Complex Function Theory

MATH 5030

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Chapter 1

Analytic Functions

We shall give a brief review of the basic results in complex functions centred around Cauchy's integral formula in its general form and its immediate consequences.

1.1 Notations

$\mathbb{C} = \{z = x + iy : |x| < \infty, |y| < \infty, i^2 = -1\}$:= complex plane;
$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:= extended complex plane or Riemann sphere;
$B(z_0, r) = \{z : |z - z_0| < r\}$:= open disk;
$\overline{B}(z_0, r) = \{z : |z - z_0| \le r\}$:= closed disk;
$\Re(z)$:= real part of z;
$\Im(z)$:= imaginary part of z.

Definition 1.1.1. 1. A set $S \in \mathbb{C}$ is *connected* if for any two points lying in S, there exist a polygonal curve lying entirely in S and connecting the points.

2. A *region* $G \in \mathbb{C}$ is an open connected set.

1.2 Cauchy-Riemann Equations

Definition 1.2.1. Let G be an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$. Then f is *differentiable at* $a \in G$ if the limit

$$\lim_{h\to\infty} \frac{f(a+h) - f(a)}{h}$$

exists; the value of the limit is denoted by $f'(a)$ which is called the *derivative of* f *at* a. If f is differentiable at each point of G, then we say f is *differentiable on* G.

Definition 1.2.2. A function $f : G \to \mathbb{C}$ is *analytic* if f is continuously differentiable on G i.e., f' is continuous at every point of G.

We shall show later (see Remark 1.11) that analyticity of f alone (i.e., without the continuity assumption) implies the continuity of f' (in a neighbourhood). That is, the function must be continuously differentiable. This is certainly not the case in real function theory; there exist many real functions such that their derivatives are not continuous. (e.g. $|x|$)

It is an easy exercise to show (from the definition) that if $f(z) = u(x, y) + iv(x, y)$ is analytic, then u and v satisfy the *Cauchy-Riemann equations* at z:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that the partial derivatives are continuous and the converse is also true.

Theorem 1.2.3. *Let u and v be real-valued functions defined on a region G and suppose that they have continuous derivatives there. Then $f : G \to \mathbb{C}$, $f = u + iv$ is analytic if and only if both u and v satisfy the Cauchy-Riemann equations.*

Proof. See Conway p.41-42. □

1.3 Line Integrals

Definition 1.3.1. A *path* in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a,b] \to G$ $(a < b)$. A path is *smooth* if γ' exists and also continuous on $[a,b]$. Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition on $[a,b]$, then a path $\gamma : [a,b] \to G$ is *piecewise smooth* if it is smooth on each subinterval $[t_{i-1}, t_i], i = 1, \ldots, n$.

Remark. We note that if $\gamma'(t) \neq 0$ implies that γ has a tangent at t. Some authors will simply assume, in addition to the existence and the continuity for the smooth curve γ, to have $\gamma' \neq 0$.

Definition 1.3.2. We define the *length* of a piecewise smooth curve to be

$$l(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

This is clearly a well-defined number. Suppose that $f : G \to \mathbb{C}$ is continuous and $\gamma[a,b] \subset G$, we define the *line integral* along γ to be the number

$$\int_\gamma f = \int_a^b f\, d\gamma = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

In fact, it can be shown that the integral always exists (see Conway p.60-62) and it is independent of any particular parametrization (see Conway p.63-64).

Definition 1.3.3. Let f and γ be defined as above. Then we define the *line integration of f along γ with respect to the arc length* as

$$\int_\gamma f\, |dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt. \tag{1.1}$$

The integral clearly exists since f is continuous, and γ is piecewise continuous. It is easy to verify that

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f|\, |dz|.$$

Remark. The (1.1) becomes $l(\gamma)$ if $f(t) \equiv 1$.

Theorem 1.3.4. *Let $\gamma : [a, b] \to G$ be a piecewise smooth path in a region G with initial and end points α and β. Suppose $f : G \to \mathbb{C}$ is continuous with primitive $F : G \to \mathbb{C}$ (i.e. $F' = f$), then*

$$\int_\gamma f = F(\beta) - F(\alpha). \quad (\gamma(a) = \alpha, \gamma(b) = \beta)$$

Proof. By definition of line integral above,

$$\begin{aligned} \int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F \circ \gamma)'(t)\,dt \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(\beta) - F(\alpha). \end{aligned}$$

by the Fundamental Theorem of Calculus. □

Definition 1.3.5. A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.

We deduce immediately from the above theorem that

$$\int_\gamma f = 0$$

when γ is a closed piecewise smooth path and with f as in the above theorem.

Remark. (i) All of the above definitions and results about piecewise smooth paths can be generalized to *rectifiable paths*. We shall restrict ourselves to piecewise smooth paths in the rest of the course. See Conway for more details.

(ii) Although the treatment here (and in most books) about line integral is short, complex line integral is considered to be a very important contribution from Cauchy (in a paper dated 1825).

1.4 Local Cauchy Integral Formula

Theorem 1.4.1 (Local Cauchy Integral Formula)**.** *Let* $f : G \to \mathbb{C}$ *be analytic and that* $\overline{B}(a,r) \subset G$, $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. *Then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for any $z \in B(a,r)$.

To prove this theorem, we require

Proposition 1.4.2. *Let* $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ *be a continuous function. Define* $g : [c,d] \to \mathbb{C}$ *by*

$$g(t) = \int_a^b \varphi(s,t)\, ds.$$

Then g *is continuous. Moreover, if* $\dfrac{\partial \varphi}{\partial t}$ *exists and is a continuous function on* $[a,b] \times [c,d]$, *then* g *is continuously differentiable on* $[c,d]$ *and*

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\, ds. \tag{1.2}$$

Proof. Since $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ is continuous and hence it just be uniformly continuous on its domain. It follows easily that g, as defined above, must be continuous on $[c,d]$. In order to prove (1.2), it suffices to show that

$$\frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \frac{\partial \varphi}{\partial t}(s, t_0)\, ds$$

can be made arbitrarily small.

Since $\varphi_t(s,t) = \dfrac{\partial \varphi}{\partial t}(s,t)$ is continuous on $[a,b] \times [c,d]$, it must be uniformly continuous there. Thus, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi_t(s',t') - \varphi_t(s,t)| < \epsilon$$

whenever $(s'-s)^2+(t'-t)^2<\delta^2$. In particular,

$$|\varphi_t(s,t)-\varphi_t(s,t_0)|<\epsilon$$

if $a\le s\le b$ and $|t-t_0|<\delta$. Hence for $|t-t_0|<\delta$, we have

$$\left|\int_{t_0}^{t}\varphi_t(s,\tau)-\varphi_t(s,t_0)\,d\tau\right|<\epsilon|t-t_0|.$$

But the integrand of the last inequality equals, with a fixed s,

$$\begin{aligned}(\varphi(s,t)-t\varphi_t(s,t_0))&-(\varphi(s,t_0)-t_0\varphi_t(s,t_0))\\ &=\varphi(s,t)-\varphi(s,t_0)-(t-t_0)\varphi_t(s,t_0).\end{aligned}$$

Hence

$$|\varphi(s,t)-\varphi(s,t_0)-(t-t_0)\varphi_t(s,t_0)|<\epsilon|t-t_0|$$

whenever $a\le s\le b$ and $|t-t_0|<\delta$. But this is precisely

$$\left|\frac{g(t)-g(t_0)}{t-t_0}-\int_a^b\varphi_t(s,t_0)\,ds\right|<\epsilon|b-a|$$

after integration with respect to s on both sides. This proves $g'(t)=\int_a^b\varphi_t(s,t)\,ds$. But φ_t is continuous and so g' must also be continuous. □

Example 1.4.3. Show that

$$\int_0^{2\pi}\frac{e^{is}}{e^{is}-z}\,ds=2\pi$$

whenever $|z|<1$.

Solution. Since $\varphi(s,t)=\dfrac{e^{is}}{e^{is}-tz}$, for $0\le t\le 1$, $0\le s\le 2\pi$, is continuously differentiable, it follows from Prop 1.4.2 that

$$g(t)=\int_0^{2\pi}\varphi(s,t)\,ds=\int_0^{2\pi}\frac{e^{is}}{e^{is}-tz}\,ds.$$

But

$$\int_0^{2\pi} \frac{ze^{is}}{(e^{is}-tz)^2}\,ds = \left.\frac{-iz}{e^{is}-tz}\right|_0^{2\pi} = \frac{-iz}{e^{2\pi i}-tz} - \frac{-iz}{e^0-tz} = 0.$$

for all $t \in [0,1]$. Hence $g(t) =$ constant, and in particular,

$$g(0) = \int_0^{2\pi} \frac{e^{is}}{e^{is}-0}\,ds = 2\pi.$$

For $t = 1$, we have the required equality. □

Now, we are sufficiently prepared to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. For any $\overline{B}(a,r) \subset G$, we are required to show

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

where $\gamma(t) = a + re^{it}$, $t \in [0,2\pi]$.

Without loss of generality, it is clear that we may consider $a = 0$ and $r = 1$ only. Since the translation $f(a + rz)$ will take that $B(0,1)$ to any preassigned $B(a,r)$. Thus we aim to show

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z}\,ds, \quad z \in B(0,1).$$

Consider

$$\varphi(s,t) = \frac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z} - f(z),$$

where $t \in [0,1]$, $s \in [0,2\pi]$, $|z| < 1$. Clearly φ is continuously differentiable. Hence

$$g(t) = \int_0^{2\pi} \varphi(s,t)\,ds$$

is also continuously differentiable, and

$$
\begin{aligned}
g'(t) &= \int_0^{2\pi} \frac{\partial}{\partial t}\left(\frac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z)} - f(z)\right) ds \\
&= \int_0^{2\pi} \frac{(e^{is}-z)f'(z+t(e^{is}-z))e^{is}}{e^{is}-z} ds \\
&= \int_0^{2\pi} f'(z+t(e^{is}-z))e^{is}\, ds \\
&= \frac{1}{it} f(z+t(e^{is}-z))\Big|_0^{2\pi} \\
&= 0
\end{aligned}
$$

for each $t \in [0,1]$. Hence $g(t) = \text{constant}$. Then

$$
\int_0^{2\pi}\left(\frac{f(z)e^{is}}{e^{is}-z} - f(z)\right) ds = g(0) = g(1) = \int_0^{2\pi}\left(\frac{f(e^{is})e^{is}}{e^{is}-z} - f(z)\right) ds.
$$

But

$$
\int_0^{2\pi}\left(\frac{f(z)e^{is}}{e^{is}-z} - f(z)\right) ds = f(z)\int_0^{2\pi}\left(\frac{e^{is}}{e^{is}-z} - 1\right) ds = 0
$$

by the Example 1.4.3 above. Hence $g(1) = 0$. And this is precisely

$$
2\pi f(z) = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z} ds = \frac{1}{i}\int_\gamma \frac{f(w)}{w-z} dw.
$$

The result follows. □

1.5 Consequences

We now investigate some consequences of the local Cauchy Integral formula.

Theorem 1.5.1. *Let f be analytic on $B(a,R)$. Then*

$$
f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n
$$

for $z \in B(a, R)$ where $a_n = \dfrac{f^{(n)}(a)}{n!}$ and the series has radius of convergence at least R.

Proof. Let $r > 0$ such that $\overline{B}(a, r) \subset B(a, R)$. Suppose $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. Define $M = \max_{z\in\gamma[0,2\pi]} |f(z)|$ since $\gamma[0, 2\pi]$ is compact and f is continuous on $\gamma[0, 2\pi]$. By Theorem 1.4.1, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad \zeta = \gamma(t) = a + re^{it}.$$

We claim that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a + a - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)}\, d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \sum_{k=0}^{\infty} \left(\frac{z - a}{\zeta - a}\right)^k d\zeta \\
&= \sum_{k=0}^{\infty} (z - a)^k \cdot \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{k+1}}\, d\zeta := \sum_{k=0}^{\infty} a_k (z - a)^k.
\end{aligned}$$

This is because

$$\left|\frac{z - a}{\zeta - a}\right| < 1 \quad \text{and} \quad \left|\frac{f(\zeta)}{\zeta - a}\left(\frac{z - a}{\zeta - a}\right)^k\right| \le \frac{M}{r}\left(\frac{|z - a|}{r}\right)^k.$$

So the series $\sum \dfrac{f(\zeta)}{\zeta - a}\left(\dfrac{z - a}{\zeta - a}\right)^k$ converges uniformly by applying M-test.

Thus we could interchange the integral and summation signs in the above computation. But the series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k$$

can be differentiated indefinitely within its radius of convergence, and the derivatives are given by

$$f^{(n)}(z) = \sum_{k=0}^{\infty} n(n-1)\cdots(n-k+1)a_k(z-a)^{k-n}, \quad n = 1,\, 2,\, 3,\, \cdots$$

so that

$$f^{(n)}(a) = n!a_n.$$

Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta = a_n = \frac{f^n(a)}{n!}$$

for each $n \geq 0$. This completes the proof. □

We deduce immediately from the above theorem that

Theorem 1.5.2. *Suppose $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a,r) \subset G$. Then*

(i) f is infinitely differentiable; and

(ii)

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta, \quad \gamma(t) = a + re^{it.}$$

The next theorem is another very important result in complex analysis. It will be derived from Theorem 1.5.1 above. However, some authors prefer to derive it directly and deduce the Cauchy Integral formula as a consequence.

Theorem 1.5.3. *Let f be analytic on $B(a,R)$ and suppose γ is any closed piecewise smooth curve in $B(a,R)$. Then f has a primitive and*

$$\int_\gamma f = 0.$$

Proof. Suppose $z \in B(a, R)$ and $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ by Theorem 1.5.1. It can be easily verified that the function defined by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$$

has the same radius of convergence as that of $f(z)$. Clearly F is differentiable, and $F'(z) = f(z)$. Hence, F is a primitive of f in $B(a, R)$.

Suppose $\gamma : [a, b] \to \mathbb{C}$ is as in the assumption, then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b F'(\gamma(t)\gamma'(t)\,dt \\
&= \int_a^b \frac{d}{dt}F(\gamma(t))\,dt \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= 0
\end{aligned}$$

since γ is closed. □

1.6 Liouville's Theorem

Definition 1.6.1. We say a function f that is analytic everywhere in $\mathbb{C}$ an *entire function.*

Clearly, any entire function has the power series representation in $B(a, r)$ for any $a \in \mathbb{C}$ and any $r > 0$. So the power series must have an infinite radius of convergence.

Proposition 1.6.2. *Let G be an region. If $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is a constant on G.*

Proof. Let $z_0 \in G$ and $f(z_0) = w_0$. Set

$$A = \{z \in G : f(z) = w_0\} \subset G.$$

We aim to show that $A = G$ by proving that A is both open and closed. Then a standard topological argument gives $A = G$. Hence, f is constant on G.

Let $\{z_n\}$ be a sequence in A and $z_n \to z$ as $n \to \infty$. Then by the continuity of f, we have

$$w_0 = \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(z).$$

Hence z belongs to A. This proves that A is closed.

Let $a \in A$, $B(a,\epsilon) \subset G$ and $z \in B(a,\epsilon)$. Let

$$g(t) = f(tz + (1-t)a), \quad 0 \le t \le 1.$$

Then

$$\begin{aligned}\frac{g(t)-g(s)}{t-s} &= \frac{f(tz+(1-t)a) - f(sz+(1-s)a)}{tz+(1-t)a-(sz+(1-s)a)} \\ &\quad \cdot \frac{tz+(1-t)a-(sz+(1-s)a)}{t-s} \\ &\to f'(sz+(1-s)a)\cdot(z-a) \quad \text{(Chain rule)} \\ &= 0\cdot(z-a) = 0,\end{aligned}$$

as $t \to s$. That is $g'(s) = 0$. So $f(z) = g(1) = g(0) = f(a) = w_0$. Since $z \in B(a,\epsilon)$ is arbitrary, we conclude that $B(a,\epsilon) \subset A$. Hence A is open. This completes the proof. □

Theorem 1.6.3 (Liouville's Theorem)**.** *Any bounded entire function must reduce to a constant. That is, there is no non-constant entire function.*

Proof. Let $z \in B(z,r) \subset \mathbb{C}$. Then Theorem 1.5.2 implies

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta, \quad \gamma = z + re^{it}, \quad t \in [0,2\pi]$$

So

$$\begin{aligned}|f'(z)| &\le \frac{1}{2\pi i}\int_\gamma \frac{|f(z)|}{|\zeta - z|^2}|ire^{it}|\,dt\\ &\le \frac{\text{upper bound of } |f|}{r} \to 0 \quad \text{as} \quad r \to \infty.\end{aligned}$$

Hence $f'(z) = 0$ for every $z \in \mathbb{C}$.

Alternatively,

$$\begin{aligned}|a_n| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta\right|\\ &\le \frac{\text{upper bound of } |f|}{r^n} \to 0 \quad \text{as} \quad r \to \infty\end{aligned}$$

for each $n \ge 1$. Hence

$$f(z) = \sum a_n(z-a)^n = a_0 = \text{constant}.$$

□

Definition 1.6.4. Let $f : G \to \mathbb{C}$ and $a \in G$ such that $f(a) = 0$. Then a is a *zero of f with multiplicity $m \ge 1$* if there is an analytic function g such that $f(z) = (z-a)^m g(z)$ and $g(a) \ne 0$.

We deduce the following important theorem from the Louville Theorem.

Theorem 1.6.5 (Fundamental Theorem of Algebra)**.** *Every polynomial $P(z) = a_n z^n + \cdots + a_0$ can be factored as*

$$P(z) = c(z-b_1)^{k_1}\cdots(z-b_m)^{k_m},$$

where c is a constant, $b_1, \ldots, b_m$ are the zeros of P and $k_1+\cdots+k_m = n$.

