MATH4822E FOURIER ANALYSIS AND ITS APPLICATIONS

10. Fourier integrals

10.1. **Introduction: Extending to infinite period.** In this section we shall study Fourier integrals as a limiting case of the Fourier series.

We first assume that the function $f(x)$ is defined on the $x-$axis and is piecewise continuous on $[-l,\, l]$, for each l. Suppose

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big),$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\, du, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\, du, \tag{10.1}$$

for $k = 0, 1, 2, \ldots$ and $b_0 = 0$. We remark that the above Fourier series equals to the value

$$\frac{f(x+0) + f(x-0)}{2}$$

if f has a discontinuity point at x. We now assume, in addition, that f is absolutely integrable on the whole $x-$axis, that is, the integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

exists.

We now substitute the expression of a_k and b_k into the Fourier series above and let l tends to infinity:

$$\begin{aligned}
f(x) &= \lim_{l\to\infty} f(x) = \lim_{l\to\infty}\Big[\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(\frac{\pi k x}{l}\Big) + b_k \sin\Big(\frac{\pi k x}{l}\Big)\Big] \\
&= \lim_{l\to\infty}\Bigg[\frac{1}{2l}\int_{-l}^{l} f(u)\, du + \sum_{k=1}^{\infty}\frac{1}{l}\Bigg(\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k u}{l}\Big)\cos\Big(\frac{\pi k x}{l}\Big)\, du \\
&\qquad + \int_{-l}^{l} f(u)\sin\Big(\frac{\pi k u}{l}\Big)\sin\Big(\frac{\pi k x}{l}\Big)\, du\Bigg)\Bigg] \\
&= 0 + \lim_{l\to\infty}\Big[\sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(u)\cos\Big(\frac{\pi k(u-x)}{l}\Big)\, du\Big] \\
&= \lim_{l\to\infty}\Big[\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\pi}{l}\int_{-l}^{l} f(u)\cos[\lambda_k(u-x)]\, du\Big] \\
&= \lim_{l\to\infty}\frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\lambda_k\int_{-l}^{l} f(u)\cos[\lambda_k(u-x)]\, du = \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\cos\big(\lambda(u-x)\big)\, du.
\end{aligned}$$

where we have set

$$\lambda_k = \frac{k\pi}{l}, \qquad \Delta\lambda_k = \lambda_{k+1} - \lambda_k \quad , k = 1,\, 2,\, 3, \ldots.$$

Although the above reasoning needs further justification, it does indicate what is possible. We further notice that the following possibility

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)du \\ &= \int_0^\infty d\lambda\, \big(a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\big), \end{aligned}$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u\; du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u\; du$$

which is known as a prototype of *Fourier Integral Theorem*. We shall later justify that it actually holds for absolutely integrable function f on $(-\infty, +\infty)$.

We call the improper integral

$$F(\lambda) = \int_0^\infty f(u)\, \cos\lambda(u-x)\; du \tag{10.2}$$

the *Fourier cosine transform* of f.

10.2. **Preparation for the Fourier cosine integral theorem.**

Definition. Let $F(x,\lambda)$ be a function of two variables, and suppose that the integral

$$\int_a^\infty F(x,\lambda)\; dx \tag{10.3}$$

exists for every λ, $\alpha \le \lambda \le \beta$. Then we say that the above integral is *uniformly convergent* for λ in $\alpha \le \lambda \le \beta$ if for every $\epsilon > 0$, there is L such that

$$\Big|\int_l^\infty F(x,\lambda)\; dx\Big| \le \epsilon,$$

whenever $l \ge L$ and for all λ, $\alpha \le \lambda \le \beta$.

Lemma 10.1. *Let* x_k *be a sequence such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n < \dots \tag{10.4}$$

and that

$$\lim_{k\to\infty} x_k = \infty.$$

Then a necessary and sufficient condition for the integral (10.3) to be uniformly convergent over $[\alpha,\, \beta]$ *is that the series*

$$\int_a^\infty F(x,\lambda)\ dx = \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\,\lambda)\ dx$$

is uniformly convergent on λ, $\alpha \le \lambda \le \beta$, *as a function of* λ, *and for every sequence* x_k *defined by (10.4) above.*

