MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

10. Fourier integrals

10.1. Introduction: Extending to infinite period. In this section we shall study Fourier integrals as a limiting case of the Fourier series.

We first assume that the function $f(x)$ is defined on the $x-$axis and is piecewise continuous on $[-l,\, l]$, for each l. Suppose

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big),$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\, du, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\, du, \tag{10.1}$$

for $k = 0, 1, 2, \dots$ and $b_0 = 0$. We remark that the above Fourier series equals to the value

$$\frac{f(x+0)+f(x-0)}{2}$$

if f has a discontinuity point at x. We now assume, in addition, that f is absolutely integrable on the whole $x-$axis, that is, the integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

exists.

We now substitute the expression of a_k and b_k into the Fourier series above and let l tends to infinity:

$$\begin{aligned}
f(x) &= \lim_{l\to\infty} f(x) = \lim_{l\to\infty} \Big[\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big)\Big] \\
&= \lim_{l\to\infty} \Bigg[\frac{1}{2l}\int_{-l}^{l} f(u)\, du + \sum_{k=1}^{\infty} \frac{1}{l}\Bigg(\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\cos\Big(\frac{\pi k x}{l}\Big)\, du \\
&\qquad + \int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\sin\Big(\frac{\pi k x}{l}\Big)\, du\Bigg)\Bigg] \\
&= 0 + \lim_{l\to\infty}\Big[\sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k(u-x)}{l}\Big)\, du\Big] \\
&= \lim_{l\to\infty}\Big[\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\pi}{l}\int_{-l}^{l} f(u)\cos[\lambda_k(u-x)]\, du\Big] \\
&= \lim_{l\to\infty}\frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\lambda_k\int_{-l}^{l} f(u)\cos[\lambda_k(u-x)]\, du = \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\cos\big(\lambda(u-x)\big)\, du.
\end{aligned}$$

where we have set

$$\lambda_k = \frac{k\pi}{l}, \qquad \Delta\lambda_k = \lambda_{k+1} - \lambda_k \quad , k = 1,\, 2,\, 3, \ldots .$$

Although the above reasoning needs further justification, it does indicate what is possible. We further notice that the following possibility

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty f(u)\cos\lambda(u-x)du \\ &= \int_0^\infty d\lambda\, \big(a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\big), \end{aligned}$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\lambda u\ du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\lambda u\ du$$

which is known as a prototype of *Fourier Integral Theorem*. We shall later justify that it actually holds for absolutely integrable function f on $(-\infty, +\infty)$.

We call the improper integral

$$F(\lambda) = \int_0^\infty f(u)\ \cos\lambda(u-x)\ du \tag{10.2}$$

the *Fourier cosine transform* of f.

10.2. **Preparation for the Fourier cosine integral theorem.**

Definition. Let $F(x,\lambda)$ be a function of two variables, and suppose that the integral

$$\int_a^\infty F(x,\lambda)\ dx \tag{10.3}$$

exists for every λ, $\alpha \le \lambda \le \beta$. Then we say that the above integral is *uniformly convergent* for λ in $\alpha \le \lambda \le \beta$ if for every $\epsilon > 0$, there is L such that

$$\Big|\int_l^\infty F(x,\lambda)\ dx\Big| \le \epsilon,$$

whenever $l \ge L$ and for all λ, $\alpha \le \lambda \le \beta$.

Lemma 10.1. *Let x_k be a sequence such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n < \dots \tag{10.4}$$

and that

$$\lim_{k\to\infty} x_k = \infty.$$

Then a necessary and sufficient condition for the integral (10.3) to be uniformly convergent over $[\alpha,\, \beta]$ *is that the series*

$$\int_a^\infty F(x,\lambda)\ dx = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\,\lambda)\ dx$$

is uniformly convergent on λ, $\alpha \le \lambda \le \beta$, *as a function of* λ, *and for every sequence* x_k *defined by (10.4) above.*

Proof. We first suppose that the integral (10.3) is uniformly convergent. That is, given any $\epsilon > 0$, there is L such that

$$\Big|\int_l^\infty F(x,\lambda)\ dx\Big| \le \epsilon$$

whenever $l \ge L$, for λ in $\alpha \le \lambda \le \beta$. Since $x_k \to \infty$, we can find a $N > 0$ such that $x_k > L$ when $k > N$. Thus

$$\begin{aligned}\Big|\int_a^\infty F(x,\lambda)\ dx - \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx\Big| &= \Big|\int_a^\infty F(x,\lambda)\ dx - \int_a^{x_k} F(x,\lambda)\ dx\Big| \\ &= \Big|\int_{x_k}^\infty F(x,\lambda)\ dx\Big| \le \epsilon,\end{aligned}$$

for $\alpha \le \lambda \le \beta$. Hence the series $\sum\limits_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$ is uniformly convergent over $[\alpha,\, \beta]$.