Proof. It suffices to show that P has at least one zero if it is non-constant, so that we have $P(z) = (z-a)g(z)$, and then obtain the general form via induction on the degree of P.

So let us suppose that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$F(z) := \frac{1}{P(z)}$$

is an entire function on $\mathbb{C}$. But $F(z) \to 0$ uniformly as $z \to \infty$ along all possible paths, so we can find an $M' > 0$ and $R > 0$ such that $|F(z)| < M$ for $z \in \mathbb{C} \setminus B(0, R)$.

Notice that F is also continuous on $\overline{B}(0, R)$ since P has no zeros there. Hence we may find a $M'' > 0$ such that $|F| < M''$ on $\overline{B}(0, R)$ since the closed disk is a compact set and F is continuous on it.

Let $M = \max\{M', M''\}$, we see that $|F| < M$ for all $z \in \mathbb{C}$. So F, and hence P, must reduce to a constant by Louville's theorem. It contradicts to the assumption that P is an arbitrary polynomial. □

1.7 Maximum Modulus Theorem

Theorem 1.7.1 (Isolated Zero Theorem)**.** *Let G be a region, $f : G \to \mathbb{C}$ be analytic. If the set $Z := \{z \in G : \ f(z) = 0\}$ has a limit point in G, then $f \equiv 0$ in G.*

Proof. Let a be a limit point of $Z := \{z \in G : \ f(z) = 0\}$. Then we can find a sequence $\{z_n\}$ in G, $z_n \to a$ and $f(z_n) = 0$. Since

$$0 = \lim_{n \to \infty} f(z_n) = f(a),$$

so $f(a) = 0$. Theorem 1.5.1 implies that for some $R > 0$ such that $B(a,\ R) \subset G$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a_k)^k$$

Suppose that there is an integer N where $0 = a_0 = a_1 = \cdots = a_{N-1}$ but $a_N \neq 0$. Then we can write

$$f(z) = (z - a)^N g(z),$$

in $B(a, R)$ where g is analytic there and $g(a) \neq 0$. But since g is analytic and hence continuous in $B(a, R)$, we can find $0 < r < R$ such that $g(z) \neq 0$ in $B(a, r)$. But since a is a limit point, so there is a $b \in B(a, r)$ different from a such that $0 = f(b) = (b-a)^N g(b) \neq 0$. A contradiction. So no such integer N can be found. Thus, the set

$$A := \{z \in G : \ f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

is non-empty.

We next show that A is both closed and open. Let z belongs to the closure of A, and $\{z_k\} \subset A$ converges to z. Since each $f^{(n)}$ is continuous, it follows that $0 = \lim_{k\to\infty} f^{(n)}(z_k) = f^{(n)}(z)$. Hence $z \in A$ and A is closed.

Let $a \in A$ and $B(a, R) \subset G$. Then $f(z) = \sum a_k (z-a)^k$ in $B(a, R)$, and $f^{(n)}(a) = 0$ for each n. So $f(z) = 0$ in $B(a, R)$. Then clearly $B(a, R) \subset A$. Hence A is open. Since A is non-empty, so $A = G$. □

Corollary 1.7.1.1 (Identity Theorem)**.** *If $f = g$ on a sequence of points having a limit point in G, then $f \equiv g$ on G.*

Theorem 1.7.2 (Maximum Modulus Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ is analytic. If there exists a point $a \in G$ such that $|f(z)| \leq |f(a)|$ for all $z \in G$, then f is constant.*

Proof. Let z_0 be an arbitrary point in G such that $|f(z_0)| = |f(a)|$, $B(z_0, r) \subset G$, $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

By Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt. \end{aligned}$$

We may suppose that $|f|$ is non-constant on $\partial B(z_0, r)$ for some $r > 0$. Hence, there exists a $t_0 \in [0, 2\pi]$ and $\delta > 0$ such that

$$|f(z_0 + re^{it})| < M = |f(a)| \quad \text{on } [t_0 - \delta, t_0 + \delta].$$

Hence

$$\begin{aligned} M = |f(z_0)| &\leq \left| \frac{1}{2\pi} \int_{t \in [0,2\pi] \setminus [t_0 - \delta, t_0 + \delta]} f(z_0 + re^{it})\, dt \right| \\ &\quad + \left| \frac{1}{2\pi} \int_{t \in [t_0 - \delta, t_0 + \delta]} f(z_0 + re^{it})\, dt \right| \\ &< \frac{M}{2\pi}(2\pi - 2\delta) + \frac{M}{2\pi} 2\delta = M. \end{aligned}$$

A contradiction since $M \not< M$. Hence $|f| \equiv M$ in $B(z_0, r)$, then f is constant in $B(z_0, r)$ (Use $f' = u_x + iv_x$ and Proposition 1.6.2). Now, since $B(z_0, r)$ is non-empty open subset of G, then by the Identity Theorem, f is constant on G. □

Theorem 1.7.3 (Minimum Modulus Theorem)**.** *Let $f : G \to \mathbb{C}$ be analytic and G is a region. If there exists $a \in G$ such that $|f(z)| \geq |f(a)|$ for all $z \in G$, then either f is a constant or $f(a) = 0$ i.e. a is zero of f.*

Proof. Exercise. □

1.8 Branch of the Logarithm

Definition 1.8.1. Let G be a region and $f : G \to \mathbb{C}$ is continuous. We call $f(z)$, *a branch of the logarithm* if $e^{f(z)} = z$ for every $z \in G$.

If $e^w = z$, then we write $w = \log z = f(z)$. But $e^{w+2\pi ik} = e^w = z$ for every integer k. Hence for each z, the equation $e^w = z$ has an infinite number of solution for $w = \log|z| + i(\arg z + 2\pi k)$. Let $G = \mathbb{C} \setminus \{x : x \leq 0\}$ and $-\pi < \arg z < \pi$. The function

$$f(z) = \log|z| + i \arg z, \quad z \in G$$

is called the *principal branch of the logarithm.* The *other branches* of the logarithm are given by

$$f_k(z) = \log|z| + i \arg z$$

for $(2k-3)\pi < \arg z < (2k-1)\pi.$, $k \in \mathbb{Z} \setminus \{1\}$. (Principal branch $f = f_1$, i.e., $k = 1$)

The principal branch of the logarithm is analytic on $\mathbb{C} \setminus \{x : x \leq 0\}$.

Proposition 1.8.2. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed piecewise smooth curve and assume that $a \notin \gamma$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - a} \in \mathbb{Z}.$$

This proposition seems trivial since

$$\int_\gamma \frac{d\zeta}{\zeta - a} = \int_\gamma d(\log(\zeta - a)) = \int_\gamma d(\log|\zeta - a|) + i \int_\gamma d(\arg(\zeta - a)).$$

When γ has described a complete revolution, $\gamma(t)$ returns to its initial position, so the first integral $\int_\gamma d(\log|\zeta - a|) = 0$; and $i \int_\gamma d(\arg(\zeta - a))$ gives $2\pi i k$, where k is the number of the complete revolutions that γ around a. However, the function $\arg(\zeta - a)$ is *not uniquely determined* (multi-valued), so the above argument is not precise.

Proof. One of the easiest proofs available is to consider the function

$$g(t) = \int_0^t \frac{\zeta'(t)}{\zeta(t) - a}\, dt.$$

Note that

$$g(1) = \int_0^1 \frac{\zeta(t)}{\zeta(t) - a}\, dt = \int_\gamma \frac{d\zeta}{\zeta - a}.$$

We aim to show that $\dfrac{e^{g(t)}}{\zeta(t)-a}$ is constant on $[0,1]$. Consider

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^{g(t)}}{\zeta(t)-a}\right) &= \frac{g'(t)e^{g}}{\zeta(t)-a} - \frac{\zeta'(t)e^{g}}{(\zeta(t)-a)^2} \\ &= e^{g}\left(\frac{\zeta'(t)}{(\zeta(t)-a)^2} - \frac{\zeta'(t)}{(\zeta(t)-a)^2}\right) \\ &= 0\end{aligned}$$

for $t \in [0,1]$. Thus

$$\frac{e^{g(0)}}{\zeta(0)-a} = \frac{e^{g(1)}}{\zeta(1)-a} \implies e^{g(0)} = e^{g(1)}.$$

But $g(0) = 0$, so $e^{g(1)} = 1$.
Hence

$$g(1) = \int_0^1 \frac{\zeta'(t)}{\zeta(t)-a}\,dt = \int_\gamma \frac{d\zeta}{\zeta - a} = 2\pi i k$$

for some integer k. Then the result follows.

□

Definition 1.8.3. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed and piecewise smooth curve, and $a \notin \gamma$. We define

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - a}$$

to be the *index of* γ *with respect to* a or the *winding number of* γ *around* a.

Suppose $\gamma(t) : [0,1] \to \mathbb{C}$ is a curve, we define $-\gamma(t) = \gamma(1-t)$. If $\sigma : [0,1] \to \mathbb{C}$ is another curve such that $\gamma(1) = \sigma(0)$, then $\gamma + \sigma$ means

$$(\gamma+\sigma)(t) = \begin{cases}\gamma(2t), & 0 \le t \le \frac{1}{2} \\ \sigma(2t-1), & \frac{1}{2} < t \le 1.\end{cases}$$

It is left as an exercise to verify that

(i) $n(-\gamma; a) = -n(\gamma; a)$

(ii) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$.

Proposition 1.8.4. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed and piecewise smooth curve, and $a \notin \gamma$. Then $n(\gamma; a)$ is constant for any a belongs to a bounded component of $\mathbb{C} \setminus \gamma$, and zero for a belongs to the unbounded component.*

Remark. There is only one unbounded component since γ is a compact set.

Proof. Let a and b belong to the same component D of $\mathbb{C} \setminus \gamma$. Since $n(\gamma; a)$ and $n(\gamma; b)$ both equal to some integers, it suffices to prove $n(\gamma; a)$ is continuous on D. (Then, $n(\gamma; D)$ is connected, and since $n(\gamma; D) \subset \mathbb{Z}$, $n(\gamma; D)$ is a constant integer only.)

Let $d = \min_{\zeta \in \gamma}\{|\zeta - a|, |\zeta - b|\}$. Then, by definition,

$$\begin{aligned}
|n(\gamma; a) - n(\gamma; b)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{\zeta - a} - \frac{1}{\zeta - b} \right) d\zeta \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{a - b}{(\zeta - a)(\zeta - b)} d\zeta \right| \\
&\leq \frac{1}{2\pi} \int_\gamma \frac{|a - b|}{|(\zeta - a)(\zeta - b)|} |d\zeta| \\
&\leq \frac{|a - b|}{2\pi d^2} \int_\gamma |d\zeta| \\
&= \frac{|a - b|}{2\pi d^2} l(\gamma) \to 0,
\end{aligned}$$

as $|a - b| \to 0$. Hence $n(\gamma; a)$ is continuous on any components of $\mathbb{C} \setminus \gamma$.

For a belongs to the unbounded component of $\mathbb{C} \setminus \gamma$, let $d = \min_{\zeta \in \gamma}\{|\zeta - a|\}$ By the above argument, we have

$$\begin{aligned}
|n(\gamma; a)| &= \frac{1}{2\pi} \left| \int_\gamma \frac{d\zeta}{\zeta - a} \right| \\
&\leq \frac{1}{2\pi d} l(\gamma).
\end{aligned}$$

But $\min_{\zeta\in\gamma}\{|\zeta - a|\} \to \infty$ as $a \to \infty$. Hence, $|n(\gamma; a)| \to 0$ as $a \to \infty$. Since $n(\gamma; a)$ is constant and so $n(\gamma; a) = 0$ in this unbounded component because $n(\gamma; a)$ was proved to be continuous. □

1.9 Cauchy's Theorem

We next prove the general Cauchy Integral formula and Cauchy's theorem. In particular, we give conditions on $n(\gamma; a)$ so that the Cauchy theorem holds.

Proposition 1.9.1. *Let γ be a piecewise smooth curve and φ is a function continuous on γ. For each $m \geq 1$, define, for $z \notin \gamma$*

$$F_m(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^m}\, d\zeta.$$

Then, F_m is analytic on $\mathbb{C} \setminus \gamma$ and $F_m' = mF_{m+1}$.

Proof. We first show that F_m is continuous on $\mathbb{C}\setminus\gamma$. Since γ is compact and φ is continuous on γ, we may let $M = \max_{z\in\gamma} |\varphi(z)|$.

Let a and b belong to the same component (if any) of $\mathbb{C} \setminus \gamma$. Then, as in the proof for $n(\gamma; a)$,

$$\begin{aligned}|F_m(a) - F_m(b)| &= \left|\int_\gamma \left(\frac{\varphi(\zeta)}{(\zeta - a)^m} - \frac{\varphi(\zeta)}{(\zeta - b)^m}\right) d\zeta\right|\\ &\leq M \int_\gamma \left|\frac{1}{(\zeta - a)^m} - \frac{1}{(\zeta - b)^m}\right| \cdot |d\zeta|.\end{aligned}$$

So, it remains to estimate the function inside the integrand: Since

$$A^m - B^m = (A - B)(A^{m-1} + A^{m-2}B + \cdots + AB^{m-2} + B^{m-1}).$$

Putting $A = \dfrac{1}{\zeta - a}$ and $B = \dfrac{1}{\zeta - b}$, and let $d = \min_{\zeta\in\gamma}\{|\zeta - a|, |\zeta - b|\}$, gives

$$|F_m(a) - F_m(b)| \leq mM\frac{|a - b|}{d^{m+1}} l(\gamma) \to 0 \quad \text{as } a \to b.$$

Hence, F_m is continuous on $\mathbb{C}\setminus\gamma$.

Let $a,\ b\in\mathbb{C}\setminus\gamma$ and $A,\ B$ as defined above. Then

$$\begin{aligned}
\frac{F_m(a)-F_m(b)}{a-b} &= \frac{1}{a-b}\int_\gamma \varphi(\zeta)(A-B)(A^{m-1}+A^{m-2}B+\cdots+AB^{m-2}+B^{m-1})\,d\zeta \\
&= \frac{1}{a-b}\int_\gamma \varphi(\zeta)(a-b)AB(A^{m-1}+A^{m-2}B+\cdots \\
&\qquad +AB^{m-2}+B^{m-1})\,d\zeta \\
&= \int_\gamma \varphi(\zeta)(A^mB+A^{m-1}B^2+\cdots+AB^m)\,d\zeta \\
&\longrightarrow \int_\gamma \varphi(\zeta)(B^{m+1}+B^{m+1}+\cdots+B^{m+1})\,d\zeta \\
&= m\int_\gamma \varphi(\zeta)B^{m+1}\,d\zeta \\
&= m\int_\gamma \frac{\varphi(\zeta)}{(\zeta-b)^{m+1}}\,d\zeta \\
&= F_m'(b)
\end{aligned}$$

as $a\to b$.

Hence, F_m is analytic with its derivative given at the end of the above expression. □

Theorem 1.9.2 (Cauchy's Integral Formula - First version)**.** *Let G be an open subset of $\mathbb{C}$ and $f: G\to\mathbb{C}$ be analytic. If γ is a closed piecewise smooth curve in G such that $n(\gamma;w)=0$ for all $w\in\mathbb{C}\setminus G$, then for $a\in G\setminus\gamma$,*

$$n(\gamma;a)f(a)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-a}\,d\zeta.$$

Proof. Define $\varphi: G\times G\to\mathbb{C}$ by

$$\varphi(z,w)=\begin{cases}\dfrac{f(z)-f(w)}{z-w}, & \text{if } z\neq w\\ f'(z), & \text{if } z=w.\end{cases}$$

(Exercise: Show φ is continuous and $z\mapsto\varphi(z,w)$ is analytic.)

Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Then H is open since $n(\gamma; w)$ is continuous on $\mathbb{C} \setminus \gamma$ and integer-valued i.e., $\{0\}$ is open in $\mathbb{Z}$. From the definition of G and H, we deduce that $\mathbb{C} = G \cup H$ and $G \cap H \neq \emptyset$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\gamma \varphi(z, \zeta)\, d\zeta, & \text{if } z \in G \\ \int_\gamma \dfrac{f(\zeta)}{\zeta - z}\, d\zeta, & \text{if } z \in H. \end{cases}$$

Next, we verify that g is well-defined on $G \cap H$.

$$\begin{aligned} \int_\gamma \varphi(z, \zeta)\, d\zeta &= \int_\gamma \frac{f(z) - f(\zeta)}{z - \zeta}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta}{\zeta - z}\, d\zeta - f(z) \cdot 2\pi i\, n(\gamma; z) \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \end{aligned}$$

since $z \in G \cap H$. Hence, g is a well-defined function on $\mathbb{C}$.

It follows from Proposition 1.9.1 that g is an entire function, and from Proposition 1.8.4, H must contain the unbounded component of $\mathbb{C} \setminus \gamma$ (because if $n(\gamma; w) = 0$, then $w \in H$). For z belongs to the unbounded component, we have

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_\gamma f(\zeta) \lim_{z\to\infty} \frac{1}{\zeta - z}\, d\zeta = 0$$

since f is bounded on γ and $\lim_{z\to\infty} \dfrac{1}{\zeta - z} = 0$ uniformly.

So, there exists an $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$, and since g is bounded on the compact set $\overline{B}(0, R)$, then g is a bounded entire function. Hence g is constant by Liouville's Theorem. Thus, $g(z) = 0$ for all $z \in \mathbb{C}$.