Proof. We first suppose that the integral (10.3) is uniformly convergent. That is, given any $\epsilon > 0$, there is L such that

$$\Big|\int_l^\infty F(x,\lambda)\ dx\Big| \le \epsilon$$

whenever $l \ge L$, for λ in $\alpha \le \lambda \le \beta$. Since $x_k \to \infty$, we can find a $N > 0$ such that $x_k > L$ when $k > N$. Thus

$$\begin{aligned}\Big|\int_a^\infty F(x,\lambda)\ dx - \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx\Big| &= \Big|\int_a^\infty F(x,\lambda)\ dx - \int_a^{x_k} F(x,\lambda)\ dx\Big| \\ &= \Big|\int_{x_k}^\infty F(x,\lambda)\ dx\Big| \le \epsilon,\end{aligned}$$

for $\alpha \le \lambda \le \beta$. Hence the series $\sum\limits_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$ is uniformly convergent over $[\alpha,\, \beta]$.

Let us now suppose that this series is uniformly convergent over $[\alpha,\, \beta]$ and for any sequence $\{x_k\}$ in (10.4). We suppose on the contrary that the integral (10.3) is not uniformly convergent, that is, one can find an $\epsilon > 0$, and an infinite sequence $\{y_k\}$, $y_k \to \infty$ as $k \to \infty$, such that

$$\Big|\int_{y_k}^\infty F(x,\lambda)\ dx\Big| \ge \epsilon$$

for all k. But this implies that when we choose $x_k = y_k$, $k = 1,\, 2,\, \dots,$

$$\Big|\int_a^\infty F(x,\lambda)dx - \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx\Big| = \Big|\int_{x_n}^\infty F(x,\lambda)\ dx\Big| \ge \epsilon$$

for each n, contradicting the uniform convergence of $\sum\limits_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx$. □

Lemma 10.2. *We suppose $F(x,\lambda)$ is regarded as a continuous function with respect to both of its variables and if the integral*

$$\int_a^\infty F(x,\lambda)\ dx \tag{10.5}$$

is uniformly convergent with respect to λ, $\alpha \le \lambda \le \beta$, then the integral (10.5) defines a continuous function with respect to λ. In addition, we have

$$\int_\alpha^\beta\ d\lambda \int_a^\infty F(x,\lambda)dx = \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda.$$

Proof. Since the integral (10.5) converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$, the last Lemma asserts that for any sequence $\{x_k\}$, $x_k \nearrow \infty$, the series

$$\sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \tag{10.6}$$

of continuous functions of λ, converges uniformly with respect to λ, $\alpha \le \lambda \le \beta$. Hence the infinite sum is a continuous function of λ ($\alpha \le \lambda \le \beta$). Writing

$$F_k(\lambda) = \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx,$$

then

$$\begin{aligned}
\int_\alpha^\beta d\lambda \int_a^\infty F(x,\lambda)\ dx &= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&= \int_\alpha^\beta d\lambda \sum_{k=0}^\infty F_k(\lambda) \\
&=^1 \sum_{k=0}^\infty \int_\alpha^\beta F_k(\lambda)\ d\lambda \\
&= \sum_{k=0}^\infty \int_\alpha^\beta\ d\lambda \int_{x_k}^{x_{k+1}} F(x,\lambda)\ dx \\
&=^2 \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}}\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda \\
&= \int_a^\infty\ dx \int_\alpha^\beta F(x,\lambda)\ d\lambda,
\end{aligned}$$

where the [1] holds because the (10.6) says the convergence is uniform. Moreover, the [2] holds because the finite integral of continuous function. This completes the proof. □

Remark. Note that we can allow f to be piecewise continuous with respect to x.

Theorem 10.3. *Suppose that $F(x,\ \lambda)$ is continuous function of two variables, and that $\frac{\partial F}{\partial \lambda}$ is continuous. If both the integrals*

$$\int_a^\infty F(x,\ \lambda)\ dx, \qquad \int_a^\infty \frac{\partial F(x,\lambda)}{\partial \lambda}\ dx$$

exist and that the second integral is uniformly convergent for $\alpha \le \lambda \le \beta$*, then we have*

$$\frac{\partial}{\partial\lambda}\int_a^\infty F(x,\,\lambda)\;dx = \int_a^\infty \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx, \qquad \alpha \le \lambda \le \beta.$$

