MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

10. Fourier integrals

10.1. **Introduction: Extending to infinite period.** In this section we shall study Fourier integrals as a limiting case of the Fourier series.

We first assume that the function $f(x)$ is defined on the $x-$axis and is piecewise continuous on $[-l,\, l]$, for each l. Suppose

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big),$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\, du, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\, du, \tag{10.1}$$

for $k = 0, 1, 2, \dots$ and $b_0 = 0$. We remark that the above Fourier series equals to the value

$$\frac{f(x+0) + f(x-0)}{2}$$

if f has a discontinuity point at x. We now assume, in addition, that f is absolutely integrable on the whole $x-$axis, that is, the integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

exists.

We now substitute the expression of a_k and b_k into the Fourier series above and let l tends to infinity:

$$\begin{aligned}
f(x) = \lim_{l\to\infty} f(x) &= \lim_{l\to\infty}\Big[\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big)\Big]\\
&= \lim_{l\to\infty}\Bigg[\frac{1}{2l}\int_{-l}^{l} f(u)\, du + \sum_{k=1}^{\infty}\frac{1}{l}\Bigg(\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\cos\Big(\frac{\pi k x}{l}\Big)\, du\\
&\qquad + \int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\sin\Big(\frac{\pi k x}{l}\Big)\, du\Bigg)\Bigg]\\
&= \lim_{l\to\infty}\Big[\sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k(u-x)}{l}\Big)\, du\Big]\\
&= \lim_{l\to\infty}\frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\lambda_k\int_{-l}^{l} f(u)\cos[\lambda_k(u-x)]\, du,\\
&= \frac{1}{\pi}\int_{0}^{\infty} d\lambda\int_{-\infty}^{\infty} f(u)\cos\big(\lambda(u-x)\big)\, du.
\end{aligned}$$

where we have set

$$\lambda_k = \frac{k\pi}{l}, \qquad \Delta\lambda_k = \lambda_{k+1} - \lambda_k \quad , k = 1,\, 2,\, 3, \ldots.$$

Although the above reasoning needs further justification, it does indicate what is possible. We further notice that the following possibility

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty f(u)\cos\lambda(u-x)du \\ &= \int_0^\infty d\lambda\, \big(a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\big), \end{aligned}$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\lambda u\ du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\lambda u\ du$$

which is known as a prototype of *Fourier Integral Theorem*. We shall later justify that it actually holds for absolutely integrable function f on $(-\infty, +\infty)$.

We call the improper integral

$$F(\lambda) = \int_0^\infty f(u)\ \cos\lambda(u-x)\ du \tag{10.2}$$

the *Fourier cosine transform* of f.

10.2. **Preparation for the Fourier cosine integral theorem.**

Definition. Let $F(x,\lambda)$ be a function of two variables, and suppose that the integral

$$\int_a^\infty F(x,\lambda)\ dx \tag{10.3}$$

exists for every λ, $\alpha \le \lambda \le \beta$. Then we say that the above integral is *uniformly convergent* for λ in $\alpha \le \lambda \le \beta$ if for every $\epsilon > 0$, there is L such that

$$\Big|\int_l^\infty F(x,\lambda)\ dx\Big| \le \epsilon,$$

whenever $l \ge L$ and for all λ, $\alpha \le \lambda \le \beta$.

Lemma 10.1. *Let x_k be a sequence such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n < \dots \tag{10.4}$$

and that

$$\lim_{k\to\infty} x_k = \infty.$$

Then a necessary and sufficient condition for the integral (10.3) to be uniformly convergent over $[\alpha, \beta]$ is that the series

$$\int_a^\infty F(x,\lambda)\ dx = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$$

is uniformly convergent on λ, $\alpha \le \lambda \le \beta$, as a function of λ, and for every sequence x_k defined by (10.4) above.

Proof. We first suppose that the integral (10.3) is uniformly convergent. That is, given any $\epsilon > 0$, there is L such that

$$\Big| \int_l^\infty F(x,\lambda)\ dx \Big| \le \epsilon$$

whenever $l \ge L$, for λ in $\alpha \le \lambda \le \beta$. Since $x_k \to \infty$, we can find a $N > 0$ such that $x_k > L$ when $k > N$. Thus

$$\begin{aligned}
\Big| \int_a^\infty F(x,\lambda)\ dx - \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \Big| &= \Big| \int_a^\infty F(x,\lambda)\ dx - \int_a^{x_k} F(x,\lambda)\ dx \Big| \\
&= \Big| \int_{x_k}^\infty F(x,\lambda)\ dx \Big| \le \epsilon,
\end{aligned}$$

for $\alpha \le \lambda \le \beta$. Hence the series $\sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$ is uniformly convergent over $[\alpha, \beta]$.