That is, for $a \in G \setminus \gamma$,

$$\begin{aligned} 0 = g(a) &= \int_\gamma \frac{f(\zeta) - f(a)}{\zeta - a}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - a} - f(a) \cdot 2\pi i\, n(\gamma; a). \end{aligned}$$

This completes the proof. □

Theorem 1.9.3 (Cauchy's Integral Formula - Second version)**.** *Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed piecewise smooth curves in G such that*

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$$

for all $w \in \mathbb{C} \setminus G$, then for all $a \in G \setminus \gamma$ and $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$,

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - a}\, d\zeta.$$

Proof. The proof is similar to that of Theorem 1.9.2 except to define suitable φ, H and g. □

Theorem 1.9.4 (Cauchy's Theorem - First version)**.** *Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed piecewise smooth curves in G such that*

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$$

for all $w \in \mathbb{C} \setminus G$, then

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

Proof. Put $f(z)(z - a)$ instead of $f(z)$, and then apply Theorem 1.9.3. □

Remark. We note that Cauchy's theorem was published around 1825, while Goursat's theorem was around 1900.

Theorem 1.9.5 (Morera's Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ be a continuous function such that*

$$\int_T f = 0$$

for every closed triangular curve T in G, then f is analytic on G.

Remark. A closed triangular curve is a closed three sides polygon.

Proof. It suffices to show that f has a primitive on each open disks in G. In fact, we may assume $G = B(a, R)$ since G is open.

Let $z \in B(a, R)$ and define

$$F(z) = \int_{[a,z]} f.$$

Suppose $z_0 \in B(a, R)$, then

$$F(z) = \left(\int_{[a,z_0]} + \int_{[z_0.z]} \right) f.$$

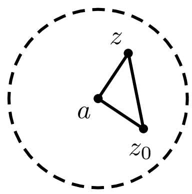

Figure 1.1: $B(a, R)$

So

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{z - z_0} \int_{[z_0,z]} f \\ &= \frac{1}{z - z_0} \int_{[z_0.z]} (f(\zeta) - f(z_0))\, d\zeta + f(z_0). \end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right| &\le \sup_{\zeta\in[z,z_0]}|f(\zeta)-f(z_0)|\cdot\left|\frac{1}{z-z_0}\right|\int_{[z_0,z]}|d\zeta|\\ &= \sup_{\zeta\in[z,z_0]}|f(\zeta)-f(z_0)|\\ &\to 0 \quad \text{as } z\to z_0.\end{aligned}$$

Hence, $F'(z_0)=f(z_0)$. But F must be infinitely differentiable, so f is analytic on $B(a,R)$. □

1.10 Homotopy version of Cauchy's Theorem

Definition 1.10.1. Let γ_0, γ_1 : $[0,1]\to G$ be two closed piecewise smooth curves in a region G. Then we say that γ_0 *is homotopic to* γ_1 is there is a continuous function $\Gamma:[0,1]\times[0,1]\to G$ such that

$$\Gamma(s,0)=\gamma_0(s),\quad \Gamma(s,1)=\gamma_1(s),\quad 0\le s\le 1;$$

$$\Gamma(0,t)=\Gamma(1,t),\quad 0\le t\le 1.$$

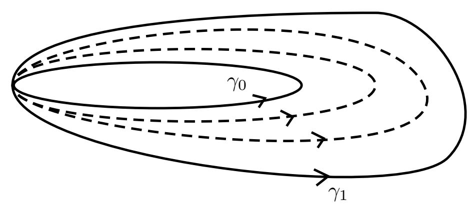

Figure 1.2: γ_0 is homotopic to γ_1

Remark. (i) If we write $\Gamma(s,t)=\gamma_t(s)$. Then the above definition does not require $\gamma_t(s)$ to be piecewise smooth.

(ii) If γ_0 is homotopic to γ_1, we write $\gamma_0\sim\gamma_1$. Note that $\sim$ defines equivalent classes on closed piecewise smooth curves in G:

(a) $\gamma_0 \sim \gamma_0$ by the identity map,

(b) If $\gamma_0 \sim \gamma_1$, then $\Lambda(s,t) = \Gamma(s, 1-t)$ would give $\gamma_1 \sim \gamma_0$,

(c) If $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ with homotopy Γ and Λ respectively, then the homotopy $\Psi : [0,1] \times [0,1] \to G$ given by

$$\Psi(s,t) = \begin{cases} \Gamma(s, 2t), & 0 \le t \le \frac{1}{2} \\ \Lambda(s, 2t-1), & \frac{1}{2} < t \le 1 \end{cases}$$

shows that $\gamma_0 \sim \gamma_1$.

Definition 1.10.2. A closed piecewise smooth curve γ is said to be *homotopic to zero* if γ is homotopic to a constant curve (written $\gamma \sim 0$).

Definition 1.10.3. A region G is *a-star shaped* if the line segment $[a,z]$ lies entirely in G for each $z \in G$. We simply call G *star shaped* if G is 0-star shaped.

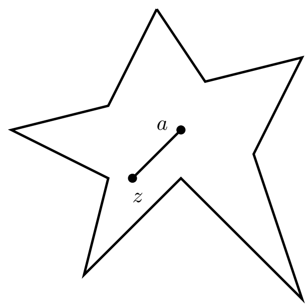

Figure 1.3: a-star shaped

Example 1.10.4. Let G be an a-star shaped region. Then every closed piecewise smooth curve γ in G is homotopic to the constant curve $\gamma_0(t) = a$.

Solution. Let

$$\begin{aligned}\Gamma(s,t) &= t\gamma_0(s) + (1-t)\gamma_1(s) \\ &= ta + (1-t)\gamma_1(s)\end{aligned}$$

for $0 \le s, t \le 1$.

It is easy to see that Γ is a homotopy between γ_1 and γ_0. □

Remark. A *convex* region is a-star shaped with respect to any a that belongs to G.

Definition 1.10.5. If $\gamma_0,\ \gamma_1 : [0,1] \to G$ are two piecewise smooth curves in a region G such that $\gamma_0(0) = a = \gamma_1(1)$, $\gamma_0(1) = b = \gamma_1(1)$. We say γ_0 *is (fixed end points) homotopic to* γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $\Gamma : [0,1]^2 \to G$ such that

$$\Gamma(s,0) = \gamma_0(s), \quad \Gamma(s,1) = \gamma_1(s), \quad 0 \le s \le 1;$$

$$\Gamma(0,t) = a, \quad \Gamma(1,t) = b, \quad 0 \le t \le 1.$$

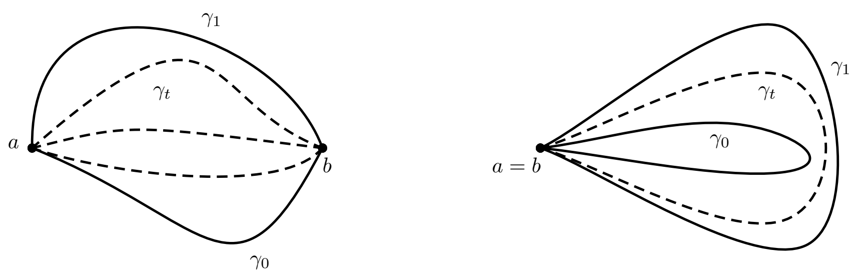

Figure 1.4: γ_0 is (fixed end points) homotopic to γ_1

Similarly, it can be verified that $\sim$ is an equivalence relation on the piecewise smooth curves satisfying the above definition. (See Conway p.93)

And note again that, the intermediate path $\gamma_s(t) = \Gamma(s,t)$ for $0 \le s \le 1$ and t fixed, need *not* be piecewise smooth.

Theorem 1.10.6 (Cauchy's Theorem - Second version)**.** *Suppose $f : G \to \mathbb{C}$ is analytic and γ is a closed piecewise smooth curve in G such that $\gamma \sim 0$, then*

$$\int_\gamma f = 0.$$

Theorem 1.10.7 (Cauchy's Theorem - Third version)**.** *Suppose $f : G \to \mathbb{C}$ is analytic and $\gamma_0,\ \gamma_1 : [0,1] \to G$ are two closed piecewise smooth curves such that $\gamma_0 \sim \gamma_1$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Let γ_0 and γ_1 be as in the hypothesis, and $\Gamma : I^2 \to G$ ($I = [0,1]$) be the corresponding continuous function. Since I^2 is compact, Γ must be uniformly continuous on I^2. Thus $\Gamma(I^2)$ is compact and is a proper subset of G. Hence

$$d(\Gamma(I^2), \mathbb{C} \setminus G) = \inf\{|x - y| : x \in \Gamma(I^2), y \in \mathbb{C} \setminus G\} = r > 0.$$

There exists an integer $n > 0$ such that

$$|\Gamma(s', t') - \Gamma(s, t)| < r$$

whenever $|(s', t') - (s, t)|^2 < \frac{4}{n^2}$ and (s', t'), $(s, t) \in I^2$.

Set

$$J_{jk} = [\frac{j}{n}, \frac{j+1}{n}] \times [\frac{k}{n}, \frac{k+1}{n}] \quad (0 \leq j, k \leq n-1)$$

(this forms a partition of $I \times I$) and

$$\zeta_{jk} = \Gamma(\frac{j}{n}, \frac{k}{n}) \quad (0 \leq j, k \leq n).$$

As the diameter (= diagonal) of J_{jk} is $\sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} < \frac{2}{n}$, we must have $\Gamma(J_{jk}) \subset B(\zeta_{jk}, r)$ for $0 \leq j, k \leq n-1$. ($\cup_{jk} B(\zeta_{jk}, r)$ forms an open cover of $\Gamma(I^2)$; also it is a proper subset of G by the choice of $r > 0$.)

Let

$$Q_k = [\zeta_{0k}, \zeta_{1k}, \ldots, \zeta_{nk}]$$

be the closed polygon (since $\zeta_{0k} = \zeta_{nk}$) for $0 \leq k \leq n$.

We will first show that

$$\int_{\gamma_0} f = \int_{Q_0} f$$

and

$$\int_{Q_n} f = \int_{\gamma_1} f,$$

then

$$\int_{Q_k} f = \int_{Q_{k+1}} f \quad (0 \le k \le n-1).$$

Thus

$$\int_{\gamma_0} f = \int_{Q_0} f = \cdots = \int_{Q_k} f = \cdots = \int_{Q_n} f = \int_{\gamma_1} f.$$

Let

$$P_{jk} = [\zeta_{jk}, \zeta_{j+1,k}, \zeta_{j+1,k+1}, \zeta_{j,k+1}, \zeta_{jk}]$$

be a closed polygon. (See Figure 1.5)

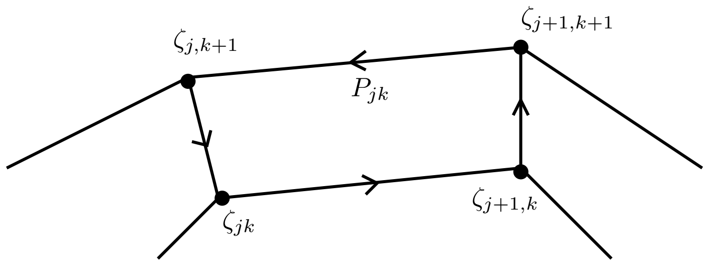

Figure 1.5: P_{jk}

But $\Gamma(J_{jk}) \subset B(\zeta_{jk}, r)$, hence $P_{jk} \subset B(\zeta_{jk}, r)$ in which f is analytic. So

$$\int_{P_{jk}} f = 0 \quad (0 \le j, k \le n-1)$$

by Theorem 1.5.3.

We now show $\int_{\gamma_0} f = \int_{Q_0} f$, where

$$Q_0 = [\zeta_{00}, \zeta_{10}, \ldots, \zeta_{n0}].$$

Let $\sigma_j(s) = \gamma_0(s)$ for $\frac{j}{n} \le s \le \frac{j+1}{n}$, $(0 \le j \le n-1)$. (See Figure 1.6)

Clearly $\sigma_j + [\zeta_{j+1,0}, \zeta_{j0}]$ is a closed piecewise smooth curve in $B(\zeta_{j0}, r)$ and so

$$\int_{\sigma_j + [\zeta_{j+1,0}, \zeta_{j0}]} f = 0.$$

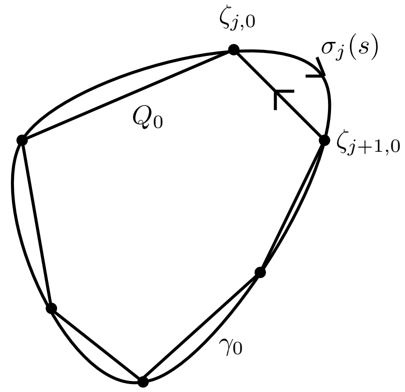

Figure 1.6: $\sigma_j(s)$

That is

$$\int_{\sigma_j} f = -\int_{[\zeta_{j+1,0},\zeta_{j0}]} f = \int_{[\zeta_{j0},\zeta_{j+1,0}]} f$$

So

$$\int_{\gamma_0} f = \sum_{j=0}^{n-1} \int_{\sigma_j} f = \sum_{j=0}^{n-1} \int_{[\zeta_{j0},\zeta_{j+1,0}]} f = \int_{Q_0} f.$$

Similarly, we can prove $\int_{\gamma_1} f = \int_{Q_n} f$. Finally, we show $\int_{Q_k} f = \int_{Q_{k+1}} f \quad (0 \leq k \leq n-1)$. Clearly we have $0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f$.

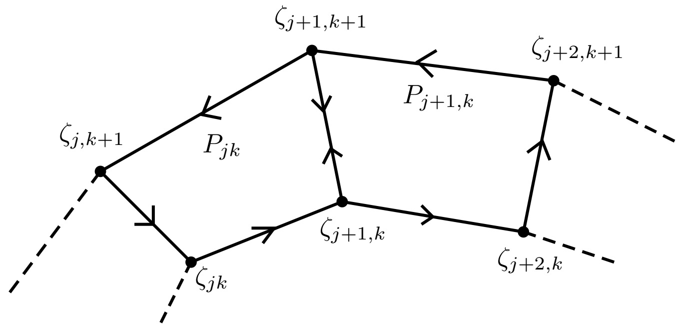

Figure 1.7: P_{jk} and $P_{j+1,k}$

It follows from the Figure 1.7 that

$$\int_{[\zeta_{j+1,k},\zeta_{j+1,k+1}]} f$$

of $\int_{P_{jk}} f$ cancels the

$$\int_{[\zeta_{j+1,k+1},\zeta_{j+1,k}]} f$$

of $\int_{P_{j+1,k}} f$. Thus

$$0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f = \int_{Q_k} f - \int_{Q_{k+1}} f.$$

□

Theorem 1.10.8. *Let γ be a closed piecewise smooth curve in G with $\gamma \sim 0$. Then $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus G$.*

Proof. The proof follows from Theorem 1.10.6. Since $\dfrac{1}{z-a}$ is analytic on G if $a \in \mathbb{C} \setminus G$,

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta = 0.$$

□

We note that the converse of Theorem 1.9.8 is not true. That is, there exist a γ such that $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus G$ but it is not true that $\gamma \sim 0$. (See exercise). Thus Theorem 1.9.2 and 1.9.3 are more general than Theorem 1.10.6 and 1.10.7.

Theorem 1.10.9. *If γ_0 and γ_1 are two piecewise smooth curves joining a to b and $\gamma_0 \sim \gamma_1$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Since $\gamma_0 \sim \gamma_1$, so there exists a continuous map $\Gamma : I^2 \to \mathbb{C}$ such that

$$\Gamma(s, 0) = \gamma_0(s), \quad \Gamma(s, 1) = \gamma_1(s), \quad 0 \le s \le 1;$$

$$\Gamma(0, t) = a, \quad \Gamma(1, t) = b, \quad 0 \le t \le 1.$$

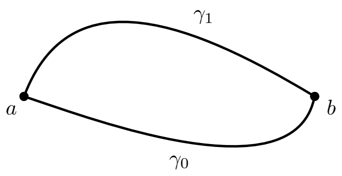

Figure 1.8: $\gamma_0 \sim \gamma_1$

Because $\gamma_0 - \gamma_1$ is a closed piecewise smooth curve, we define

$$\gamma(s) = \begin{cases} \gamma_0(3s), & 0 \le s \le \frac{1}{3} \\ b, & \frac{1}{3} < s \le \frac{2}{3} \\ \gamma_1(3-3s), & \frac{2}{3} < s \le 1. \end{cases}$$

Next we show $\gamma \sim 0$ by claiming that $\Lambda : I^2 \to G$ is a suitable function:

$$\Lambda(s,t) = \begin{cases} \Gamma(3s(1-t), t), & 0 \le s \le \frac{1}{3} \\ \Gamma(1-t, 3s-1+2t-3st), & \frac{1}{3} < s \le \frac{2}{3} \\ \gamma_1((3-3s)(1-t)), & \frac{2}{3} < s \le 1. \end{cases}$$

Note that

$$\Lambda(s,t) = \gamma_t(s), \quad \Lambda(s,0) = \gamma_0 - \gamma_1, \quad \Lambda(s,1) = a = b.$$

It is easy to see that Λ is continuous at $s = \frac{1}{3}$; and at $s = \frac{2}{3}$ because $\Gamma(1-t,1) = \gamma_1(1-t)$. So, Λ is continuous on I^2.

Hence

$$0 = \int_\gamma f = \int_{\gamma_0} f - \int_{\gamma_1} f.$$

□

Definition 1.10.10. An open set G is called *simply connected* if it is connected and every closed curve in G is homotopic to zero (i.e., $\gamma \sim 0$).

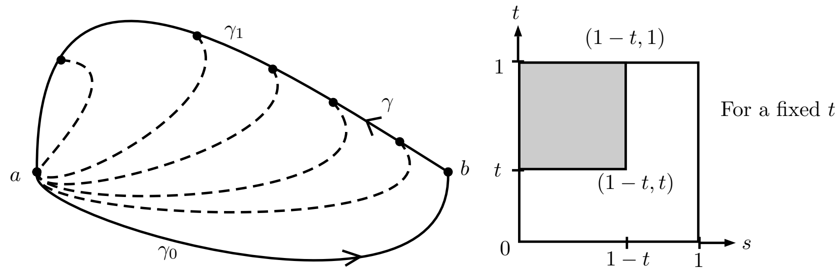

Figure 1.9: $\Lambda(s,t)$ and $[0,1-t] \times [t,1]$

So we have the following version of Cauchy's Theorem.