Proof. Since the second integral in the statement of the Theorem is uniformly convergent, so the sum

$$\int_a^\infty \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx = \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} \frac{\partial F(x,\lambda)}{\partial\lambda}\;dx$$

is uniformly convergent as a function of λ, $\alpha \le \lambda \le \beta$. Theorem 2.10 (iii) shows that

$$\begin{aligned}
\frac{d}{d\lambda}\int_a^\infty F(x,\,\lambda)\;dx &= \frac{d}{d\lambda}\sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} F(x,\,\lambda)\;dx\\
&= \sum_{k=0}^\infty \frac{d}{d\lambda}\int_{x_k}^{x_{k+1}} F(x,\,\lambda)\;dx\\
&= \sum_{k=0}^\infty \int_{x_k}^{x_{k+1}} \frac{\partial F(x,\,\lambda)}{\partial\lambda}\;dx\\
&= \int_a^\infty \frac{\partial F(x,\,\lambda)}{\partial\lambda}\;dx.
\end{aligned}$$

□

Theorem 10.4. *Suppose for* λ, $\alpha \le \lambda \le \beta$,

$$|F(x,\,\lambda)| \le f(x)|$$

where F *is continuous with respect to both variables, and that*

$$\int_a^\infty |f(x)|\;dx < \infty.$$

Then

$$\int_a^\infty F(x,\,\lambda)\;dx$$

is uniformly convergent for λ, $\alpha \le \lambda \le \beta$.

Proof. The uniform convergence of the integral follows easily from the Weierstrass M-test. □

We now extend the usual Riemann-Lebesgue lemma.

Lemma 10.5. *If* $f(a)$ *is absolutely integrable on* $[a,\,\infty)$*, then*

$$\lim_{l\to\infty}\int_a^\infty f(u)\sin lu\;du = 0.$$

Proof. Since f is absolutely integrable on $[a,\infty)$, so given $\epsilon>0$, there is $b>0$, $b>a$, such that

$$\Big|\int_b^\infty f(u)\sin lu\ du\Big| \le \int_b^\infty |f(u)|\ du \le \frac{\epsilon}{2}.$$

But the usual Riemann-Lebesgue Lemma implies that

$$\Big|\int_a^b f(b)f(u)\sin lu\ du\Big| < \frac{\epsilon}{2}$$

when l is chosen to be sufficiently large. Combining the above considerations gives the desired result. □

Remark. The above result obviously works for "$\cos lu$", as well as for the integration in the range $\int_{-\infty}^a$ or $\int_{-\infty}^\infty$.

Lemma 10.6. *If $f(x)$ is absolutely integrable on the whole $x-$axis, and if $f(x+0)$ and $f(x-0)$ both exist at x, then*

$$\lim_{l\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin lu}{u}\ du = \frac{f(x+0)+f(x-0)}{2}.$$

Remark. We compare the above formula with the previous formula:

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})}\ du = \frac{f(x+0)+f(x-0)}{2}. \tag{10.7}$$

Proof. We first divide the interval $(-\infty,+\infty)$ into $(-\infty,\ -\delta)$, $(-\delta,\ \delta)$, $(\delta,\ +\infty)$ where δ is some positive number. It is easy to see that the function $\dfrac{f(x+u)}{u}$ is absolutely integrable on $-\infty<u<\delta$ and $\delta\le u<\infty$. Thus, the Riemann-Lebesgue Lemma 10.5 (and the following remark) implies that

$$\lim_{l\to\infty}\int_\delta^\infty f(x+u)\frac{\sin lu}{u}\ du = 0 = \lim_{l\to\infty}\int_{-\infty}^{-\delta} f(x+u)\frac{\sin lu}{u}\ du.$$

Now we write (10.7) with $m=n+\frac{1}{2}$

$$\begin{aligned}
\frac{f(x+0)+f(x-0)}{2} &= \lim_{n\to\infty}\Big[\frac{1}{\pi}\int_{-\pi}^{-\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du \\
&\qquad + \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}du + \frac{1}{\pi}\int_{\delta}^{\pi} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du\Big] \\
&= 0 + \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{2\sin\frac{u}{2}}\ du + 0 \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin mu}{u}\ du \\
&\qquad + \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\Big(\frac{1}{2\sin\frac{u}{2}}-\frac{1}{u}\Big)\sin mu\ du \\
&= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du + 0
\end{aligned} \tag{10.8}$$

since the factor $\frac{1}{2\sin\frac{u}{2}} - \frac{1}{u} \sim 0$ as $u \to 0$ making $f(x+u)(\frac{1}{2\sin\frac{u}{2}} - \frac{1}{u})$ absolutely integrable over $[-\delta, \delta]$ (and so the Riemann-Lebesgue Lemma implies again).