Let us now suppose that this series is uniformly convergent over $[\alpha,\, \beta]$ and for any sequence $\{x_k\}$ in (10.4). We suppose on the contrary that the integral (10.3) is not uniformly convergent, that is, one can find an $\epsilon > 0$, and an infinite sequence $\{y_k\}$, $y_k \to \infty$ as $k \to \infty$, such that

$$\Big|\int_{y_k}^\infty F(x,\lambda)\ dx\Big| \ge \epsilon$$

for all k. But this implies that when we choose $x_k = y_k$, $k = 1,\, 2,\, \dots,$

$$\Big|\int_a^\infty F(x,\lambda) dx - \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx\Big| = \Big|\int_{x_n}^\infty F(x,\lambda)\ dx\Big| \ge \epsilon$$

for each n, contradicting the uniform convergence of $\sum\limits_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$. $\square$

Lemma 10.2. *We suppose $F(x,\lambda)$ is regarded as a continuous function with respect to both of its variables and if the integral*

$$\int_a^\infty F(x,\lambda)\ dx \tag{10.5}$$

is uniformly convergent with respect to λ, $\alpha \le \lambda \le \beta$, then the integral (10.5) defines a continuous function with respect to λ. In addition, we have

$$\int_\alpha^\beta d\lambda \int_a^\infty F(x,\lambda)dx = \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda.$$

Proof. Since the integral (10.5) converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$, the last Lemma asserts that for any sequence $\{x_k\}$, $x_k \nearrow \infty$, the series

$$\sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \tag{10.6}$$

of continuous functions of λ, converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$. Hence the infinite sum is a continuous function of λ ($\alpha \le \lambda \le \beta$). Writing

$$F_k(\lambda) = \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx,$$

then

$$\begin{aligned}
\int_\alpha^\beta d\lambda \int_a^\infty F(x,\lambda)\ dx &= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty F_k(\lambda) \\
&=^1 \sum_{k=0}^\infty \int_\alpha^\beta F_k(\lambda)\ d\lambda \\
&= \sum_{k=0}^\infty \int_\alpha^\beta\ d\lambda \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&=^2 \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}}\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda \\
&= \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda,
\end{aligned}$$

where the ^{1} holds because the (10.6) says the convergence is uniform. Moreover, the ^{2} holds because the finite integral of continuous function. This completes the proof. □

Remark. Note that we can allow f to be piecewise continuous with respect to x.

Theorem 10.3. *Suppose that $F(x,\ \lambda)$ is continuous function of two variables, and that $\frac{\partial F}{\partial \lambda}$ is continuous. If both the integrals*

$$\int_a^\infty F(x,\ \lambda)\ dx, \qquad \int_a^\infty \frac{\partial F(x,\lambda)}{\partial \lambda}\ dx$$

exist and that the second integral is uniformly convergent for $\alpha \le \lambda \le \beta$, *then we have*

$$\frac{\partial}{\partial\lambda}\int_a^\infty F(x,\,\lambda)\;dx = \int_a^\infty \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx, \qquad \alpha \le \lambda \le \beta.$$