Theorem 1.10.11 (Cauchy's Theorem - Fourth version)**.** *If G is simply connected, then $\int_\gamma f = 0$ for every closed piecewise smooth curve and every analytic f.*

The notion of simply connected region lies much deeper than it appears. We shall study this in a more detailed way in a later chapter (pending). Here we chiefly want to prove some immediate consequences of analytic function defined on simply connected region.

Theorem 1.10.12. *Suppose the region G is simply connected, and $f : G \to \mathbb{C}$ is analytic. Then f has a primitive on G.*

Proof. Let $a \in G$ and $\gamma : [0,1] \to G$ be a piecewise smooth curve (if closed, then by Theorem 1.5.3 immediately) in G where $\gamma(0) = a$.

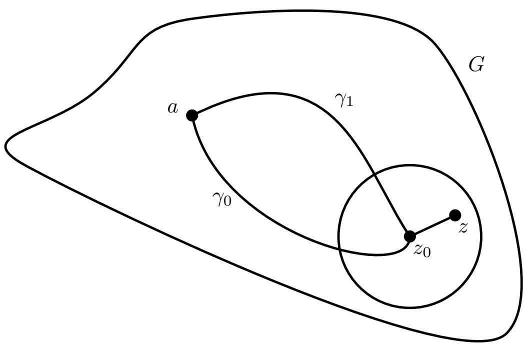

Figure 1.10: $\gamma_0 - \gamma_1$

Define an expression $F(z) = \int_\gamma f(\zeta)\, d\zeta$. We first verify that F is well-defined.

Since $\gamma_0 - \gamma_1 \sim 0$, Cauchy's Theorem implies that

$$\int_{\gamma_0-\gamma_1} f\, d\zeta = \int_{\gamma_0} f\, d\zeta - \int_{\gamma_1} f d\zeta = 0.$$

Hence F is independent on the choice of γ. Thus F is a well-defined function.

To show F is analytic and $F' = f$, we consider $r > 0$ so small such that $B(z_0, r) \subset G$. Replace γ by $\gamma + [z_0, z]$ in F:

$$F(z) = \int_{\gamma+[z_0,z]} f.$$

Then we have

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, d\zeta.$$

By the similar argument in the proof of Morera's Theorem, we can deduce that $F' = f$ and F is analytic. □

The next result lies deeper.

Theorem 1.10.13. *Let G be simply connected and $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = e^{g(z)}$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, then we may choose g such that $g(z_0) = w_0$. So simply connected region implies every non-vanishing analytic function can have a logarithm.*

Proof. Since f has no zeros, and $\frac{f'}{f}$ is analytic on G. By Theorem 1.10.12, we let g to be a primitive of $\frac{f'}{f}$. Consider

$$\frac{d}{dz}\left(\frac{f}{e^g}\right) = \frac{f' - g'f}{e^g} = 0.$$

Thus $f = (\text{constant})e^g = e^{g+c}$, where c is a constant.

So, if $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may find a suitable integer k such that $w_0 = g(z_0) + c + 2\pi k$. Now define $\tilde{g} = g + c + 2\pi k$, which is the required function. □

Remark. The converse of the above statement also hold, namely that, G is a simply connected region if every non-vanishing analytic function f can be represented as $f = e^g$ for same analytic function g on G. We refer to [1] or [3] for the detail.

1.11 Open Mapping Theorem

Definition 1.11.1. If γ is a closed piecewise smooth curve in G such that $n(\gamma; w) = 0$ for each $w \in \mathbb{C} \setminus G$. We call such curve *homologous to zero* ($\gamma \approx 0$).

The following contour shows that although $\gamma \sim 0$ implies $\gamma \approx 0$, the converse is not true. One can verify that following figure has $\gamma \approx 0$ but $\gamma \not\sim 0$ since $n(\gamma;\, a) = 0 = n(\gamma;\, b)$. The contour was first written down independently by C. Jordan (1887) and L. Pochhammer (1890).

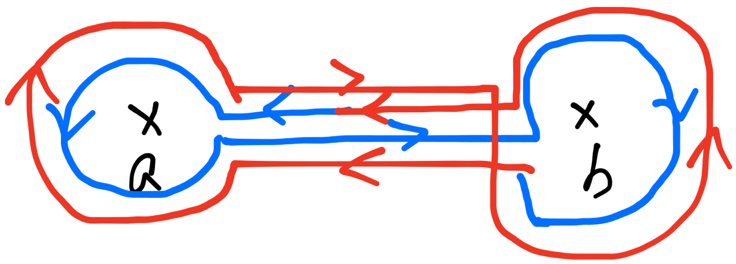

Figure 1.11: Pochhammer contour

Remark. The *Beta function* is defined by the integral

$$B(x,\, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \tag{1.3}$$

where $\Re(x)$ and $\Re(y) > 0$ so that the integral converges. However, if we remove the restriction$\Re(x)$ and $\Re(y) > 0$, then we can still compute the beta function via Pochhammer contour to

$$B(x,\, y) = \int_{\text{(Pochhammer)}}^{(1+,\,0+,\,1-,\,0-)} t^{x-1}(1-t)^{y-1}\,dt = \frac{-e^{\pi i(x+y)}4\pi^2}{\Gamma(1-x)\Gamma(1-y)\Gamma(x+y)}\,dt. \tag{1.4}$$

See [9] for the detail.

By using Cauchy's Theorem, we shall see below some topological results of different natures.

Theorem 1.11.2. *Let G be a region and $f : G \to \mathbb{C}$ analytic on G with zeros $a_1, \ldots, a_m$ (counted with multiplicity). If γ is a closed piecewise smooth curve in G such that $a_k \notin \gamma$ for each k, and if $\gamma \approx 0$ in G, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\,d\zeta = \sum_{k=1}^{m} n(\gamma; a_k).$$

Proof. According to previous discussion,

$$f(z) = (z-a_1)\cdots(z-a_m)g(z), \quad g(z) \neq 0, \quad z \in G.$$

Then for $z \neq a_1, \ldots, a_m$, we have

$$\frac{f'}{f}(z) = \frac{1}{z-a_1} + \cdots + \frac{1}{z-a_m} + \frac{g'}{g}.$$

So

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\,d\zeta &= \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta-a_1} + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta-a_m} + \int_\gamma \frac{g'}{g}\,d\zeta \\ &= n(\gamma; a_1) + \cdots + n(\gamma; a_m) + \int_\gamma \frac{g'}{g}\,d\zeta.\end{aligned}$$

Since $\gamma \approx 0$ and $\dfrac{g'}{g}$ is analytic on G, by the Cauchy Theorem - First version, we have $\int_\gamma \dfrac{g'}{g}\,d\zeta = 0$. This completes the proof. □

Corollary 1.11.2.1. *Let f, G and γ be as in the preceding theorem except that $a_1, \ldots, a_m$ are the roots of $f(z) = \alpha$ (counted according to multiplicity). Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)-\alpha}\,d\zeta = \sum_{k=1}^{m} n(\gamma; a_k).$$

We next prove the important Open Mapping Theorem. But we first need the following theorem.

Theorem 1.11.3. *Let $f : G \to \mathbb{C}$ be analytic where $f(a) = \alpha$. Suppose $f - \alpha$ has a zero of multiplicity m. Then we can find an $\epsilon > 0$ and a $\delta > 0$ such that for all ξ in $0 < |\zeta - a| < \delta$, the equation $f(z) = \xi$ has exactly m simple roots in $0 < |z - a| < \epsilon$. (A simple root of $f(z) = \xi$ is a zero of $f - \xi$ with multiplicity 1.)*

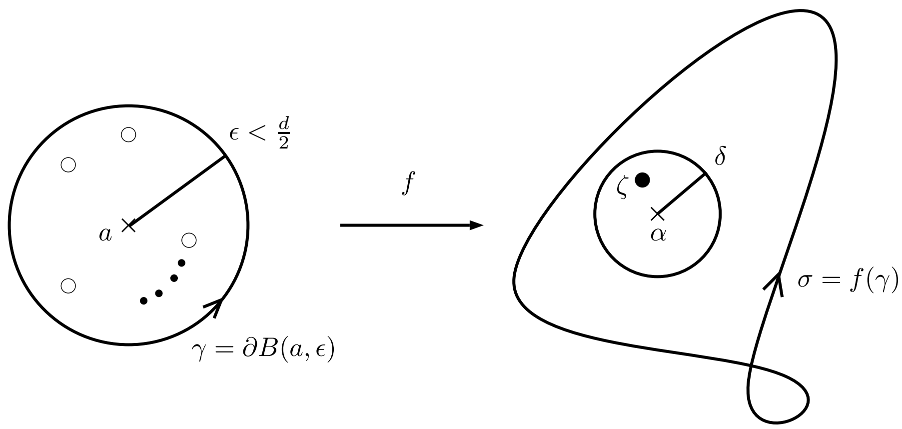

Figure 1.12: $f : G \to \mathbb{C}$, $f(a) = \alpha$

Proof. Let

$$d = \inf_{w \in \mathbb{C}\setminus G}\{|a - w|\}.$$

Since the zero a of $f - \alpha$ is isolated, we may choose $\epsilon < \dfrac{d}{2}$ such that $f(z) - \alpha \neq 0$ in $0 < |z - a| < \epsilon$. Then we have the representation

$$F(z) = f(z) - \alpha = (z - a)^m g(z)$$

over the disk $B(a, \epsilon)$, where g is analytic and $g \neq 0$ there.

Let γ be the boundary of $B(a, \epsilon)$, and write $\sigma = f(\gamma)$. Since $\mathbb{C} \setminus \sigma$ is open, we can find a component of $\mathbb{C} \setminus \sigma$ containing α, and a number $\delta > 0$ such that $B(\alpha, \delta)$ is a proper subset of this component.

Consider

$$\begin{aligned} n(\sigma; \alpha) &= \frac{1}{2\pi i} \int_\sigma \frac{dw}{w - \alpha} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{F'(\zeta)}{F(\zeta)}\, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{m}{\zeta - a}\, d\zeta + \frac{1}{2\pi i} \int_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta \\ &= m + 0 = m \end{aligned}$$

since γ is closed and $g \neq 0$ on $B(a, \epsilon)$. (So $\dfrac{g'}{g}$ has a primitive.)

According to Proposition 1.8.4, $n(\sigma; \zeta)$ is a constant on this component for each $\xi \in B(\alpha, \delta) \setminus \{\alpha\}$. Theorem 1.11.2 gives

$$\begin{aligned} n(\sigma; \xi) &= \frac{1}{2\pi i} \int_\sigma \frac{dw}{w - \xi} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - \xi}\, d\zeta \\ &= \sum_{k=1}^{n} n(\gamma; a_k) \end{aligned}$$

where a_k for $k = 1, \ldots, n$ are the zero of $f - \xi$ in $B(a, \epsilon)$. But γ is a circle, so $n(\gamma; a_k) = 1$ for $1 \leq k \leq n$. But then we must have $m = n$. Theorem 1.11.2 again implies that each of these zeros a_k is a simple root of $f - \xi$. This completes the proof. □

We deduce immediately the following important result.

Theorem 1.11.4 (Open Mapping Theorem)**.** *Let f be a non-constant analytic function defined on a region G. Then f is an* open mapping*, i.e. f maps open sets onto open sets.*

Proof. Suppose $U \subset G$ is open. To show $f(U)$ is open, it suffices to find a $\delta > 0$ for each $\xi \in f(U)$ such that $B(\xi, \delta) \subset f(U)$. But this follows easily from Theorem 1.11.3 that there exist ϵ, $\delta > 0$ such that $B(a, \epsilon) \subset U$, $B(\alpha, \delta) \subset f(B(a, \epsilon))$. In fact, only part of the conclusion in Theorem 1.11.3 is used. □

We now can give a second proof for the maximum modulus theorem.

Theorem 1.11.5 (Maximum Modulus Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ is analytic. If there exists a point a in G such that $|f(z)| \leq |f(a)|$ for all $z \in G$, then f is constant.*

Second proof (Topological argument). Suppose $\alpha \in f(G)$ and $f(a) = \alpha$, $a \in G$. Then we can find a $\delta > 0$ such that $B(\alpha, \delta) \subset f(G)$ by open mapping theorem. Hence there exist points in $B(\alpha, \delta)$ with modulus strictly longer than $|\alpha|$. Hence $\max |f(z)|$ cannot occur at an interior of G. □

We now consider the definition of an analytic function. Since the main result we use is Morera's Theorem, we could do this immediately after the proof of Morera's Theorem.

Recall that $f : G \to \mathbb{C}$ is *analytic* on G if f is continuously differentiable.

Theorem 1.11.6. *Let G be an open set and $f : G \to \mathbb{C}$ is differentiable. Then f must be analytic on G. That is, f is differentiable if and only if f is continuous differentiable.*

Proof. According to the statement of the theorem, it suffices to show f' is continuous. But by using Morera's Theorem, we can show that f is analytic directly. See, for examples, [1], [3], [4] for a proof of Goursat's Theorem. □

Remark. It follows from Theorem 1.11.6 that we could define analytic function simply that f is merely differentiable (without continuity) at each point of an open set G.

1.12 Isolated Singularities

We have proved that every zero of an analytic function must be isolated; and as indicated that this property is not shared by real functions. The next natural question is about the singularities of analytic functions, i.e., the nature of points a such that $f(a)$ undefined, such as $f(a) = \infty$. The following is a list of examples:

1.
$$\sqrt{z-1}$$
has a (square-root) branch point at $z = 1$.

2.
$$\ln(z-1)$$
has a logarithmic branch point at $z = 1$.

3.
$$e^{1/(z-1)}$$
has an essential singularity at $z = 1$ (see below).

4.
$$\tan[\ln(z-1)]$$
has a non-isolated essential singularity at $z = 1$ (see below).

We can deal with a small selection of singularities in this course. In the case where $f(a) = \infty$, the standard way to investigate the problem is to consider $F(z) = \dfrac{1}{f}$ at a i.e. $F(a) = \dfrac{1}{\infty} = 0$. Since any zeros are isolated, we may assume F has no zeros in $0 < |z-a| < \delta$ for some $\delta > 0$. So F has only one zero at a i.e. any singularities of f with $f(a) = \infty$ must be isolated (just like the zeros). It turns out that there are only a few types of singularities for analytic functions, and the easiest way to study them is by considering the power series expansions of the functions around the singularities.

Theorem 1.12.1 (Laurent Series, 1843)**.** *Let $f(z)$ be analytic function in an annulus $\Gamma(a; R_1, R_2) = \{z : R_1 < |z-a| < R_2\}$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

and the series converges uniformly in $\Gamma(a; R_1, R_2) = \{z : R_1 < |z-a| < R_2\}$. The coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

where γ is any circle in $\Gamma(a; R_1, R_2)$ centred at a, and for all integers n.

Proof. Let r_1 and r_2 be two real numbers such that $R_1 < r_1 < r_2 < R_2$, and σ be a straight line segment joining the boundary of $\Gamma(a; r_1, r_2)$ and passing through a. Let $\gamma_1(t) = a + r_1 e^{it}$, and $\gamma_2(t) = a + r_2 e^{it}$ for $t \in [0, 2\pi]$, then any closed curve inside $\gamma := \gamma_2 + \sigma - \gamma_1 - \sigma$ is ~ 0. By Cauchy's formula we obtain, for $z \in \Gamma(a; r_1, r_2)$,

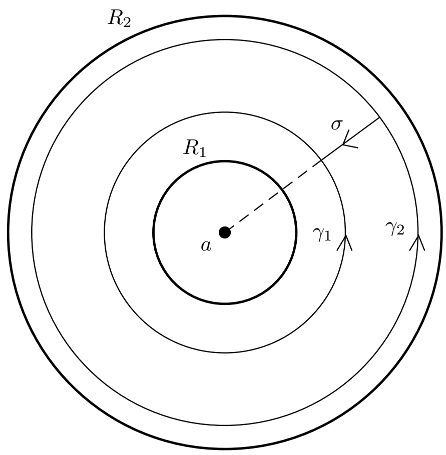

Figure 1.13: $\gamma := \gamma_2 + \sigma - \gamma_1 - \sigma$

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_2} + \int_\sigma - \int_{\gamma_1} - \int_\sigma\right)\frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(z - a)\left(1 - \frac{\zeta-a}{z-a}\right)}\,d\zeta \\
&= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \\
&\qquad + \sum_{n=0}^{\infty}(z-a)^{-n+1}\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta-a)^n\,d\zeta \\
&= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{-\infty}^{-1}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta-a)^{-n-1}\,d\zeta \\
&= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{-\infty}^{-1} a_n(z-a)^n \quad \text{(uniform convergence)}
\end{aligned}
$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \quad \text{for } n \geq 0$$

and

$$a_n = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \quad \text{for } n \leq -1.$$

Let $\gamma = a + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r_1 < r < r_2 < R_2$. By constructing suitable contours involving γ, we may bring the above two line integrals over γ_2 and γ_1 respectively to the common curve γ. Thus we obtain the formula for a_n as stated in the theorem. □

Remark. We remark that Laurent expansion of an analytic function in a punctured disk gives a beautiful generalization of Taylor expansion of analytic function.

Looking at the Laurent expansion of the functions in the above theorem, there are several possibilities:

(i) $a_k = 0$ for all $k \leq -n$ for some integer $n > 0$; the point a is called *a pole of order* n;

(ii) there are infinitely many $a_k \neq 0$, $k \leq -1$; the point a is called *an essential singularity* of f at a;

(iii) $a_k = 0$ for all $k \leq -1$, then a is called a *removable singularity* of f at a.