It remains to extend (10.8) to arbitrary number l instead of $m = n + \frac{1}{2}$, n integer. But we may write $l = m + \theta$, $m \le l < m+1$, $0 \le \theta < 1$. Applying the mean value theorem yields

$$\frac{\sin lu - \sin mu}{(l-m)u} = \frac{\sin lu - \sin mu}{\theta u} = \cos hu$$

for some h, $m < h < l$. Thus,

$$\begin{aligned}
&\Big|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin lu}{u}du - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du\Big| \\
&= \frac{1}{\pi}\Big|\int_{-\delta}^{\delta} f(x+u)\cdot\theta\cdot\cos hu\ du\Big| \\
&< \frac{1}{\pi}\int_{-\delta}^{\delta} |f(x+u)|\ du < \frac{\epsilon}{2}
\end{aligned}$$

for any l when we choose δ to be sufficiently small. Thus for all l to sufficiently large,

$$\begin{aligned}
&\Big|\frac{f(x+0)+f(x-0)}{2} - \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\quad \le \Big|\frac{f(x+0)+f(x-0)}{2} - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du\Big| \\
&\quad + \Big|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin mu}{u}\ du - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\quad + \Big|\frac{1}{\pi}\int_{-\infty}^{-\delta} f(x+u)\frac{\sin lu}{u}\ du\Big| + \Big|\frac{1}{\pi}\int_{\delta}^{\infty} f(x+u)\frac{\sin lu}{u}\ du\Big| \\
&\quad \to 0.
\end{aligned}$$

as l or $m \to \infty$. This completes the proof. □

We are ready to consider

Theorem 10.7. *Let f be an absolutely integrable function on the $x-$axis $\mathbb{R}$. Then*

$$\frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\ \cos\lambda(u-x)\ du = \begin{cases} \dfrac{f(x+0)+f(x-0)}{2}, & x \text{ is a jump discontinuity for } f \\ f(x), & f \text{ is continuous at } x. \end{cases}$$

Remark. Note that we may rewrite the integral in the form

$$\int_0^{\infty} \big(a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\big)\ d\lambda,$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u\ du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u\ du.$$

Proof. Since $f(u)$ is absolutely integrable over $\mathbb{R}$ and

$$|f(u)\cos\lambda(u-x)| \le |f(u)|$$

over $\mathbb{R}$, Theorem 10.4 asserts that

$$\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)du$$

is uniformly convergent with respect to λ, $-\infty < \lambda < +\infty$. Then Lemma 10.2 implies, for a fixed x, the above integral is continuous with respect to λ. Moreover,

$$\begin{aligned}\int_0^l d\lambda \int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\ du &= \int_{-\infty}^{\infty} du \int_0^l f(u)\cos\lambda(u-x)\ d\lambda \\ &= \int_{-\infty}^{\infty} f(u)\frac{\sin l(u-x)}{u-x}\ du \\ &= \int_{-\infty}^{\infty} f(x+u)\frac{\sin lu}{u}\ du,\end{aligned}$$

after a change of variable. The result now follows from letting $l \to \infty$ and Lemma 10.7. □

Remark. (1) If $f(u)$ is absolutely integrable on $\mathbb{R}$, then the inequality

$$|f(u)\sin\lambda(u-x)| \le |f(u)|$$

implies that the integral

$$\int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\ du$$

converges uniformly for $-\infty < \lambda < +\infty$, and hence represents a continuous function of λ which is *odd*. Hence

$$\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\ du = 0.$$

Thus, we may write

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\underbrace{\cos\lambda(u-x)}_{\text{even in }\lambda}\ du + 0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\big(\cos\lambda(u-x) + i\sin\lambda(u-x)\big)\ du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(u)\, e^{i\lambda(u-x)}\ du\end{aligned}$$

which is known as the *complex form* of the Fourier Integral Theorem.