Proof. Since the second integral in the statement of the Theorem is uniformly convergent, so the sum

$$\int_a^\infty \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx = \sum_{k=0}^{\infty}\int_{x_k}^{x_{k+1}} \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx$$

is uniformly convergent as a function of λ, $\alpha \le \lambda \le \beta$. Theorem 2.10 (iii) shows that

$$\begin{aligned}\frac{d}{d\lambda}\int_a^\infty F(x,\,\lambda)\;dx &= \frac{d}{d\lambda}\sum_{k=0}^{\infty}\int_{x_k}^{x_{k+1}} F(x,\,\lambda)\;dx\\ &= \sum_{k=0}^{\infty}\frac{d}{d\lambda}\int_{x_k}^{x_{k+1}} F(x,\,\lambda)\;dx\\ &= \sum_{k=0}^{\infty}\int_{x_k}^{x_{k+1}} \frac{\partial F(x,\,\lambda)}{\partial\lambda}\;dx\\ &= \int_a^\infty \frac{\partial F(x,\,\lambda)}{\partial\lambda}\;dx.\end{aligned}$$

□

Theorem 10.4. *Suppose for* λ, $\alpha \le \lambda \le \beta$,

$$|F(x,\,\lambda)| \le f(x)|$$

where F *is continuous with respect to both variables, and that*

$$\int_a^\infty |f(x)|\;dx < \infty.$$

Then

$$\int_a^\infty F(x,\,\lambda)\;dx$$

is uniformly convergent for λ, $\alpha \le \lambda \le \beta$.

Proof. The uniform convergence of the integral follows easily from the Weierstrass M-test. □

We now extend the usual Riemann-Lebesgue lemma.

Lemma 10.5. *If* $f(a)$ *is absolutely integrable on* $[a,\,\infty)$, *then*

$$\lim_{l\to\infty}\int_a^\infty f(u)\sin lu\;du = 0.$$

Proof. Since f is absolutely integrable on $[a,\infty)$, so given $\epsilon>0$, there is $b>0$, $b>a$, such that

$$\Big|\int_b^\infty f(u)\sin lu\ du\Big| \le \int_b^\infty |f(u)|\ du \le \frac{\epsilon}{2}.$$

But the usual Riemann-Lebesgue Lemma implies that

$$\Big|\int_a^b f(b)f(u)\sin lu\ du\Big| < \frac{\epsilon}{2}$$

when l is chosen to be sufficiently large. Combining the above considerations gives the desired result. □

Remark. The above result obviously works for "$\cos lu$", as well as for the integration in the range $\int_{-\infty}^a$ or $\int_{-\infty}^\infty$.

Lemma 10.6. *If $f(x)$ is absolutely integrable on the whole $x-$axis, and if $f(x+0)$ and $f(x-0)$ both exist at x, then*

$$\lim_{l\to\infty}\frac{1}{\pi}\int_{-\infty}^\infty f(x+u)\frac{\sin lu}{u}\ du = \frac{f(x+0)+f(x-0)}{2}.$$

Remark. We compare the above formula with the previous formula:

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^\pi f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\ du = \frac{f(x+0)+f(x-0)}{2}. \tag{10.7}$$

Proof. We first divide the interval $(-\infty,+\infty)$ into $(-\infty,\,-\delta)$, $(-\delta,\,\delta)$, $(\delta,\,+\infty)$ where δ is some positive number. It is easy to see that the function $\dfrac{f(x+u)}{u}$ is absolutely integrable on $-\infty<u<\delta$ and $\delta\le u<\infty$. Thus, the Riemann-Lebesgue Lemma 10.5 (and the following remark) implies that

$$\lim_{l\to\infty}\int_\delta^\infty f(x+u)\frac{\sin lu}{u}\ du = 0 = \lim_{l\to\infty}\int_{-\infty}^{-\delta} f(x+u)\frac{\sin lu}{u}\ du.$$

Now we write (10.7) with $m=n+\frac{1}{2}$

$$\begin{aligned}
\frac{f(x+0)+f(x-0)}{2} &= \lim_{n\to\infty}\Big[\frac{1}{\pi}\int_{-\pi}^{-\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du \\
&\qquad + \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}du + \frac{1}{\pi}\int_\delta^\pi f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du\Big] \\
&= 0 + \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du + 0 \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin mu}{u}\ du \\
&\qquad + \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\Big(\frac{1}{2\sin\frac{u}{2}}-\frac{1}{u}\Big)\sin mu\ du \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du + 0
\end{aligned} \tag{10.8}$$

since the factor $\dfrac{1}{2\sin\frac{u}{2}} - \dfrac{1}{u} \sim 0$ as $u \to 0$ making $f(x+u)(\dfrac{1}{2\sin\frac{u}{2}} - \dfrac{1}{u})$ absolutely integrable over $[-\delta, \delta]$ (and so the Riemann-Lebesgue Lemma implies again).