- If f has a pole of order n, then

$$f(z) = \sum_{k=1}^{n} \frac{a_k}{(z-a)^k} + \sum_{k=0}^{\infty} a_k (z-a)^k$$

  where the sum $\sum_{k=1}^{n} a_k/(z-a)^k$ is called the *principal part of* f *at* a, and $|f| \to \infty$ in the manner of $O(|z-a|^{-n})$ as $z \to a$.

- If f has a removable singularity at a, then $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$ in $0 < |z-a| < \delta$ (some $\delta > 0$). But we clearly have $f \to a_0$ as $z \to a$, thus we may define a new function at a by $g(z) = f(z)$ for $0 < |z-a| < \delta$ and $g(z) = a_0$ at $z = a$. Then g is an analytic function in $|z-a| < \delta$. Thus f is almost analytic at a if it has a removable singularity at a and so from this point of view, this case is less interesting.

We shall discuss the implication of pole later. The behaviour of f near an essential singularity is very different. It is *not* true that $|f| \to \infty$ as $z \to a$.

Example 1.12.2. 1. The $\sin z/z$ has a removable singularity at $z = 0$.

2. The Euler-Gamma function $\Gamma(z)$ has simple poles at each of negative integers (see a later chapter).

3. The Weierstrass function $\wp(z)$ has double poles at the vertices of its fundamental period parallelograms (see a later chapter).

4. The $e^{1/z}$, $\sin\big(1/z\big)$ and $\cos(1/z)$ all have an essential singularity at $z = 0$.

5. Show the following Laurent expansion

$$e^{\frac{1}{2}(z-1/z)} = \sum_{-\infty}^{\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\theta - \sin\theta)\, d\theta.$$

Theorem 1.12.3 (Casorita-Sokhotskii-Weierstrass-1864)**.** *Suppose f has an essential singularity at a. Then for every $\delta > 0$, $\overline{f(\Gamma(a;0,\delta))} = \mathbb{C}$.*

The statement of this theorem is equivalent to given any ρ, $\epsilon > 0$ and any $c \in \mathbb{C}$, there is a point z inside $0 < |z-a| < \rho$ in which $|f(z) - c| < \epsilon$. That is to say, given any c, f tends to c as the limit as z tends to a through a suitable sequence of complex numbers.

Proof. We first show that f is unbounded on any punctured disks $\Gamma(a;0,\delta)$.

Suppose $|f(z)| \le M$ for all $z \in \Gamma(a;0,\delta)$. Let $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$, then

$$\begin{aligned}
|a_n| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta\right| \quad \text{for } n \le -1\\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(a+Re^{it})}{(Re^{it})^{n+1}} iRe^{it}\, dt\right|\\
&\le MR^{-n}\\
&\to 0 \quad \text{as } R \to 0.
\end{aligned}$$

Hence $a_n = 0$ for all $n \leq -1$ and f has a removable singularity at most. A contradiction.

Let us now assume that $\delta > 0$ is chosen so small that $f - c$ has no zero in $\Gamma(a; 0, \delta)$. Then the function $\phi(z) = \dfrac{1}{f-c}$ is analytic in $\Gamma(a; 0, \delta)$. We claim that ϕ has an essential singularity at a. For if ϕ has a pole at a, then $f = \dfrac{1}{\phi} + c$ would be analytic at a; while if ϕ has a removable singularity, then f either has pole or analytic at a. This is a contradiction.

We now apply the result obtained above to ϕ i.e. ϕ is unbounded on $\Gamma(a; o, \delta)$, so $|f - c| = 0$ on $\Gamma(a; 0, \delta)$. That is, given $\varepsilon > 0$, there exists $z \in \Gamma(a; 0, \delta)$ such that

$$|\phi(z)| > \ 1/\varepsilon,$$

i.e.,

$$|f(z) - c| = |1/\phi(z)| < \varepsilon.$$

So we could find a sequence $\varepsilon_n = 1/n$ and $\{\delta_n\}$ such that $\delta_n \to 0$ and $z_n \in \Gamma(a; 0, \delta_n)$ so that $z_n \to a$ for and $f(z_n) \to c$. This completes the proof. □

1.13 Rouché's theorem

This is an application of the argument principle discussed earlier.

Theorem 1.13.1 (E. Rouché)**.** *Let $f(z)$ and $g(z)$ be analytic in the domain D containing the closed, piece-wise smooth curve γ. Suppose*

$$|f(z)| > |g(z)|, \qquad \textit{for all} \ \ z \in \gamma.$$

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicity, in the domain enclosed by γ.

Proof. It is evident from the assumption that $|f(z)| > |g(z)|$, for all $z \in \gamma$ that both $f(z)$ and $f(z) + g(z)$ do not have zeros on γ.
The argument principle assets that

$$\begin{aligned}\Delta_\gamma \arg\big(f(z) + g(z)\big) &= \Delta_\gamma \arg\Big[f(z)\Big(1 + \frac{g(z)}{f(z)}\Big)\Big] \\ &= \Delta_\gamma \arg f(z) + \Delta_\gamma \arg\Big(1 + \frac{g(z)}{f(z)}\Big).\end{aligned}$$

But since

$$1 > \left|\frac{g(z)}{f(z)}\right| = \left|\Big(\frac{g(z)}{f(z)} + 1\Big) - 1\right|,$$

on γ. It follows that $1 + \frac{g(z)}{f(z)}$ can never circle around $w = 0$. Hence

$$\Delta_\gamma \arg(f + g) = \Delta_\gamma \arg f(z) + 0.$$

Thus

$$N_{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{(f + g)'(z)}{f(z) + g(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N_f$$

inside γ, as required. □

Example 1.13.2. If $f(z)$ has zero of order two at a, and a pole of order 3 at b, where both a and b are inside γ, then

$$\Delta_\gamma \arg f(z) = 2\pi(2 - 3) = -2\pi.$$

Example 1.13.3. Determine the number of roots of

$$z^7 - 4z^3 + z - 1 = 0$$

in $|z| < 1$.
On $|z| = 1$, we write

$$f(z) = -4z^3, \qquad g(z) = z^7 + z - 1.$$

Then $|f(z)| = 4$ and $|g(z)| \leq |z|^7 + |z| + 1 = 3$ Hence $|f(z)| > |g(z)|$ on $|z| = 1$. Thus Rouché's theorem asserts that $f + g$ has the same number of zeros as that of $f = -4z^3$ in $|z| < 1$. Thus there are 3 zeros inside $|z| < 1$.

Exercise 1.13.1. Prove the open mapping theorem for analytic function by applying Rouché's theorem.

See next chapter for an hint.

Chapter 2

Conformal mappings

2.1 Stereographic Projection

One known problem with numbers in the complex plane $\mathbb{C} = \{(x,\, y) : -\infty < x,\, y < +\infty\}$ do not have an ordering like the real numbers on the real-axis $\mathbb{R}$. Riemann's (1826-1866) idea is to add an ideal point, denoted by ∞, to $\mathbb{C}$ to obtain an extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This construction can get around the problem of ordering. The resulting $\hat{\mathbb{C}}$ is compact which can be vasualised by the following construction.

We show that there is an one-to-one correspondence between

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The S is called the Riemann sphere.

Let $N = (0, 0, 1)$ and $z \in \mathbb{C}$. If we join the straight line between N and z, the straight line intersects the sphere S at $Z = (x_1, x_2, x_3)$ say. The construction clearly exhibits an one-to-one correspondence between $S \setminus \{N\}$ and $\mathbb{C}$. Note that $Z \to N$ as $|z| \to \infty$. We may associate N with ∞ and obtain the bijection between S and $\widehat{\mathbb{C}}$. This is known as the Stereographic projection.

Suppose $P(x_1, x_2, x_3) = Z \in S$ associates with $z = (x,\, y) \in \widehat{\mathbb{C}}$. Then we may associate z the notation P with coordinate $(x,\, y,\, 0)$.

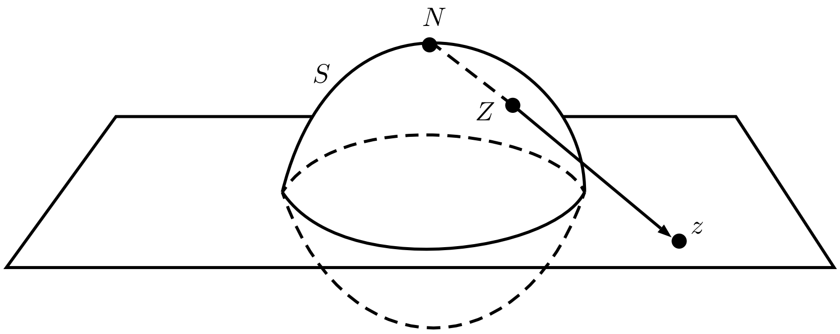

Figure 2.1: Riemann sphere

Then we have, by considering similar triangles formed by the line segment NP and projecting onto the $x-$, $y-$ and $z-$axes respectively,

$$\frac{|NP|}{|NZ|} = \frac{x}{x_1} = \frac{y}{x_2} = \frac{1}{1-x_3}, \tag{2.1}$$

so that

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}.$$

Then

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3},$$

hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Then

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2}, \qquad x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)}.$$

This clearly shows a one-one correspondence between $S\backslash(0, 0, 1)$ and $\mathbb{C}$ with the $N = (0, 0, 1)$ corresponds to ∞. We also note that the upper hemisphere where $x_3 > 0$ corresponds to $|z| > 1$ and the lower hemisphere of S corresponds to $|z| < 1$. An advantage with this Riemann sphere model is that it puts all complex numbers including '∞' in equal footing since any number can be rotated to N and vice-verse.

From a geometrical viewpoint, it is evident that every (infinite) straight line in the $z-$plane is transformed into a circle on S that passes through the North pole N, and conversely. Hence, every circle (straight line included) on the $z-$plane corresponds to a circle/straight line on S.

Theorem 2.1.1. *A circle on the Riemann sphere is mapped under the Stereographic projection into a circle (including a straight line) of the* $\mathbb{C}$*, and conversely.*

Proof. Show that

1. a circle equation that lies on the Riemann sphere is an equation of the form
$$ax_1 + bx_2 + cx_3 = d$$
subject to $0 \leq c < 1$ and $a^2 + b^2 + c^2 = 1$ (this is the intersection of the plane and the unit sphere).

2. the above equation can be rewritten in the form
$$a(z + \bar{z}) - ib(z - \bar{z}) + c(|z|^2 - 1) = d(|z|^2 + 1)$$

3. the above equation can be further rewritten into the form
$$(d - c)(x^2 + y^2) - 2ax - 2by + d + c = 0,$$

which is clearly a circle equation in the $\mathbb{C}$ and it becomes a straight line. equation if and only if $c = d$. □

That is, a circle on the Riemann sphere S corresponds to either a circle or a straight line on $\mathbb{C}$. In the case the circle on S passes through the North pole $N = (0, 0, 1)$, then the corresponding straight line (also considered as an unbounded circle passes through) to ∞.

Exercise 2.1.1. Show that if z and w are two points in $\mathbb{C}$ so that their images lie on two diametrically opposite points on the Riemann sphere, then
$$w\bar{z} + 1 = 0.$$

Theorem 2.1.2. *The stereographic projection is isogonal (i.e., the mapping preserves angles).*

Proof. The statement of the theorem means that the tangents of two curves in the $\mathbb{C}$ intersect at point z_0 is equal to the angle made by two tangents at the corresponding intersection point of two image curves on the Riemann sphere. We shall make two assumptions:

1. that the Stereographic projection preserves tangents. We skip the detail verification of this fact. But this is not difficult to see since the Stereographic projection is a smooth map,

2. that without loss of generality that the two curves in $\mathbb{C}$ are (infinite) straight lines.

Suppose the two straight line equations are given by

$$\begin{aligned} a_1x + a_2y + a_3 &= 0 \quad (x_3 = 0); \\ b_1x + b_2y + b_3 &= 0. \quad (x_3 = 0) \end{aligned} \tag{2.2}$$

It follows from (2.1) that the two plane equations become respectively,

$$\begin{aligned} a_1X_1 + a_2X_2 + a_3(X_3 - 1) &= 0; \\ b_1X_1 + b_2X_2 + b_3(X_3 - 1) &= 0. \end{aligned}$$

In the limiting case when $X_3 = 1$, we have the two tangent plane equations

$$\begin{aligned} a_1X_1 + a_2X_2 &= 0; \\ b_1X_1 + b_2X_2 &= 0. \end{aligned} \tag{2.3}$$

at $N(0, 0.1)$ parallel to the $\mathbb{C}$. Clearly the angle between the two curves in (2.2) is the same angle between the two lines in (2.3).

Note that any two intersecting circles in general positions on S can be rotated so that the intersection point passes through the North pole N. This consideration takes care of the preservation of the angle of intersection of two curves in general position in $\mathbb{C}$ under the Stereographic projection. □

Theorem 2.1.3. *Let z_1, z_2 be two points in $\mathbb{C}$ and Z_1, Z_2 be their images on the Riemann sphere S under the Stereographic projection. We denote $a(Z_1, Z_2)$ to be arc length between Z_1 and Z_2. Then*

$$\lim_{z_2\to z_1} \frac{a(Z_1, Z_2)}{|z_1 - z_2|} = \frac{2}{1+|z_1|^2}. \tag{2.4}$$

That is, the ratio depends on position only. So the Stereographic projection is called a pure magnification.

We easily deduce from the above theorem that

Theorem 2.1.4. *Let $C = \{z = z(s) : \ 0 \le s \le L\}$ be a piecewise smooth curve in $\mathbb{C}$. Let Γ be the image curve of C on the Riemann sphere under the Stereographic projection. Then the length $\ell(\Gamma)$ of Γ is given by*

$$\ell(\Gamma) = \int_0^L \frac{2|dz(s)|}{1+|dz(s)|^2}.$$

Let $d(Z_1, Z_2)$ denote the *chordal distance* between Z_1 and Z_2 on S. We also write

$$\chi(z_1, z_2) := d(Z_1, Z_2).$$

where z_1, z_2 are the corresponding points in $\mathbb{C}$.

Theorem 2.1.5. *Let $z_1, z_2 \in \mathbb{C}$. Then*

$$\chi(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}. \tag{2.5}$$

Since

$$\chi(z_1, z_2) := d(Z_1, Z_2) \approx a(Z_1, Z_2)$$

as $z_1 \to z_2$. So the Theorem 2.1.3 follows from the equation (2.5) in the limit $z_2 \to z_1$.

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ and none equal to ∞. We construct a plane passing through the following three points:

$$(0, 0, 1), \quad (x_1, y_1, 0), \quad (x_2, y_2, 0).$$

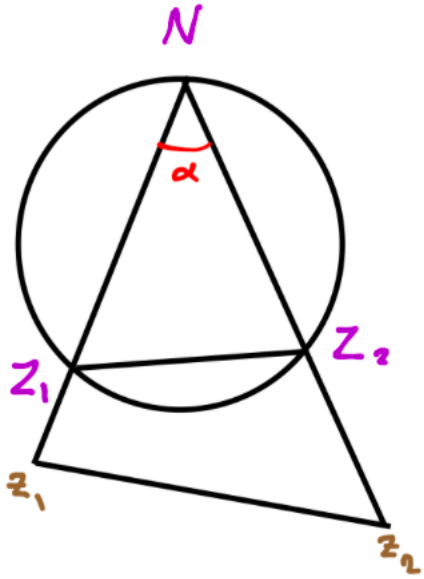

Figure 2.2: Riemann sphere slide

Then we have the above figure.

We deduce from the Riemann sphere S that

$$d(N,\, z_1) = \sqrt{1+|z_1|^2}, \qquad d(N,\, z_2) = \sqrt{1+|z_2|^2}.$$

One can see from similar triangles consideration on the Riemman sphere S that

$$\frac{x_1}{x} = \frac{1-x_3}{1} = \frac{x_2}{y}.$$

Hence

$$1+|z|^2 = 1+x^2+y^2 = 1+\frac{x_1^2}{(1-x_3)^2}+\frac{x_2^2}{(1-x_3)^2}$$
$$= \frac{2(1-x_3)}{(1-x_3)^2} = \frac{2}{1-x_3}.$$

and

$$\frac{d(N,\, Z)}{d(N,\, z)} = \frac{1-x_3}{1} = \frac{2}{1+|z|^2}.$$

holds. This gives raise to

$$d(N, Z_1) = \frac{2}{\sqrt{1+|z_1|^2}}, \qquad d(N, Z_2) = \frac{2}{\sqrt{1+|z_2|^2}}.$$

We conclude that

$$d(N, z_1)d(N, Z_1) = 2 = d(N, z_2)d(N, Z_2).$$

Hence the triangles $\Delta N z_1 z_2$ and $\Delta N Z_1 Z_2$ are similar. Hence

$$\frac{d(Z_1, Z_2)}{d(z_1, z_2)} = \frac{d(N, Z_2)}{d(N, z_1)}.$$

It follows from the above consideration that

$$d(Z_1, Z_2) = d(z_1, z_2) \cdot \frac{d(N, Z_2)}{d(N, z_1)} = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

as required. □

We are ready to prove Theorem 2.1.3.
We observe the relation

$$\frac{d(Z_1, Z_2)}{a(Z_1, Z_2)} = \frac{\sin\alpha}{\alpha},$$

holds, where α is the angle between the line segments NZ_1 and NZ_2 from the above figure. Hence

$$\frac{a(Z_1, Z_2)}{|z_1 - z_2|} = \frac{d(Z_1, Z_2)}{|z_1 - z_2|} \approx \frac{\chi(z_1, z_2)}{|z_1 - z_2|} \to \frac{2}{1+|z_1|^2}$$

as $z_2 \to z_1$.
We also note that

$$\chi(z_1, \infty) = \lim_{z_2 \to \infty} \chi(z_1, z_2) = \frac{2}{\sqrt{1+|z_1|^2}},$$

which follows from the Riemann sphere (geometric) or the Theorem 2.1.5 (algebraic) considerations. Thus we define the chordal distance to be

$$\chi(z,z') = \begin{cases} \dfrac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}, & z,z'\in\mathbb{C} \\ \dfrac{2}{\sqrt{1+|z|^2}}, & z'=\infty. \end{cases}$$

Alternative derivation

of the chordal distance. Suppose $(x_1, x_2, x_3) \in S$ associates with $z = (x,\, y) \in \widehat{\mathbb{C}}$ and $(x_1', x_2', x_3') \in S$ associates with $z' \in \widehat{\mathbb{C}}$.