Let us now suppose that this series is uniformly convergent over $[\alpha, \beta]$ and for any sequence $\{x_k\}$ in (10.4). We suppose on the contrary that the integral (10.3) is not uniformly convergent, that is, one can find an $\epsilon > 0$, and an infinite sequence $\{y_k\}$, $y_k \to \infty$ as $k \to \infty$, such that

$$\Big| \int_{y_k}^\infty F(x,\lambda)\ dx \Big| \ge \epsilon$$

for all k. But this implies that when we choose $x_k = y_k$, $k = 1, 2, \dots,$

$$\Big| \int_a^\infty F(x,\lambda) dx - \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \Big| = \Big| \int_{x_n}^\infty F(x,\lambda)\ dx \Big| \ge \epsilon$$

for each n, contradicting the uniform convergence of $\sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$. $\square$

Lemma 10.2. *We suppose $F(x,\lambda)$ is regarded as a continuous function with respect to both of its variables and if the integral*

$$\int_a^\infty F(x,\lambda)\ dx \tag{10.5}$$

is uniformly convergent with respect to λ, $\alpha \le \lambda \le \beta$, then the integral (10.5) defines a continuous function with respect to λ. In addition, we have

$$\int_\alpha^\beta d\lambda \int_a^\infty F(x,\lambda)dx = \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda.$$

Proof. Since the integral (10.5) converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$, the last Lemma asserts that for any sequence $\{x_k\}$, $x_k \nearrow \infty$, the series

$$\sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \tag{10.6}$$

of continuous functions of λ, converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$. Hence the infinite sum is a continuous function of λ ($\alpha \le \lambda \le \beta$). Writing

$$F_k(\lambda) = \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx,$$

then

$$\begin{aligned}
\int_\alpha^\beta d\lambda \int_a^\infty F(x,\lambda)\ dx &= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty F_k(\lambda) \\
&=^1 \sum_{k=0}^\infty \int_\alpha^\beta F_k(\lambda)\ d\lambda \\
&= \sum_{k=0}^\infty \int_\alpha^\beta\ d\lambda \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&=^2 \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}}\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda \\
&= \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda,
\end{aligned}$$

where the ^{1} holds because the (10.6) says the convergence is uniform. Moreover, the ^{2} holds because the finite integral of continuous function. This completes the proof. □

Remark. Note that we can allow f to be piecewise continuous with respect to x.

Theorem 10.3. *Suppose that $F(x,\ \lambda)$ is continuous function of two variables, and that $\frac{\partial F}{\partial \lambda}$ is continuous. If both the integrals*

$$\int_a^\infty F(x,\ \lambda)\ dx, \qquad \int_a^\infty \frac{\partial F(x,\lambda)}{\partial \lambda}\ dx$$

exist and that the second integral is uniformly convergent for $\alpha \le \lambda \le \beta$, then we have

$$\frac{\partial}{\partial \lambda} \int_a^\infty F(x,\, \lambda)\; dx = \int_a^\infty \frac{\partial F(x, \lambda)}{\partial \lambda}\; dx, \qquad \alpha \le \lambda \le \beta.$$

Proof. Since the second integral in the statement of the Theorem is uniformly convergent, so the sum

$$\int_a^\infty \frac{\partial F(x, \lambda)}{\partial \lambda}\; dx = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \frac{\partial F(x, \lambda)}{\partial \lambda}\; dx$$

is uniformly convergent as a function of λ, $\alpha \le \lambda \le \beta$. Theorem 2.10 (iii) shows that

$$\begin{aligned}
\frac{d}{d\lambda} \int_a^\infty F(x,\, \lambda)\; dx &= \frac{d}{d\lambda} \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\, \lambda)\; dx \\
&= \sum_{k=0}^{\infty} \frac{d}{d\lambda} \int_{x_k}^{x_{k+1}} F(x,\, \lambda)\; dx \\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \frac{\partial F(x,\, \lambda)}{\partial \lambda}\; dx \\
&= \int_a^\infty \frac{\partial F(x,\, \lambda)}{\partial \lambda}\; dx.
\end{aligned}$$

□

Theorem 10.4. *Suppose for λ, $\alpha \le \lambda \le \beta$,*

$$|F(x, \lambda) \le f(x)|$$

where F is continuous with respect to both variables, and that

$$\int_a^\infty |f(x)|\; dx < \infty.$$

Then

$$\int_a^\infty F(x,\, \lambda)\; dx$$

is uniformly convergent for λ, $\alpha \le \lambda \le \beta$.

Proof. The uniform convergence of the integral follows easily from the Weierstrass M-test. □

We now extend the usual Riemann-Lebesgue lemma.

Lemma 10.5. *If $f(a)$ is absolutely integrable on $[a, \infty)$, then*

$$\lim_{l \to \infty} \int_a^\infty f(u) \sin lu\; du = 0.$$

Proof. Since f is absolutely integrable on $[a,\infty)$, so given $\epsilon > 0$, there is $b > 0$, $b > a$, such that

$$\Big|\int_b^\infty f(u)\sin lu\ du\Big| \le \int_b^\infty |f(u)|\ du \le \frac{\epsilon}{2}.$$

But the usual Riemann-Lebesgue Lemma implies that

$$\Big|\int_a^b f(b)f(u)\sin lu\ du\Big| < \frac{\epsilon}{2}$$

when l is chosen to be sufficiently large. Combining the above considerations gives the desired result. $\square$

Remark. The above result obviously works for "$\cos lu$", as well as for the integration in the range $\int_{-\infty}^a$ or $\int_{-\infty}^\infty$.

To be continued ...