (which will always be the case if the terms of the series (7.4) are bounded) and that the limit

$$\lim_{r \to 1} \sigma(r) = \sigma$$

exists. If this is the case, then we say that the series (7.4) is *summable by Abel's method* to the value σ.

Abel's method can be used to sum certain divergent series. For example, the series

$$1 - 1 + 1 - 1 + \ldots$$

already encountered in the beginning of this chapter is summable by by the method of arithmetic means and by Abel's method to the value $\sigma = \dfrac{1}{2}$. In fact, the latter is given by case

$$\sigma(r) = 1 - r + r^2 - r^3 + \cdots = \frac{1}{1+r}$$

and therefore

$$\lim_{r \to 1} \sigma(r) = \frac{1}{2},$$

which agrees with the Cesàro sum.

An important question is if the series (7.4) converges, will Abel's method of summation gives the sum of the series (7.5) that agrees with (7.4) as $r \to 1$.

Theorem 7.7. *If the series (7.4) converges and its sum equals σ, then the series is summable by Abel's method to the same number σ.*

To prove this result, first we need the following lemma.

Lemma 7.8. *Let (7.4) and (7.5) be the series defined above. If the series (7.4) is a convergent series (with real or complex terms), then (7.5) converges for $0 \le r \le 1$, and its sum $\sigma(r)$ is continuous on the interval* $[0, 1]$.

Proof. If

$$\sigma = u_0 + u_1 + \cdots + u_n + \ldots,$$

then for every $\epsilon > 0$, there exists an integer N such that if $n \geq N$

$$|\sigma - \sigma_n| \leq \frac{\epsilon}{2},$$

where σ_n is the partial sum of the series (7.4). Consider the remainder of the series (7.4)

$$R_n = \sigma - \sigma_n = u_{n+1} + u_{n+2} + \cdots + u_n + \cdots, \tag{7.6}$$

Since $|\sigma - \sigma_n| \leq \epsilon/2$, we have

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+m}| = |R_n - R_{n+m}| \leq |R_n| + |R_{n+m}| \leq \epsilon$$

for $m = 1, 2, \ldots$. Thus every partial sum of the series (7.6), just like its entire sum, is less than ϵ in absolute value. Since the numbers

$$r^{n+1}, r^{n+2}, \ldots$$

decrease monotonically to zero as $n \to \infty$ for any r $(0 \leq r < 1)$, Abel's lemma (Lemma 6.1) is applicable to the series

$$R_n(r) = u_{n+1}r^{n+1} + u_{n+2}r^{n+2} + \ldots$$

It follows that this series, and hence the series (7.5) converges. Moreover,

$$|R_n(r)| \leq \epsilon r^{n+1} \leq \epsilon$$

for $0 \leq r < 1$. Now let $\sigma_n(r)$ denote the partial sum of the series (7.5) converges. Then we have

$$|\sigma(r) - \sigma_n(r)| = |R_n(r)| \leq \epsilon \tag{7.7}$$

for $0 \leq r < 1$. Noting that $\sigma(1) = \sigma$ and $\sigma_n(1) = \sigma_n$, and because $|\sigma - \sigma_n| \leq \epsilon/2$, we can assert that the inequality (7.7) holds everywhere on the interval $0 \leq r \leq 1$. This proves that the series (7.5) converges uniformly on $0 \leq r \leq 1$ and hence implies that the function $\sigma(r)$ is continuous on $0 \leq r \leq 1$. □

We now prove the theorem.

Proof. If the series (7.4) converges, then by the Lemma 7.8, the series (7.5) converges and its sum $\sigma(r)$ is continuous on the interval $0 \leq r \leq 1$. This means that

$$\lim_{r \to 1} \sigma(r) = \sigma(1) = \sigma,$$

which proves the theorem. □

Poisson Kernel. Consider

$$\frac{1}{2}+\sum_{k=1}^{\infty} r^k \cos k\phi \qquad (0 \le r < 1).$$

We can consider the above series as the real part of the series

$$\frac{1}{2}+\sum_{k=1}^{\infty} z^k = \frac{1}{2}+\sum_{k=1}^{\infty} r^k(\cos k\phi + i \sin k\phi).$$

where $z = re^{i\phi} = r(\cos\phi + i\sin\phi)$. We have

$$\begin{aligned}\frac{1}{2}+\sum_{k=1}^{\infty} z^k &= \frac{1}{2}+\frac{z}{1-z} = \frac{1+z}{2(1-z)}\\ &= \frac{1+r\cos\phi + ir\sin\phi}{2(1-r\cos\phi - ir\cos\phi)} = \frac{1-r^2+2ir\sin\phi}{2(1-2r\cos\phi+r^2)}\end{aligned}$$