Then the distance or the length of the chord joining (x_1, x_2, x_3) and (x_1', x_2', x_3') on S is given by

$$\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}.$$

On the other hand,

$$(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2 = 2-2(x_1x_1'+x_2x_2'+x_3x_3').$$

Exercise 2.1.2. Show that

$$\begin{aligned} x_1x_1' &+ x_2x_2' + x_3x_3' \\ &= \frac{(z+\bar z)(z'+\bar z')-(z-\bar z)(z'-\bar z')+(|z|^2-1)(|z'|-1)}{(1+|z|^2)(1+|z'|^2)} \\ &= \frac{(1+|z|^2)(1+|z'|^2)-2|z-z'|^2}{(1+|z|^2)(1+|z'|^2)} \end{aligned}$$

Exercise 2.1.3. Verify the formaula for chordal distance using the above formala.

Exercise 2.1.4. Verify that $\chi(z_1,\, z_2) = \chi(\bar z_1,\, \bar z_2) = \chi(1/z_1,\, 1/z_2)$.

Exercise 2.1.5. Describe a $\varepsilon-$neighbourhood of a pont z_0 in the chordal metric.

Metric space

The chordal distance $\chi(z_1, z_2)$ defines a metric on $\hat{\mathbb{C}}$. This is because

1. $\chi(z_1, z_2) \geq 0$ and with equality if and only if $z_1 = z_2$;
2. $\chi(z_1, z_2) = \chi(z_2, z_1)$;
3. $\chi(z_1, z_3) \leq \chi(z_1, z_2) + \chi(z_2, z_3)$,

where the third item follows from

Exercise 2.1.6. Let a, b, $c \in \mathbb{C}$. Then

$$(a - b)(1 - \bar{c}c) = (a - c)(1 + \bar{c}b) + (c - b)(1 + \bar{c}a).$$

Exercise 2.1.7. Show that the above metric space is complete.

2.2 Analyticity revisited

Local properties of one-one analytic functions

We recall that if $f : E \to P$ and there correspond only one point in E for every point in P under this f, then we say the map f is **injective**. This defines a function g on P, denoted by $z = g(w)$, called the **inverse function or inverse mapping** of f. In particular, we see that $g[f(z)] = z$.

Let $w = f(z) = u(x, y) + iv(x, y)$. Then one can view f as a mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

What is a criterion that guarantee the existence of an inverse mapping for the above mapping?

Standard material from calculus courses asserts that if

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \neq 0, \qquad \text{at } z_0 = (x_0,\, y_0),$$

then the Implicit function theorem asserts that an inverse function of f exists there. That is, if the *Jacobian* is non-zero at z_0. But then the Cauchy-Riemann equations give

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |f'(z_0)|^2.$$

This leads to the following statement.

Theorem 2.2.1. *Let $f(z)$ be an analytic function on a domain D such that $f'(z_0) \neq 0$. Then there is an analytic function $g(w)$ defined in a neighbourhood $N(w_0)$ of $w_0 = f(z_0)$ such that $g(f(z)) = z$ throughout this neighbourhood.*

Proof. Since

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = |f'(z_0)|^2 \neq 0,$$

so the Implicit Function theorem asserts that is a neighbourhood $N(w_0)$ of $w_0 = f(z_0)$ in which f has a local inverse at w_0

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} x(u,\, v) \\ y(u,\, v) \end{pmatrix}.$$

Moreover, the analytic Implicit Function theorem asserts that the stronger conclusion that since f is analytic at z_0 so the $g(w)$ is analytic at w_0. □

We prove that a strong form of converse of the above statement also holds. Please note we could apply the Theorem 1.11.3 to prove the theorem. But we prefer to apply the Rouché theorem instead.

Theorem 2.2.2. *Let $f(z)$ be an one-one analytic function on a domain D. Then $f'(z) \neq 0$ on D.*

Proof. We suppose on the contrary that $f'(z_0) = 0$ for some z_0 and we write $f(z_0) = w_0$. We first notice that $f'(z) \not\equiv 0$. For otherwise, $f(z)$ is identically a constant, contradicting to the assumption that $f(z)$ is one-one on D.
Since the zeros of $f'(z)$ are isolated , so there is a $\rho > 0$ such that $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$. Because of the assumption that f is one-one, so

$$f(z) \neq f(z_0) \quad \text{on} \quad |z - z_0| = \rho.$$

On the other hand, $|f(z)|$ is continuous on the compact set $|z - z_0| = \rho$ so that we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| \geq \delta > 0 \quad \text{on} \quad |z - z_0| = \rho.$$

Let w' be an arbitrary point in $\{w : 0 < |w' - w_0| < \delta\}$. Then the inequality

$$|f(z) - w_0| \geq \delta > |w' - w_0|$$

holds, so that the Rouché theorem again implies that the function $f(z) - f(z_0) = f(z) - w_0$ and the function

$$[f(z) - f(z_0)] + [f(z_0) - w'] = f(z) - w'$$

have the same number of zeros inside $\{z : |z - z_0| < \rho\}$. But $f'(z_0) = 0$ so $f(z) - f(z_0)$ has at least two zeros (counting multiplicity). Hence $f(z) - w'$ also has at least two zeros (counting multiplicity) in $\{z : |z - z_0| < \rho\}$. But $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$, so there are at least two different zeros z_1 and z_2 in $\{z : |z - z_0| < \rho\}$ so that $f(z_1) = w'$ and $f(z_2) = w'$, thus contradicting to the assumption that $f(z)$ is one-one. □

2.3 Angle preserving mappings

We consider geometric properties of an analytic function $f(z)$ at z_0 such that $f'(z_0) \neq 0$. Let $\gamma = \{\gamma(t): \ a \leq t \leq b\}$ a piece-wise smooth path such that $z_0 = \gamma(t_0)$ where $a \leq t_0 \leq b$ and $z'(t_0) \neq 0$, and

$$\Gamma := \{w = f(z(t)): \ a \leq t \leq b\}.$$

That is, $\Gamma = f(\gamma)$.
It is clear that the assumption $z'(t_0) \neq 0$ above means that the path γ must have a tangent at t_0. Thus,

$$\begin{aligned} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0} &= \left.\frac{df(z)}{dz}\right|_{z=z_0} \cdot \left.\frac{dz}{dt}\right|_{t=t_0} \\ &= f'(z_0) \cdot z'(t_0) \neq 0 \end{aligned}$$

since $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$. We deduce

$$\operatorname{Arg} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0} = \operatorname{Arg} \left.\frac{df(z)}{dz}\right|_{z=z_0} + \operatorname{Arg} \left.\frac{dz}{dt}\right|_{t=t_0}.$$

Let $\theta_0 = z'(t_0)$ denote the inclination angle of the tangent to γ at z_0 and positive real axis, and let $\varphi_0 := \operatorname{Arg} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0}$ denote the inclination angle of the tangent to Γ at $w_0 = f(z_0)$. Thus

$$\operatorname{Arg} f'(z_0) = \varphi_0 - \theta_0.$$

Now let $\gamma_1(t): \ z_1(t): \ a \leq t \leq b$ and $\gamma_2(t): \ z_2(t): \ a \leq t \leq b$ be two paths such that they intersect at z_0. Then

$$\varphi_1 - \theta_1 = \operatorname{Arg} f'(z_0) = \varphi_2 - \theta_2.$$

That is,

$$\varphi_2 - \varphi_1 = \theta_2 - \theta_1.$$

This shows that the difference of tangents of $\Gamma_2 = f(\gamma_2)$ and $\Gamma_1 = f(\gamma_1)$ at w_0 is equal to difference of tangents of γ_2 and γ_1 at z_0.

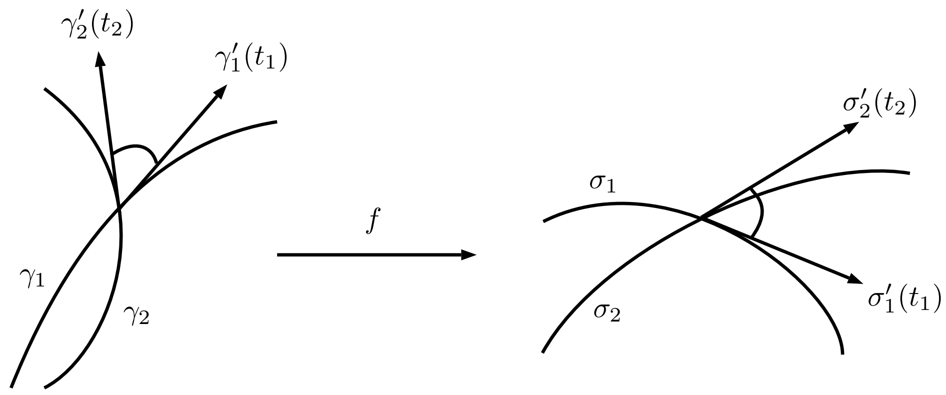

Figure 2.3: Conformal map at z_0

Definition 2.3.1. *An analytic $f : D \to \mathbb{C}$ is called* ***conformal at*** z_0 *if $f'(z_0) \neq 0$. f is called* ***conformal in*** D *if f is conformal at each point of the domain D.*

We call $|f'(z_0)|$ the **scale factor** of f at z_0.

Theorem 2.3.2. *Let $f(z)$ be analytic at z_0 and that $f'(z_0) \neq 0$. Then*

1. *$f(z)$ preserves angles (i.e.,* isogonal*) and its sense at z_0;*
2. *$f(z)$ preserves scale factor, i.e., a pure magnification at z_0 in the sense that it is* independent of directions of approach *to z_0.*

We consider a converse to the above statement.

Theorem 2.3.3. *Let $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined in a domain D with continuous u_x, u_y, v_x, v_y such that they do not vanish simultaneously. If either*

1. *f is isogonal (preserve angles) at every point in D,*
2. or *f is a pure magnification at each point in D,*

then either f or $\bar{f}$ is analytic in D.

Proof. Let $z = z(t)$ be a path passing through the point $z_0 = z(t_0)$ in D. We write $w(t) = f(z(t))$. Then

$$w'(t_0) = \frac{\partial f}{\partial x}\, x'(t_0) + \frac{\partial f}{\partial y}\, y'(t_0),$$

That is,

$$w'(t_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) z'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \overline{z'(t_0)}. \tag{2.6}$$

That is,

$$\frac{w'(t_0)}{z'(t_0)} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \frac{\overline{z'(t_0)}}{z'(t_0)}$$

where we have adopted new notation

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

If f is isogonal, then the $\arg \frac{w'(t_0)}{z'(t_0)}$ is independent of $\arg z'(t_0)$ in the above expression. This renders the expression (2.6) to be independent of $\arg z'(t_0)$. Therefore, the only way for this to hold in(2.6) is that

$$0 = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

which represent the validity of the Cauchy-Riemann equations at z_0. Thus f is analytic at z_0. This establishes the first part.

We note that the right-hand side of (2.6) represents a circle of radius

$$\Big|\frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)\Big|$$

centered at $\partial f/\partial z$. Suppose now that we assume that f is a pure magnification. Then the (2.6) representation this circle must either have its radius vanishes which recovers the Cauchy-Riemann equations, or the centre is at the origin, i.e.,

$$0 = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

or the equivalently $\overline{f(z)}$ is analytic at z_0 and hence over D. □

Remark. If $\overline{f(z)}$ is analytic at z_0, then it means that f preserves the size of the angle but reverse its sense.

Example 2.3.4. Consider $w = f(z) = e^z$ on $\mathbb{C}$. Clearly $f'(z) = e^z \neq 0$ so that the exponential function is conformal throughout $\mathbb{C}$. Observe

$$w = e^z = e^x + e^{iy} := Re^{i\phi},$$

so that the line $x = a$ in the is mapped onto the circle $R = e^a$ in the $w-$plane, while the horizontal line $y = b$ $(-\infty < x < \infty)$ is mapped to the line $\{Re^{ib} : 0 < R < +\infty\}$. One sees that the lines $x = a$ and $y = b$ are at right-angle to each other. Their images, namely the concentric circles centred at the origin and infinite ray at angle b from the $x-$axis from the origin are also at right angle at each other. The infinite horizontal strip

$$G = \{z = x + iy : \ |y| < \pi, \ -\infty < x < \infty\}$$

is being mapped onto the slit-plane $\mathbb{C}\backslash\{z : \ z \leq 0\}$. Moreover, the image of any vertical shift of G by integral multiple of 2π under f will cover the slit-plane again. So the $f(\mathbb{C})$ will cover the slit-plane an infinite number of times.

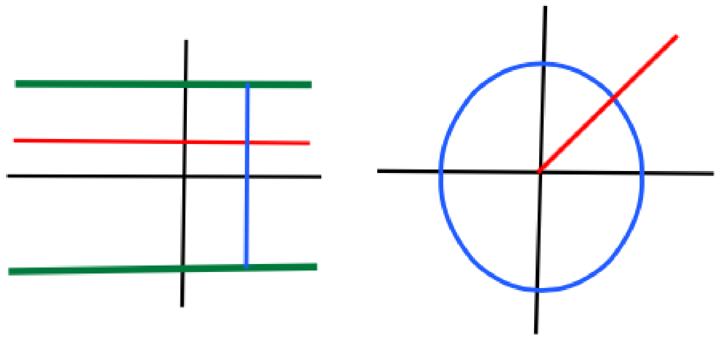

Figure 2.4: Exponential map

2.4 Möbius transformations

We study mappings initiated by A. F. Möbius (1790–1868) on the $\mathbb{C}$ that map $\mathbb{C}$ to $\mathbb{C}$ or even between $\hat{\mathbb{C}}$. Möbious considered

The mapping

$$w = f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0$$

is called a **Möbius transformation**, a **linear fractional transformation**, a **homographic transformation**. In the case when $c = 0$, then a Möbious transformation reduces to a linear function $f(z) = az + b$ which is a combination of a translation $f(z) = z + b$ and a rotation/magnification $f(z) = az$. If $ad - bc = 0$, then the mapping degenerates into a constant.

We recall that a function f having a pole of order m at z_0 is equivalent to $1/f$ to have a zero of order m at z_0 . Similarly, a function have a pole of order m at ∞ means that $1/f(\frac{1}{z})$ to have a zero of order m at $z = 0$.

The mapping w is defined on $\mathbb{C}$ except at $z = -d/c$, where $f(x)$ has a simple pole. On the other hand,

$$f(1/\zeta) = \frac{a/\zeta + b}{c/\zeta + d} = \frac{a + b\zeta}{c + d\zeta} = \frac{a}{c}$$

when $\zeta = 0$. That is, $f(\infty) = a/c$. So $f(z)$ is a one-one map between $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. One can easily check that the inverse f^{-1} of f is given by

$$f^{-1}(w) = -\frac{wd - b}{cw - a}, \quad w \neq \frac{a}{c}.$$

Thus $f^{-1} : \frac{a}{c} \mapsto \infty,\ \infty \mapsto -\frac{d}{c}$ (Since

$$f^{-1}\Big(\frac{1}{\eta}\Big) = -\frac{d/\eta - b}{c/\eta - a} = -\frac{d - b\eta}{c - a\eta} = -\frac{d}{c}$$

as $\eta = 0$. Thus $f^{-1}(\infty) = -\frac{d}{c}$. Similarly, since

$$\frac{1}{f^{-1}(w)}\Big|_{a/c} = -\frac{cw - a}{dw - b}\Big|_{w=a/c} = 0.$$

Thus $f^{-1}\Big(\frac{a}{c}\Big) = \infty$.)

Theorem 2.4.1. *The above Möbius map is conformal on the Riemann sphere.*

Proof. Let $c \neq 0$. Then

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

whenever $z \neq -\frac{d}{c}$. Hence $f(z)$ is conformal at every point except perhaps when $z = -d/c$ where f has a simple pole. So we should check if $\frac{1}{f(z)}$ is conformal at $z = -d/c$. But

$$\begin{aligned}\Big(\frac{1}{f(z)}\Big)'\Big|_{z=-d/c} &= -\frac{f'(z)}{f(z)^2}\Big|_{-d/c} = \frac{ad - bc}{(cz + d)^2} \times \Big(\frac{cz + d}{az + b}\Big)^2 \\ &= -\frac{ad - bc}{(az + b)^2}\Big|_{-d/c} = -\frac{(ad - bc)c^2}{(ad - bc)^2} = \frac{-c^2}{ad - bc} \neq 0.\end{aligned}$$

Hence f is conformal at $-d/c$, whenever $c \neq 0$.

Similarly, in order to check if f is conformal at ∞, we consider, when $c \neq 0$

$$\Big(f\Big(\frac{1}{\zeta}\Big)\Big)' = \Big(\frac{a+b\zeta}{c+d\zeta}\Big)' = \frac{bc-ad}{(c+d\zeta)^2} = \frac{bc-ad}{c^2} \neq 0$$

when $\zeta = 0$ and whenever $c \neq 0$. Hence f is conformal at ∞ if $c \neq 0$.