(2) Recall that

$$f(x) = \int_0^\infty a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x \; d\lambda$$

where

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u \; du, \qquad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u \; du.$$

If $f(u)$ is even, then $b(\lambda) = 0$. If f is odd, then $a(\lambda) = 0$. Thus, if f is defined on $[0, \infty)$, then we may get either an odd or even extension of f onto the $\mathbb{R}$ corresponding to the two representations of f above.

10.3. **The Fourier Transform.**

Let us recall from first remark of the previous section that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(u)\, e^{i\xi(u-x)} du \tag{10.9}$$

If we write

$$F(x) = \int_{-\infty}^{\infty} f(u)\, e^{-ixu} \; du$$

then (10.9) can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{ixu} \; du \tag{10.10}$$

The above $F(x)$ is called the *Fourier transform* of $f(x)$, and (10.9) is called *inverse Fourier transform* of $F(x)$. We note that one may regard that the F above is the solution to the integral equation (10.9).

We now list some properties of $F(x)$ which is often denoted by $\hat{f}(x)$:

(i) If $f(x)$ is absolutely integrable on the $\mathbb{R}$, then the Fourier transform $F(x)$ is continuous for all x and converges to zero as $|x| \to \infty$. Since f is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |f(u)|\, du$$

exists, and

$$|f(u)\, e^{-ixu}| = |f(u)|$$

for all x, so that

$$|\hat{f}(x)| = |F(x)| \le \int_{-\infty}^{\infty} |f(u)| \; du.$$

Hence Theorem 10.4 implies that F is uniformly convergent and so continuous with respect to x. We now write

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-ixu}\, du \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\cos(xu)\, du - i\int_{-\infty}^{\infty} f(u)\sin(xu)\, du \\ &\to 0+0, \end{aligned}$$

as $x \to +\infty$, by the Riemann-Lebesgue Lemma.

(ii) If $x^n f(x)$ is absolutely integrable on the whole $\mathbb{R}$ (n is positive integer), then $F^{(k)}(x)$, $k = 1, \ldots, n$ exists, and

$$F^{(k)}(x) = \frac{i^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)u^k\, e^{-ixu}\, du, \qquad k = 1,\, 2,\, \ldots,\, n,$$

and each of these derivatives converges to zero as $|x| \to \infty$.

We first note that $F^{(k)}(x)$, $k = 1, \ldots, n$ is uniformly convergent because of

$$|f(u)\, u^k\, e^{-ixu}| = |f(u)\, u^k|$$

for all x and the Theorem 10.4. Theorem 10.3 asserts that we can differentiate under the integral sign of F since each time we obtain an integral which is uniformly convergent again (One may verify this by separating the real and imaginary parts and then differentiating each term if we want to avoid complex differentiation.) Now the Riemann-Lebesgue Lemma again gives each $F^{(k)}(x) \to 0$, as $|x| \to \infty$.

We switch to more popular notation, that is, we write

$$\hat{f}(\xi) = \mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \tag{10.11}$$

for the *Fourier transform* of f for absolutely integrable functions.

Lemma 10.8. *Let f be absolutely integrable on $\mathbb{R}$. Then*

(i) *For any real a,*

$$\mathfrak{F}[f(x-a)] = e^{-ia\xi}\, \hat{f}(\xi) \quad \text{and} \quad \mathfrak{F}[e^{iax} f(x)] = \hat{f}(\xi - a);$$

(ii) *If f is continuous and piecewise smooth with absolutely integrable derivative, then*

$$[\hat{f'}](\xi) = i\xi\, \hat{f}(\xi),$$

(iii) *If $xf(x)$ is absolutely integrable, then*

$$\mathfrak{F}[xf(x)] = i\, [\hat{f}]'(\xi).$$

Proof.

$$\begin{aligned}\mathfrak{F}[f(x-a)] &= \int_{-\infty}^{\infty} f(x-a)\, e^{-i\xi x}\ dx = \int_{-\infty}^{\infty} e^{-i\xi(x+a)} f(x)\ dx \\ &= e^{-ia\xi} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\ dx = e^{-ia\xi}\hat{f}(\xi).\end{aligned}$$