It remains to extend (10.8) to arbitrary number l instead of $m = n + \frac{1}{2}$, n integer. But we may write $l = m + \theta$, $m \le l < m+1$, $0 \le \theta < 1$. Applying the mean value theorem yields

$$\frac{\sin lu - \sin mu}{(l-m)u} = \frac{\sin lu - \sin mu}{\theta u} = \cos hu$$

for some h, $m < h < l$. Thus,

$$\begin{aligned}
&\Big|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin lu}{u} du - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du\Big| \\
&= \frac{1}{\pi}\Big|\int_{-\delta}^{\delta} f(x+u)\cdot\theta\cdot\cos hu\ du\Big| \\
&< \frac{1}{\pi}\int_{-\delta}^{\delta} |f(x+u)|\ du < \frac{\epsilon}{2}
\end{aligned}$$

for any l when we choose δ to be sufficiently small. Thus for all l to sufficiently large,

$$\begin{aligned}
\Big|\frac{f(x+0)+f(x-0)}{2} &- \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\le \Big|\frac{f(x+0)+f(x-0)}{2} - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du\Big| \\
&\quad + \Big|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\quad + \Big|\frac{1}{\pi}\int_{-\infty}^{-\delta} f(x+u)\frac{\sin lu}{u}\ du\Big| + \Big|\frac{1}{\pi}\int_{\delta}^{\infty} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\quad \to 0.
\end{aligned}$$

as l or $m \to \infty$. This completes the proof. □

We are ready to consider

Theorem 10.7. *Let f be an absolutely integrable function on the $x-$axis $\mathbb{R}$. Then*

$$\frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\ \cos\lambda(u-x)\ du = \begin{cases} \dfrac{f(x+0)+f(x-0)}{2}, & x \text{ is a jump discontinuity for } f \\ f(x), & f \text{ is continuous at } x. \end{cases}$$

Remark. Note that we may rewrite the integral in the form

$$\int_0^{\infty} \big(a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\big)\ d\lambda,$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u\ du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u\ du.$$

Proof. Since $f(u)$ is absolutely integrable over $\mathbb{R}$ and

$$|f(u)\cos\lambda(u-x)| \le |f(u)|$$

over $\mathbb{R}$, Theorem 10.4 asserts that

$$\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)du$$

is uniformly convergent with respect to λ, $-\infty < \lambda < +\infty$. Then Lemma 10.2 implies, for a fixed x, the above integral is continuous with respect to λ. Moreover,

$$\begin{aligned}\int_0^l d\lambda \int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\ du &= \int_{-\infty}^{\infty} du \int_0^l f(u)\cos\lambda(u-x)\ d\lambda\\ &= \int_{-\infty}^{\infty} f(u)\frac{\sin l(u-x)}{u-x}\ du\\ &= \int_{-\infty}^{\infty} f(x+u)\frac{\sin lu}{u}\ du,\end{aligned}$$

after a change of variable. The result now follows from letting $l \to \infty$ and Lemma 10.7. □

Remark. (1) If $f(u)$ is absolutely integrable on $\mathbb{R}$, then the inequality

$$|f(u)\sin\lambda(u-x)| \le |f(u)|$$

implies that the integral

$$\int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\ du$$

converges uniformly for $-\infty < \lambda < +\infty$, and hence represents a continuous function of λ which is *odd*. Hence

$$\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\ du = 0.$$

Thus, we may write

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\underbrace{\cos\lambda(u-x)}_{\text{even in }\lambda}\ du + 0\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\big(\cos\lambda(u-x) + i\sin\lambda(u-x)\big)\ du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\,e^{i\lambda(u-x)}\ du\end{aligned}$$

which is known as the *complex form* of the Fourier Integral Theorem.

(2) Recall that

$$f(x) = \int_0^\infty a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x \; d\lambda$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u \; du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u \; du.$$

If $f(u)$ is even, then $b(\lambda) = 0$. If f is odd, then $a(\lambda) = 0$. Thus, if f is defined on $[0,\infty)$, then we may get either an odd or even extension of f onto the $\mathbb{R}$ corresponding to the two representations of f above.

To be continued ...