Comparing the real parts yield

$$\frac{1}{2}+\sum_{k=1}^{\infty} r^k \cos k\phi = \frac{1}{2}\frac{1-r^2}{1-2r\cos\phi+r^2}, \qquad 0 \le r < 1.$$

which is called the *Poisson kernel*. We note that the kernel is in fact positive for all ϕ and $0 \le r < 1$ since

$$1-2r\cos\phi+r^2 = (1-r)^2 + 4r\sin^2\phi/2 > 0.$$

Relationship with Fourier series. We assume f to be absolutely integrable. Suppose

$$f(x) \sim \frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos k + b_k \sin kx).$$

We define

$$f(x,\, r) = \frac{a_0}{2}+\sum_{k=1}^{\infty} r^n(a_k\cos k + b_k\sin kx), \qquad 0 \le r < 1.$$

Since f is assumed to be absolutely integrable, then $a_k,\, b_k \to 0$ as $k \to \infty$. So one can find a $M > 0$ such that

$$|a_k| \le M, \quad |b_k| \le M, \qquad k \in \mathbb{N}.$$

Hence

$$\sum_{k=1}^{\infty} |r^n(a_k\cos k + b_k\sin kx)| \le \sum_{k=1}^{\infty} 2Mr^k < \infty$$

because $0 \le r < 1$. Writing the Fourier coefficients of f in integral form allows us to rewrite

$$\begin{aligned}
f(x,\, r) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} r^n (a_k \cos k + b_k \sin kx) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt + \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{-\pi}^{\pi} f(t)\, r^n (\cos kt \cos kx + \sin kt \sin kx)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt + \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{-\pi}^{\pi} f(t)\, r^n \cos k(t-x)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} f(t)\, r^n \cos k(t-x)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{\infty} r^n \cos k(t-x)\right)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1-r^2}{1-2r\cos\phi + r^2}\ dt, \qquad 0 \le r < 1,
\end{aligned}$$

where we have interchanged the summation and the integration signs above because the corresponding series converges uniformly. The integral

$$f(x,\, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1-r^2}{1-2r\cos(t-x) + r^2}\ dt, \qquad 0 \le r < 1, \tag{7.8}$$

is called *the Poisson integral* of f. In particular, we have, when $f(x) \equiv 1$, then $a_0/2 = 1$, $a_k = 0 = b{+}k$ for all k and $f(x,\, r) \equiv 1$. Hence

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t-x) + r^2}\ dt, \qquad 0 \le r < 1. \tag{7.9}$$

Theorem 7.9. *Let $f(x)$ be an absolutely integrable function of period π. Then*

$$\lim_{r\to 1} f(x,\, r) = f(x),$$

at every continuity point, and

$$\lim_{r\to 1} f(x,\, r) = \frac{f(x+0) + f(x-0)}{2}$$

at every point of jump discontinuity.

Proof. Let $u = t - x$ in (7.8). Then, as in a very similar in a previous consideration that we have

$$f(x,\, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)\frac{1-r^2}{1-2r\cos u + r^2}\ du, \qquad 0 \le r < 1, \tag{7.10}$$

in view of the integral in (7.10) being periodic of period 2π. We note that at a continuity point x,

$$f(x) = \frac{f(x+0) + f(x-0)}{2}.$$

. Hence it is sufficient to consider the

$$\lim_{r\to 1} \frac{1}{\pi} \int_0^{\pi} f(x+u) \frac{1-r^2}{1-2r\cos u + r^2}\, du = \frac{f(x+0)}{2},$$

and

$$\lim_{r\to 1} \frac{1}{\pi} \int_{-\pi}^{0} f(x+u) \frac{1-r^2}{1-2r\cos u + r^2}\, du = \frac{f(x-0)}{2}$$

separately. But it would be sufficient to consider only the first equation since the since the second one can be dealt with similarly. '