If $c = 0$, then we consider $f(z) = \dfrac{az+b}{d} = \alpha z + \beta$ instead. Since $f'(z) = \alpha \neq 0$ for all $z \in \mathbb{C}$, so f is conformal everywhere. It remains to consider

$$\frac{1}{f\big(1/\zeta\big)} = \frac{1}{\alpha/\zeta + \beta} = \frac{\zeta}{\alpha + \beta\zeta}.$$

Hence $f(\infty) = \infty$. We now consider the conformality at ∞:

$$\Big(\frac{1}{f(1/\zeta)}\Big)'_{\zeta=0} = \frac{\alpha}{(\alpha+\beta\zeta)^2}\Big|_{\zeta=0} = \frac{1}{\alpha} \neq 0,$$

as required. □

Exercise 2.4.1. Complete the above proof by considering the case when $c = 0$.

Exercise 2.4.2. Show that

1. the composition of two Möbius transformations is still a Möbius transformation.
2. For each Möbius transformation f, there is an inverse f^{-1}.
3. If we denote I be the identity map, then show that the set of all Möbius transformations M forms a group under composition.

Theorem 2.4.2. *Let* $w = f(z) = \dfrac{az+b}{cz+d}$*. Then* $f(z)$ *maps any circle in the* $z-$*plane to a circle in the* $w-$*plane.*

Remark. *We regard any straight lines to be circles having infinite radii* $(+\infty)$*.*

Proof. We note that any $\dfrac{az+b}{cz+d}$ can be written as

$$\begin{aligned} w =& \frac{a}{c}\Big[\frac{z+b/a}{z+d/c}\Big] = \frac{a}{c}\Big[1 + \frac{b/a - d/c}{z+d/c}\Big] \\ =& \frac{a}{c}\Big[1 + \Big(\frac{bc/a - d}{1}\Big)\frac{1}{cz+d}\Big] \\ =& \frac{a}{c} + \Big(\frac{bc-ad}{c}\Big)\Big(\frac{1}{cz+d}\Big), \end{aligned}$$

Showing that w can be decomposed by transformations of the basic types:

1. $w = z + b$ (translation),
2. $w = e^{i\theta_0} z$ (rotation),
3. $w = kz$ ($k > 0$, scaling),
4. $w = 1/z$ (inversion).

In fact, we can write the $T(z)$ as a compositions of four consecutive mappings in the forms

$$w_1 = cz + d, \qquad w_2 = \frac{1}{w_1}, \qquad w_3 = \Big(\frac{bc-ad}{c}\Big)w_2, \quad w_4 = \frac{a}{c} + w_3,$$

From the geometric view point, the translation $z + b$ or rotation $w = e^{i\theta_0} z$ all presences circles (lines). So it remains to consider scaling

$w = kz$ $(k > 0)$ and inversion $w = 1/z$.
Let us consider the circle equation (centred at $z_0 = (x_0,\, y_0)$ with radius R). Then

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

That is,

$$x^2 + y^2 - 2x_0x - 2y_0y + (x_0^2 + y_0^2 - R^2) = 0.$$

Substituting $z = x + iy,\ \overline{z} = x - iy$

$$z\overline{z} + \frac{-2}{2}(z_0 + \overline{z}_0)\frac{1}{2}(z + \overline{z}) - \frac{2}{2i}(z_0 - \overline{z}_0)\,\frac{1}{2i}(z - \overline{z}) + z_0\overline{z}_0 - R^2.$$

This can be rewritten as

$$z\overline{z} + \overline{B}z + B\overline{z} + D = 0,$$

where $B = -z_0,\ D = x_0^2 + y_0^2 - R^2$.

Conversely, suppose $B = -z_0,\ |B|^2 - D = R^2 > 0$, then the above equation represents a circle equation centred at $-B = z_0$ with radius

$$R = \sqrt{|B|^2 - D}.$$

In fact, $|z - (-B)| = \sqrt{|B|^2 - D}$. We consider the scaling : $w = kz$. The circle equation becomes

$$\frac{1}{k^2}\,w\overline{w} + \frac{\overline{B}}{k}w + \frac{B}{k}\overline{w} + D = 0.$$

Thus

$$w\overline{w} + k\overline{B}w + kB\overline{w} + k^2D = 0$$

Clearly, k^2D is a real number, and $\sqrt{k^2\,|B|^2 - k^2D} = k\sqrt{|B|^2 - D} > 0$. Hence the above equation is a circle equation in the $w-$plane.
It remains to consider inversion $w = 1/z$. Then the equation becomes

$$\frac{1}{w\overline{w}} + \frac{\overline{B}}{w} + \frac{B}{\overline{w}} + D = 0,$$

or

$$w\overline{w} + \frac{B}{D}w + \frac{\overline{B}}{D}\overline{w} + \frac{1}{D} = 0.$$

clearly $1/D$ is a real number, and $|B/D|^2 - 1/D = \frac{1}{D^2}(|B|^2 - D) > 0$. So the equation is a circle equation in the $w-$plane. □

2.5 Cross-ratios

Let

$$T(z) = \frac{az+b}{cz+d} \tag{2.7}$$

be a Möbius transformation, and let w_1, w_2, w_3, w_4 be the respectively images of the points z_1, z_2, z_3, z_4. Then it is routine to check that

$$w_j - w_k = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k), \quad j,\, k = 1,\, 2,\, 3,\, 4.$$

Then

$$(w_1 - w_3)(w_2 - w_4) = \frac{(ad-bc)^2}{\Pi_{j=1}^4 (cz_j+d)}(z_1 - z_3)(z_2 - z_4) \tag{2.8}$$

Similarly, we have

$$(w_1 - w_4)(w_2 - w_3) = \frac{(ad-bc)^2}{\Pi_{j=1}^4 (cz_j+d)}(z_1 - z_4)(z_2 - z_3). \tag{2.9}$$

Dividing the (2.8) by (2.9) yields

$$\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{2.10}$$

Definition 2.5.1. Let z_1, z_2, z_3, z_4 be four distinct numbers in $\mathbb{C}$. Then

$$(z_1,\, z_2,\, z_3,\, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} \tag{2.11}$$

is called the **cross-ratio** of the four points. If, however, when any one of z_1, z_2, z_3, z_4 is ∞, then the cross-ratio becomes

$$\begin{aligned}(\infty,\, z_2,\, z_3,\, z_4) &:= \frac{z_2 - z_4}{z_2 - z_3},\\ (z_1,\, \infty,\, z_3,\, z_4) &:= \frac{z_1 - z_3}{z_1 - z_4},\\ (z_1,\, z_2,\, \infty,\, z_4) &:= \frac{z_2 - z_4}{z_1 - z_4},\\ (z_1,\, z_2,\, z_3,\, \infty) &:= \frac{z_1 - z_3}{z_2 - z_3},\end{aligned}$$

respectively.

The equation (2.10) implies that we have already proved the following theorem.

Theorem 2.5.2. *Let T be any Möbius transformation. Then*

$$(Tz_1,\, Tz_2,\, Tz_3,\, Tz_4) = (z_1,\, z_2,\, z_3,\, z_4). \tag{2.12}$$

Remark. The above formula means that the cross-ratio of four points is preserved under any Möbius transformation $T(z)$.

Example 2.5.3. We note that the cross-ratio when written as

$$(z,\, z_2,\, z_3,\, z_4) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)} = \frac{z - z_3}{z - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$$

is a Möbius transformation of z that maps the points z_2, z_3, z_4 to 1, 0, ∞ respectively.

Theorem 2.5.4. *Let z_1, z_2, z_3 and w_1, w_2, w_3 be two sets of three arbitrary complex numbers. Then there is a unique Möbius transformation $T(z)$ that satisfies $T(z_i) = w_j$, $j = 1, 2, 3$.*

Proof. The cross-ratio formula

$$\frac{w-w_3}{w-w_4} : \frac{w_2-w_3}{w_2-w_4} = \frac{z-z_3}{z-z_4} : \frac{z_2-z_3}{z_2-z_4}$$

does the trick. □

Example 2.5.5. *Find a Möbius transformation w that maps -1, i, 1 to -1, 0, 1 respectively.*
It follows that

$$\frac{w-0}{w-1} : \frac{-1-0}{-1-1} = \frac{z-i}{z-1} : \frac{-1-i}{-1-1}.$$

So

$$\frac{2w}{w-1} = \frac{z-i}{z-1}\Big(\frac{1}{1+i}\Big).$$

Hence

$$w = \frac{1+iz}{i+z}.$$

Arrangements

The above arrangement of the four points z_1, z_2, z_3, z_4 in the construction of our cross-ratio is not special. One can try the remaining twenty three different permutations of z_1, z_2, z_3, z_4 in the construction. However, we note that

$$\lambda := (z_1,\, z_2,\, z_3,\, z_4) = (z_2,\, z_1,\, z_4,\, z_3) = (z_3,\, z_4,\, z_1,\, z_2) = (z_4,\, z_3,\, z_2,\, z_1)$$

so that the list reduces to six only. They are given by

$$(z_2,\, z_3,\, z_1,\, z_4) = \frac{\lambda-1}{\lambda}, \quad (z_3,\, z_1,\, z_2,\, z_4) = \frac{1}{1-\lambda}$$

$$(z_2,\, z_1,\, z_3,\, z_4) = \frac{1}{\lambda}, \quad (z_3,\, z_2,\, z_1,\, z_4) = \frac{\lambda}{\lambda-1}, \quad (z_1,\, z_3,\, z_2,\, z_4) = 1-\lambda.$$

The above list contains all six distinct values for the cross-ratio for distinct z_1, z_2, z_3, z_4. If, however, two of the points z_1, z_2, z_3, z_4 coincide, then the list of values will reduce further. More precisely, if $\lambda = 0$ or 1, then the list reduces to three, namely 0, 1, ∞. If $\lambda = -1$, $-1/2$ or 2, then the list reduces to three again with values -1, $1/2$, 2. There is another possibility that

$$\lambda = \frac{1 \pm i\sqrt{3}}{2}.$$

See exercise.

Moreover, if we put $z_2 = 1,\ z_3 = 0,\ z_4 = \infty$, then the cross-ratio becomes

$$(\lambda,\, 1,\, 0,\, \infty) = \lambda,$$

which means that λ is a **fixed point** of the map.

Theorem 2.5.6. *Let z_1, z_2, z_3, z_4 be four distinct points in $\hat{\mathbb{C}}$. Then their cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle (including a straight line).*

Proof. Let $Tz = (z_1, z_2, z_3, z)$.

We first prove that if z_1, z_2, z_3, z_4 lie on a circle/straight-line in $\hat{\mathbb{C}}$, then Tz is real. But by the fundamental property that T is the unique Möbius map that maps z_1, z_2, z_3 onto 0, 1, ∞. Hence T is real on $T^{-1}\mathbb{R}$. It remains to show that the whole circle/straight-line passing through z_1, z_2, z_3 has Tz real.

If Tz is real, then we have $Tz = \overline{Tz}$. Hence

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}.$$

Cross multiplying yields

$$(a\bar{c} - c\bar{a})|w^2| + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - d\bar{b} = 0$$

which is a straight-line if $a\bar{c}-c\bar{a}=0$ (and hence $a\bar{d}-c\bar{b}\neq 0$). Moreover, in the case when $a\bar{c}-c\bar{a}\neq 0$, the above equation can be written in the form

$$\Big|w+\frac{\bar{a}d-\bar{c}b}{\bar{a}c-\bar{c}a}\Big|=\Big|\frac{ad-bc}{\bar{a}c-\bar{c}a}\Big|,$$

which is an equation of a circle. □

Exercise 2.5.1. Verify that

$$(\lambda,\, 1,\, 0,\, \infty)=\lambda.$$

Then use this identity to give a different proof of the above theorem: $(z_1,\, z_2,\, z_3,\, z_4)$ is real if and only if the four points $z_1,\, z_2,\, z_3,\, z_4$ lie on a circle.

Exercise 2.5.2. Show that if one of $z_2,\, z_3,\, z_4$ is ∞, the corresponding cross-ratio still maps the triple onto $1,\, 0,\, \infty$. Namely the

$$\begin{aligned}(z,\, \infty,\, z_3,\, z_4) &:= \frac{z-z_3}{z-z_4},\\ (z,\, z_2,\, \infty,\, z_4) &:= \frac{z_2-z_4}{z-z_4},\\ (z,\, z_2,\, z_3,\, \infty) &:= \frac{z-z_3}{z_2-z_3},\end{aligned}$$

2.6 Inversion symmetry

We already know that the point z and its conjugate $\bar{z}$ are symmetrical with respect to the real-axis. If we take the real-axis into a circle C by a Möbius transformation T, then we say that the points $w=Tz$ and $w^*=T\bar{z}$ are *symmetric with respect to* C. Since the symmetry is a geometric property, so the w and w^* are independent of T. For suppose there is another Möbius transformation that maps the real-axis onto the C, then the composite map $S^{-1}T$ maps the $\mathbb{R}$ onto itself. Thus the images,

$$S^{-1}w = S^{-1}Tz, \quad S^{-1}w^* = S^{-1}T\bar{z}$$

are obviously conjugates. Hence we can define

Definition 2.6.1. Two points z and z^* are said to be *symmetrical with respect to the circle* C passing through z_1, z_2, z_3 if and only if

$$(z^*,\, z_1,\, z_2,\, z_3) = \overline{(z,\, z_1,\, z_2,\, z_3)}.$$

In order to see what is the relationship between z and z^*, we consider the following special case.

Example 2.6.2. When $z_3 = \infty$. Then the symmetry yields

$$\frac{z^* - z_2}{z - z_4} = \frac{\bar{z}_2 - \bar{z}_4}{\bar{z}_1 - \bar{z}_4}.$$

That is,

$$|z^* - z_2| = |z - z_2|$$

first showing that the z and z^* are equal distances to z_2 (which is arbitrary on C). And

$$\Im\Big(\frac{z^* - z_2}{z_1 - z_2}\Big) = -\Im\Big(\frac{z - z_2}{z_1 - z_2}\Big)$$

finally showing that the z and z^* are on *different sides* of C.

Theorem 2.6.3. *Let z and z^* be symmetrical with respect to a circle C of radius R and centred at a. Then*

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a.$$

Proof. We note that

$$(z_j - a)\overline{(z_j - a)} = R^2, \quad j = 1, 2, 3.$$

Thus we have

$$\begin{aligned}
\overline{(z,\, z_1,\, z_2,\, z_3} &= \overline{(z-a,\, z_2-a,\, z_3-a,\, z_3-a)} \\
&= \Big(\bar{z}-\bar{a},\, \frac{R^2}{z_1-a},\, \frac{R^2}{z_2-a},\, \frac{R^2}{z_3-a},\Big) \\
&= \Big(\frac{R^2}{z-a},\, z_1-a,\, z_2-a,\, z_3-a\Big) \\
&= \Big(\frac{R^2}{z-a}+a,\, z_1,\, z_2,\, z_3,\Big) \\
&:= (z^*,\, z_2,\, z_3,\, z_3)
\end{aligned}$$

as required. □

We deduce immediately that

Theorem 2.6.4. *A Möbius transformation carries a circle C_1 into a circle C_2 also transforms any pair of symmetric points of C_1 into a pair of symmetric points of C_2.*

Remark.

1. $(z^* - a)(\bar{z} - \bar{a}) = R^2$,
2. The symmetry point $a^* = \infty$ for the centre a above.
3. The expression

$$\frac{z^* - a}{z - a} = \frac{R^2}{(\bar{z} - \bar{a})(z - a)} > 0$$

implying that z and z^* lie on the same half-line from a.

We briefly mention the issue of orientation. Suppose we have a circle C. Then there is an analytic method to distinguish the inside/outside of the circle by the cross-ratio. Since the cross-ration

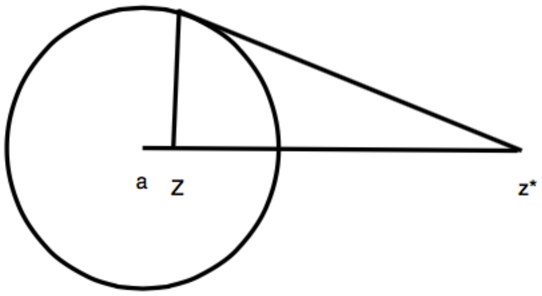

Figure 2.5: Inversion: z and z^*

is invariant with respect to any Möbius transformation, so it is sufficient to consider the inside/outside issue of the real-axis $\mathbb{R}$ since we can always map the circle C onto the $\mathbb{R}$. Let us write

$$(z_1,\, z_2,\, z_3,\, z) = \frac{az+b}{cz+d}$$

where a, b, c, d are real coefficients (since z_1, z_2, $z_3 \in \mathbb{R}$). Then

$$\Im(z,\, z_1,\, z_2,\, z_3) = \frac{ad-bc}{|cz+d|^2}\,\Im z.$$

Suppose we choose $z_1 = 1$, $z_2 = 0$ and $z_3 = \infty$. Then a previous formulai

$$(z,\, 1,\, 0,\, \infty) = z$$

implies that $\Im(z,\, 1,\, 0,\, \infty) = \Im z$, so that $\Im(i,\, 1,\, 0,\, \infty) > 0$ and $\Im(-i,\, 1,\, 0,\, \infty) < 0$. The ordered triple, namely 1, 0, ∞ clearly indicates that the point i is on the **right** of $\mathbb{R}$ (in that order) and the other point $-i$ is on the **left** of $\mathbb{R}$ (in that order). But any circle C can be brought to the real-axis $\mathbb{R}$ while keeping the cross-ratio unchanged. So we have

Definition 2.6.5. Let C be a given circle in $\hat{\mathbb{C}}$. An **orientation** of C is determined by the direction of a triple z_1, z_2, z_3 (i.e., $z_1 \mapsto z_2 \mapsto z_3$

) lying on C. Let $z \notin C$. The point z is said to lie on the **right** of C if $\Im(z,\, z_1,\, z_2,\, z_3) > 0$ of the oriented circle. The point z is said to lie on the **left** of C if $\Im(z,\, z_1,\, z_2,\, z_3) < 0$ of the oriented circle.