This resolves the first half in (i). We leave the second half as exercise. To see part (ii), we note that since f' is absolutely integrable on $\mathbb{R}$, so that the limit

$$\lim_{x\to\infty} f(x) = f(0) + \lim_{x\to\infty}\int_0^x f'(t)\ dt$$

exists and with limit 0 since f is absolutely integrable. Similarly, $\lim_{x\to-\infty} f(x) = 0$. Integrating by-parts yields

$$\begin{aligned}[\hat{f'}](\xi) &= \int_{-\infty}^{\infty} f'(x)\, e^{-i\xi x}\ dx \\ &= e^{-i\xi x} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\xi) e^{-i\xi x} f(x)\ dx \\ &= 0 + i\xi\hat{f}(\xi) = \xi\hat{f}(\xi).\end{aligned}$$

It remains to consider (iii). We first note the elementary differentiation

$$i\frac{d}{d\xi} e^{-i\xi x} = x e^{-i\xi x}.$$

It is also assumed that xf is absolutely integrable, so

$$\begin{aligned}\mathfrak{F}[xf(x)] &= \int_{-\infty}^{\infty} x f(x)\, e^{-i\xi x}\ dx \\ &= i\frac{d}{d\xi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\ dx \\ &= i[\hat{f}]'(\xi).\end{aligned}$$

□

10.4. **Convolutions.** Let $f(x)$ and $g(x)$ be both absolutely integrable functions on $\mathbb{R}$. We define their *convolution* to be

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\ dy.$$

We note that the convolution between f and g still make sense if either of f and g is bounded on $\mathbb{R}$.

Theorem 10.9. *Let $f(x)$, $g(x)$ and $h(x)$ be absolutely integrable functions on $\mathbb{R}$. Then*

(i) $f * (ag + bh) = a(f * g) + b(f * h)$ *for arbitrary constants* a, b;

(ii) $f * g = g * f$;

(iii) $f * (g * h) = (f * g) * h$.

Proof. Part (i) is obvious. Let $z = x - y$. Then

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(x-y)\, g(y)\ dy \\ &= \int_{-\infty}^{\infty} f(z)\, g(x-z)\ dz \\ &= g * f(x). \end{aligned}$$

This proves (ii). It follows that

$$\begin{aligned} (f*g)*h(x) &= \int_{-\infty}^{\infty} (f*g)(x-y)h(y)\ dy = \int_{-\infty}^{\infty} (g*f)(x-y)h(y)\ dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-y-z)f(z)h(y)\ dz\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(x-z-y)\, h(y)\ dy\, dz \\ &= \int_{-\infty}^{\infty} f(z)(g*h)(x-z)\ dz \\ &= f*(g*h)(x). \end{aligned}$$

□

Theorem 10.10. *Suppose f is differentiable and $f' * g$ and $f * g$ are well-defined. Then $f * g$ is differentiable and $(f * g)' = f' * g$.*

Proof. Let us differentiate the $f * g$ under the integral sign:

$$\begin{aligned} (f*g)'(x) &= \int_{-\infty}^{\infty} \frac{d}{dx} f(x-y)g(y)\ dy \\ &= \int_{-\infty}^{\infty} f'(x-y)\, g(y)\, dy = (f'*g)(x). \end{aligned}$$

□

Remark. If we are given g being differentiable instead of f, then we would have $(g * f)'(x) = g' * f(x)$, but $(f * g)' = (g * f)'$ so that $f' * g = g * f'$ in case both f and g are differentiable.

We can interpret convolution between two integrable functions f and g as

(1)
$$\int_{-\infty}^{\infty} f'(x-y)\, g(y)\, dy \approx \sum f(x-y_j)\, g(y_j)\, \Delta y_j,$$
which represents an approximation of the $f * g$ as a sum of areas obtained by translations $f(x - y_j)$ of $f(x)$ by y_j multiplied by $g(y_j)\Delta y_j$. So $f * g$ is a continuous sums of translates $f'(x - y)\, g(y)\, dy$.

(2) We compare it with
$$\frac{1}{b-a}\int_a^b f(y)\ dy, \qquad w(x) = 1,$$
or generally with
$$\frac{\int_a^b f(y)w(y)\ dy}{\int_a^b w(y)\ dy},$$

where we regard $w(x) = g(x-y)$ as the *weight function* with $\int_a^b g(y)\ dy = 1$. This way, we can view the $f * g$ as a kind of weighted-average of f by g over $\mathbb{R}$.