We note that the (7.9) can be written as

$$1 = \frac{1}{\pi} \int_0^{\pi} \frac{1-r^2}{1-2r\cos u + r^2}\, du, \qquad 0 \le r < 1. \tag{7.11}$$

since the integrand is an even function. But then it would be sufficient to prove

$$\lim_{r\to 1} \frac{1}{\pi} \int_0^{\pi} [f(x+u) - f(x+0)] \frac{1-r^2}{1-2r\cos u + r^2}\, du = 0. \tag{7.12}$$

Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|f(x+u) - f(x+0)| \le \epsilon, \qquad 0 < u \le \delta$$

. Hence

$$\begin{aligned}
&\frac{1}{\pi} \int_0^{\pi} [f(x+u) - f(x+0)] \frac{1-r^2}{1-2r\cos u + r^2}\, du \\
&\qquad = \frac{1}{\pi} \int_0^{\delta} [f(x+u) - f(x+0)] \frac{1-r^2}{1-2r\cos u + r^2}\, du \\
&\qquad + \frac{1}{\pi} \int_{\delta}^{\pi} [f(x+u) - f(x+0)] \frac{1-r^2}{1-2r\cos u + r^2}\, du \\
&\qquad = I_1 + I_2,
\end{aligned}$$

say. It follows from (7.11) that

$$\begin{aligned}
|I_1| &\le \frac{1}{\pi} \int_0^{\delta} \big|f(x+u) - f(x+0)\big| \frac{1-r^2}{1-2r\cos u + r^2}\, du \\
&\le \frac{\epsilon}{2} \frac{1}{\pi} \int_0^{\pi} \frac{1-r^2}{1-2r\cos u + r^2}\, du = \frac{\epsilon}{2}.
\end{aligned}$$

The estimate of the second integral I_2 is of no less difficult:

$$|I_2| \le \frac{1-r^2}{2\cdot 4\pi \sin^2 \delta/2} \int_\delta^\pi \big|f(x+u)-f(x+0)\big|\, du \to 0,$$

as $r \to 1$. Hence we can choose r sufficiently close to 1 such that

$$|I_2| \le \frac{\epsilon}{2}.$$

This proves (7.12) and hence the theorem. □

The following discusses about uniformity of Able's summation.

Theorem 7.10. *The Fourier series of an absolutely integrable function $f(x)$ of period 2π is uniformly summable by Abel's method to $f(x)$ on every interval $[\alpha, \beta]$ lying entirely within an interval of continuity $[a, b]$.*

We omit its proof. In particular, we have

Theorem 7.11. *If f is a continuous function of period 2π, then $f(x, r) \to f(x)$ as $r \to 1$, uniformly for all x.*

We recall from the Theorem 6.5 that for a continuous function $f(x)$ of period 2π to have $f''(x)$ absolutely integrable, while $f'(x)$ is continuous, then the Fourier series of f' converges to $f'(x)$. Now we have

Theorem 7.12. *If an absolutely integrable function $f(x)$ of period 2π has derivative up to order m, that is, $f^{(m)}(x)$ exists at x, then the series obtained by term-wise differentiation of the Fourier series of $f(x)$ is summable by Abel's method to the value $f^{(m)}(x)$.*

We omit its proof. However, we explore its consequence. We again consider the well-known example:

$$\frac{x}{2} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k}, \qquad -\pi < x < \pi.$$

Then the above theorem asserts that term-wise differentiation gives

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cos kx \to \frac{1}{2}$$

for which the coefficients do not tend to zero. A further differentiation yields

$$\sum_{k=1}^{\infty} (-1)^{k} k \sin kx \to 0,$$

for which the coefficients clear tend to infinity.

Exercise

Q1 Find the following sums by Cesàro's method:

(i) $\frac{1}{2} + \sum_{k=1}^{\infty} \cos kx$,

(ii) $1 + 0 - 1 + 1 + 0 - 1 + \cdots$.

Q2 Let $f(x)$ be square integrable and if a_k, and b_k are the Fourier coefficients of f. Suppose further that $\sigma_n(x)$ is the nth-Cesàro sum, prove that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [\sigma_n(x) - f(x)]^2\, dx = \frac{1}{n^2}\sum_{k=1}^{n-1} k^2(a_k^2 + b_k^2) + \sum_{k=n}^{\infty}(a_k^2 + b_k^2).$$

Q3 Let $\sum u_k$ be Cesàro summable, and let $t_n = \sum_{k=1}^{n} k u_k$. Show that

(i) $\sum u_k < \infty$; $\iff$ $t_n/n \to 0$.

(ii) If $n u_n \to 0$, then $\sum u_k < \infty$.