(3) Suppose $g(x) = 0$ for $|x| > a$ so that $g(x-y) = 0$ for $|x-y| > a$ and $f * g$ is a weighted average of f on $[x-a,\, x+a]$ only. Suppose in addition that we choose

$$g(x) = \begin{cases} 1/(2a), & |x| < a; \\ 0, & \text{otherwise.} \end{cases}$$

then we see that

$$(f * g)(x) = \frac{1}{2a} \int_{x-a}^{x+a} f(y)\ dy.$$

Theorem 10.11. *Let g be an absolutely integrable function with $\int_{-\infty}^{\infty} g(y)\ dy = 1$, and we denote $\alpha = \int_{-\infty}^{0} g(y)\ dy$, and $\beta = \int_0^{\infty} g(y)\ dy$. Suppose that f is piecewise smooth on $\mathbb{R}$ and that either f is bounded or g vanishes outside a finite interval so that $f * g$ is well-defined for all x. Let $g_\epsilon(x) = \frac{1}{\epsilon} g\big(\frac{x}{\epsilon}\big)$. Then for all x on $\mathbb{R}$*

(1)
$$\lim_{\epsilon\to 0} f * g_\epsilon(x) = \alpha f(x+0) + \beta f(x-0).$$

(2) *if in addition that f is continuous at x, then*
$$\lim_{\epsilon\to 0} f * g_\epsilon(x) = f(x).$$

(3) *Finally, if f is continuous on a bounded interval $[a,\, b]$ then the above convergence is uniform there.*

Proof. We first note that

$$\int g_\epsilon(x)\ dx = \int g\Big(\frac{x}{\epsilon}\Big)\, d\Big(\frac{x}{\epsilon}\Big) = \int g(y)\ dy, \qquad \text{and} \qquad \int_a^b g_\epsilon(x)\ dx = \int_{a/\epsilon}^{b/\epsilon} g(y)\ dy.$$

We aim to show

$$\begin{aligned} f * g_\epsilon(x) - \big(\alpha f(x+0) + \beta f(x-0)\big) = \alpha = &\int_{-\infty}^{0} [f(x-y) - f(x+0)]\, g_\epsilon(y)\ dy \\ &+ \int_0^{\infty} [f(x-y) - f(x-0)]\, g_\epsilon(y)\ dy \end{aligned}$$

can be made arbitrary small with respect to ϵ. The arguments for $\int_{-\infty}^{0}$ and $\int_0^{\infty}$ are essentially the same, so we omit the former case. So given $\delta > 0$, we can find $c > 0$ so that $|f(x-y) - f(x-0)| < \delta$ when $0 < y < c$. We first deal with

$$\begin{aligned} \Big| \int_0^c [f(x-y) - f(x-0)]\, g_\epsilon(y)\ dy \Big| &\le \delta \int_0^c |g_\epsilon(y)|\ dy \\ &\le \delta \int_0^{c/\epsilon} |g(y)|\ dy \le \delta \int_0^{\infty} |g(y)|\ dy. \end{aligned}$$

Suppose first that f is bounded ($|f| \le M$), then

$$\begin{aligned} \Big| \int_c^{\infty} [f(x-y) - f(x-0)]\, g_\epsilon(y)\ dy \Big| &\le 2M\delta \int_c^{\infty} |g_\epsilon(y)|\ dy \\ &= 2M\delta \int_{c/\epsilon}^{\infty} |g(y)|\ dy \to 0, \end{aligned}$$

as $\epsilon \to 0$. Suppose now that g vanishes outside a finite interval ($g(x) = 0$ $|x| > R$). Then clearly $g_\epsilon(x) = 0$ when $|x| > \epsilon R$ so that $g_\epsilon(x) = 0$ when $|x| > c$ or $\epsilon < c/R$. Thus the integral $\int_\epsilon^{\infty} = 0$.

We leave the remaining parts as exercises. □

Theorem 10.12. *Let $f(x)$ and $g(x)$ both be absolutely integrable functions on $\mathbb{R}$. Then*

$$\widehat{(f * g)} = \hat{f}\,\hat{g}.$$

Proof.

$$\begin{aligned}
\widehat{(f*g)}(\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\,g(y)e^{-i\xi x}\;dy\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(x-y)} f(x-y)\;e^{-i\xi y}\,g(y)\;dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi z} f(z)\,e^{-i\xi y}\,g(y)\;dz\,dy \qquad (z = x-y) \\
&= \hat{f}(\xi)\;\hat{g}(\xi).
\end{aligned}$$

□

To be continued ...