Definition 2.6.6. We define an **absolute orientation** for each finite circle with respect to ∞ in the sense that the ∞ is on its *right* (we call this *outside*), otherwise, on its *left* (we call this *inside*).

2.7 Explicit conformal mappings

Example 2.7.1. *Find a Möbius mapping that maps the upper half-plane* $\mathbb{H}$ *onto itself.*

Suppose $f(z) = \dfrac{az+b}{cz+d}$ maps the upper half-plane onto itself.

Then $f(z)$ must map any three points $\{x_1,\, x_2,\, x_3\}$ on the x-axis in the order $x_1 < x_2 < x_3$ respectively to three points $u_1 < u_2 < u_3$ on real-axis. It follows that is “no turning" on the real-axis, thus implying that

$$\arg f'(x_1) = 0 \quad \text{or} \quad f'(x_1) > 0.$$

Moreover, one can solve for the coefficients a, b, c and d by solving

$$u_i = \frac{ax_i + b}{cx_i + d}, \quad i = 1,\, 2,\, 3.$$

One notices that a, b, c and d are therefore all real constants. But

$$f'(x_1) = \frac{ad - bc}{(cx_1 + d)^2} > 0,$$

implying that $ad - bc > 0$. Since f must map $\hat{\mathbb{C}}$ one-one onto $\hat{\mathbb{C}}$, the upper half-plane onto itself. Thus we deduce that

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc > 0.$$

Conversely, suppose

$$w = f(z) = \frac{az+b}{cz+d},$$

where a, b, c and d are real and $ad - bc > 0$. Then for all real x,

$$f'(x) = \frac{ad-bc}{(cx+d)^2} > 0, \quad \text{and} \quad \arg f'(x) = 0.$$

That is, there is "no turning" on the real-axis. Therefore w must map the real-axis onto the real-axis, and hence Therefore w must map the upper half-plane onto upper half-plane.

Exercise 2.7.1. *Prove directly, that is without applying f', that it is necessary sufficient that $ad - bc > 0$ for*

1. *f maps $\mathbb{H}$ into $\mathbb{H}$;*
2. *that the above map is "onto".*

Example 2.7.2. Construct a Möbius mapping f that maps upper half-plane into upper half-plane such that $0 \mapsto 0$ and $i \mapsto 1+i$.

According to the last example, we must have

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc > 0,$$

where a, b, c and d are real. Since $f(0) = 0$ implying that $b = 0$. On the other hand,

$$1 + i = f(i) = \frac{ai}{ci+d} = \frac{i}{ei+f},$$

say. That is, $e - f = 0$ and $e + f = 1$, or $e = f = \frac{1}{2}$. Hence

$$w = \frac{2z}{z+1}.$$

Example 2.7.3. *Show that a Möbius mapping f that maps the upper half-plane $\mathbb{H}$ onto $\triangle = \{z: \ |z| < 1\}$ if and only if*

$$w = f(z) = e^{i\theta_0}\frac{z-\alpha}{z-\overline{\alpha}}, \quad \Im\alpha > 0, \quad \theta_0 \in \mathbb{R}.$$

Suppose $f : \mathbb{H} \to \triangle$. It follows that f must map the $x-$axis onto $|w| = 1$. Let us consider the images of $z = 0$, 1 and ∞. Since $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$. Thus $1 = |f(0)| = \left|\frac{b}{d}\right|$, implying $|b| = |d|$. We also require $f(\infty)$ to lie on $|w| = 1$ which is necessary finite. But we know from a previous discussion that

$$|f(\infty)| = \left|f\Big(\frac{1}{\zeta}\Big)\right|_{\zeta=0} = \left|\frac{a+b\zeta}{c+d\zeta}\right|_{\zeta=0} = \left|\frac{a}{c}\right| = 1,$$

implying that $|a| = |c|$. So

$$w = \frac{az+b}{cz+d} = \frac{a}{c} \times \frac{z+b/a}{z+d/c} = \frac{a}{c}\frac{z-z_0}{z-z_1}$$

where $|z_0| = |b/a| = |d/c| = |z_1|$. Since $|a/c| = 1$, so there exists a real θ_0 such that $\frac{a}{c} = e^{i\theta_0}$. Thus

$$w = e^{i\theta_0}\,\frac{z-z_0}{z-z_1}, \quad |z_0| = |z_1|.$$

Consider

$$1 = |f(1)| = \left|\frac{z-z_0}{z-z_1}\right|$$

implying $|z - z_0| = |z - z_1|$ or

$$(1 - z_1)(1 - \overline{z}_1) = (1 - z_0)(1 - \overline{z}_0).$$

Notice that $|z_1| = |z_0|$. Hence

$$1 - z_1 - \overline{z}_1 + |z_1|^2 = 1 - z_0 - \overline{z}_0 + |z_0|^2.$$

Thus

$$2\Re(z_1) = z_1 + \overline{z}_1 = z_0 + \overline{z}_0 = 2\Re(z_0)$$

or $\Re(z_1) = \Re(z_0)$. Hence $z_1 = z_0$ or $z_1 = \overline{z}_0$. We must have $z_1 = \overline{z}_0$, for if $z_1 = z_0$, then $f(z)$ is identically a constant. Thus

$$f(z) = e^{i\theta_0}\Big(\frac{z - z_0}{z - \overline{z_0}}\Big).$$

Since $f(z_0) = 0$ so $\Im(z_0) > 0$.
Conversely, suppose

$$f(z) = e^{i\theta}\Big(\frac{z - \alpha}{z - \overline{\alpha}}\Big), \quad z \in \mathbb{H}.$$

Then $|w| < |f| = \Big|\frac{z - \alpha}{z - \overline{\alpha}}\Big| < 1$. If z lies on the lower half-plane, then $|w| < |f| = \Big|\frac{z - \alpha}{z - \overline{\alpha}}\Big| > 1$. If z lies on the real axis, then $|w| = \Big|\frac{z - \alpha}{z - \overline{\alpha}}\Big| = 1$. Since f maps $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ in a one-one manner, so f must maps the $\mathbb{H}$ onto $|w| < 1$.

Remark. *If $\Im(z_0) = \Im(\alpha) < 0$, then f maps the upper half-plane onto the lower half-plane.*

Exercise 2.7.2. *Find a Möbius transformation $w : \mathbb{H} \to \triangle, \quad i \mapsto 0$. So*

$$w = f(z) = e^{i\theta_0}\Big(\frac{z - i}{z + i}\Big).$$

Exercise 2.7.3. *Let $\triangle = \{z : |z| < 1\}$. Show that a Möbius transformation f that $f : \triangle \to \triangle$ if and only if there exists θ_0, $|\alpha| < 1$ such that*

$$w = f(z) = e^{i\theta_0}\,\frac{z - \alpha}{1 - \overline{\alpha}z}.$$

2.8 Orthogonal circles

We follow the ideas of Riemann and Klein to visualise the effects of conformal mappings. We use the toy models of Möbius transformations to allow us to have a glimpse.

Consider the map

$$w = h(z) = k\,\frac{z-a}{z-b},$$

where k is some non-zero constant to be chosen later. The map carries $z = a$ to $w = 0$ and $z = b$ to $w = \infty$. This means that any straight-line passing through the origin in the $w-$plane has its preimage to pass through the points $z = a$ and $z = b$, and this preimage must be a circle (may be a generalised circle, i.e., a straight-line) in the $z-$plane.

On the other hand, the circles centred at the orgin in the $w-$plane are of the form $|w| = \rho$ for some $\rho > 0$. That is,

$$\left|\frac{z-a}{z-b}\right| = \rho/|k|.$$

Hence the loci of the $h^{-1}\{|w| = \rho/|k|\}$, which must also be a circle, also lies on the $z-$plane. The relation

$$|z-a| = (\rho/|k|)\,|z-b|$$

describes the loci of the point z so that the distances of it to a and b are in a constant ratio. Such circles, denoted by C_2, are called **Apollonius' circles** and the points a and b are called the **limit points**. It is clear that the family of concentric circles $|w| = \rho/|k|$ are always at right angles with any straight-line through the origin in the $w-$plane. So their preimages, denoted by C_1 are orthogonal to the Apollonius circles C_2. In general, we denoted by C_1' and by C_2' the images of $C_!$ and Apollonius circles C_2, respectively, under a Möbius transformation in the $w-$plane. Obviously, the C_1' and C_2' are orthogonal to each other at their intersections.

We have the following theorem.

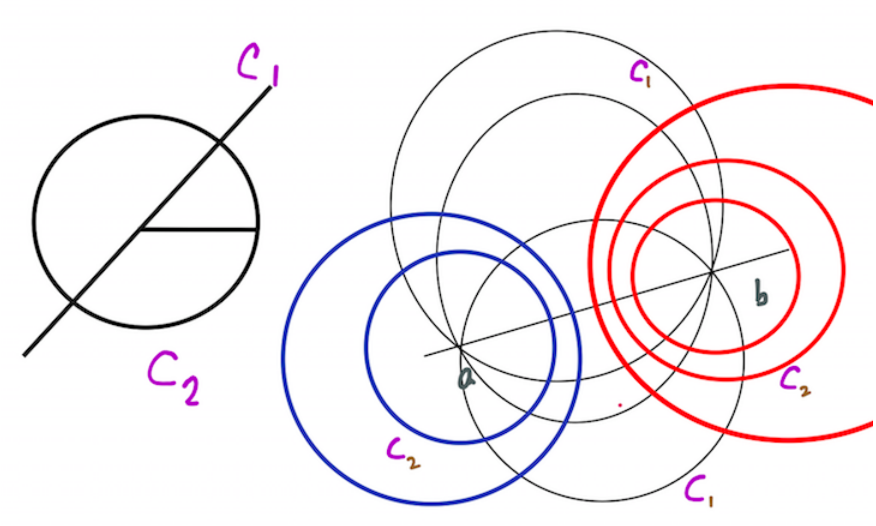

Figure 2.6: Orthogonal circles

Theorem 2.8.1. *Let a and b be two given points, C_1 and C_2 as defined above. Then*

(i) there is exactly one C_1 and one C_2 through each point in $\mathbb{C}$ except at the limit point a and b in the $z-$plane;

(ii) the tangent of each C_1 and that of each C_2 are orthogonal to each other at the points of intersections;

(iii) reflection in C_1 transforms every C_2 into itself and every C_1 into another C_1;

(iv) reflection in a C_2 transforms every C_1 into itself and every C_2 into another C_2;

(v) the limit points are symmetric with respect to each C_2, but not with respect to any other circle.

Proof. We consider the special case that $a = 0$ and $b = 0$ so that the circles passing through 0 and ∞ become straightlines passing through the origin in the $z-$plane. Then

(i) it is clear since there is only one straightline passing through any non-zero finite point and the origin, and only one circle intersecting with the straightline and orthogonal to it at that point;

(ii) follows since the C_2 are concentric circles;

(iii) also follows since it is clearly that any reflection of a concentric circle C_2 with respect to any straight line passing through the origin remains unchange. Reflection of any C_1 (straightline) with respect to a C_1 is obviously another C_1;

(iv) follows from Theorem 2.6.3 when considering symmetric points lying on a straightline is reflected upon each other lying on the same straightline with respect to a C_2. So a C_1 is mapped onto itself with respect to any C_2. Let C_2 relfect with respect to another C_2. Then parts (i) and (ii) imply that each point of the image of C_2 upon reflection must be orthogonal to each C_1 and this implies the image must be a circle. The image circle C_2 must be different from its preimage except itself because of the symmetric principle Theorem 2.6.3;

(v) this is obvious because of the choice.

Having established the special case $a = 0$ and $b = 0$, the general case (i-v) for arbitrary a and b follow since one can map a C_1 by a Möbius transformation to a straightline C_1' passing through the origin and then each corresponding C_2 becomes a circle C_2' centred at the origin so that C_2 must be orthogonal to C_1 because any Möbius transformation is conformal on $\mathbb{C}$. □

Fixed points

The general Möbius transformation T that carries a to a' and b to b' can be written as

$$\frac{w-a'}{w-b'} = k\,\frac{z-a}{z-b}$$

which is an application of cross-ratios. Suppose we impose the requirement that $a = a'$ and $b = b'$. That is, we assume that

$$z = T(z) = \frac{az + b}{cz + d},$$

which will have two **fixed points** $Ta = a$ and $Tb = b$ since we have a quadratic equation in z. In the exceptional circumstance, we have a double root from the quadratic equation so that we are left with one double root. The transformation T maps C_1 to C_1', C_2 to C_2' and a, b to a', b'.

Theorem 2.8.2. *Let $w = T(z)$ be a Möbius transformation that satisfies,*

$$\frac{w - a}{w - b} = k\,\frac{z - a}{z - b}.$$

Then

(i) the whole circular net consists of C_1 and C_2 are mapped onto itself. That is, the union of C_1' and C_2' are the same as the union of C_1 and C_2;

(ii) when the images C_1' and C_2' are plotted on the same graph as C_1 and C_2, then

(a) the $\arg k$ represents the difference of the angle made by the tangents at the point of intersections between the circles C_1 and C_1';

(b) the

$$|k| = \frac{|w - a|/|w - b|}{|z - a|/|z - b|}$$

measures the ratio of the above right-hand side concerning the Apollonius circles C_2 and C_2',

(iii) $C_1 = C_1'$ if $k > 0$ (with orientation reversed if $k < 0$), where the points on Tz on C_1 flow toward b upon increasing the value of k, and we call T ***hyperbolic****;*

(iv) *$C_2 = C_2'$ if $|k| = 1$, then as* $\arg k$ *increase, the Tz circulates along C_2, and we call T* ***elliptic****.*

Proof. Exercise. □

Definition 2.8.3. If two fixed points of a Möbius transformation T coincide, then we call the transformation **parabolic**.

Rotations of the Riemann sphere

Let us consider a subgroup R of the set of all Möbius transformation that represent the rotation of the Riemann sphere S about its centre. Let us assume that the axis of rotation passes through the antipodal points Z_0 and Z_1 whose images on $\mathbb{C}$ are z_0 and z_1. Then we know that they are z_0 and $z_1 = -1/\bar{z}_0$ since $z_0\bar{z}_1 + 1 = 0$.

Theorem 2.8.4. *The Möbius transformation*

$$\frac{w - z_0}{1 + \bar{z}_0 w} = k\,\frac{z - z_0}{1 + \bar{z}_0 z}, \quad k = \cos\alpha + i\sin\alpha \tag{2.13}$$

(i) *leaves the points z_0 and $-1/\bar{z}_0$ invariant;*

(ii) *leaves the points Z_0 and Z_1 corresponding to z_0 and $-1/\bar{z}_0$ respectively, on the Riemann sphere S invariant;*

(iii) *rotates the plane that intersects the S in a great circle passing through Z_0 and Z_1 by an angle of α.*

Proof. The statements (i) and (ii) are clear. It remains to verify the (iii). It is left as an exercise for the reader to check that if $Tz = w$,

$$\Big|\frac{w - z_0}{1 + \bar{z}_0 w}\Big| = \Big|\frac{z - z_0}{1 + \bar{z}_0 z}\Big| = \rho > 0$$

then their chordal distance is

$$\chi(z,\, z_0) = \chi(w,\, z_0) = \frac{\rho}{\sqrt{1+\rho^2}}.$$

Let Z and W be the images of z and w respectively. Then it follows from (2.13) that the T is a rotation of the Riemann sphere S through the plane containing the great circle passing through the points Z_0, Z and Z_1 to the plane containing the great circle Z_0, W and Z_1. □

2.9 Extended Maximum Modulus Theorem

Let us recall some knowledge about metric spaces. Let (X, d) be a metric space. Then $F \subset X$ is *closed* if $X \setminus F$ is open. Let $A \subset X$ be a subset, the *closure* $\overline{A}$ of A is defined by

$$\cap\{F : F \text{ is closed and } A \supset F\}.$$

The *boundary* ∂A of A is defined by $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Let G be a subset of $\widehat{\mathbb{C}}$. We write

$$\partial_\infty G = \begin{cases} \partial G & \text{if } G \text{ is bounded;} \\ \partial G \cup \{\infty\} & \text{if } G \text{ is unbounded.} \end{cases}$$

to be the extended boundary of G in $\widehat{\mathbb{C}}$. If $a = \infty$, then the $B(a, r)$ is understood in terms of chordal metric.

Example 2.9.1. Let $G = \{z : |\arg z| < \frac{\pi}{2}\}$. Then

$$\partial G = \{z = x + iy : x = 0\}, \qquad \partial_\infty G = \partial G \cup \{\infty\}.$$

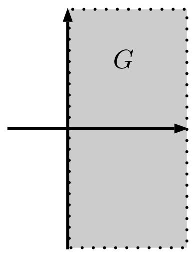

Figure 2.7: $G = \{z : |\arg z| < \frac{\pi}{2}\}$

Definition 2.9.2. Let $G \subset \mathbb{C}$ and $f : G \to \mathbb{R}$ be continuous. Suppose $a \in \partial_\infty G$, then we define

$$\limsup_{z\to a} f(z) = \lim_{r\to 0}\Big(\sup_z\{f(z) : z \in G \cap B(a,r)\}\Big) = L$$

and

$$\liminf_{z\to a} f(z) = \lim_{r\to 0}\Big(\inf_z\{f(z) : z \in G \cap B(a,\, r)\}\Big) = l.$$

If $a \neq \infty$, the above definition can be written as:
Given $\epsilon > 0$, there exists $r > 0$ such that

$$L - \epsilon < \sup_z\{f(z) : z \in G \cap B(a,r)\} < L + \epsilon.$$

In particular, $f(z) < L + \epsilon$ for all $z \in G \cap B(a,r)$.

Similarly, given $\epsilon > 0$, there exists $r > 0$ such that

$$l - \epsilon < \inf_z\{f(z) : z \in G \cap B(a,r)\} < l + \epsilon.$$

In particular, $f(z) > l - \epsilon$ for all $z \in G \cap B(a,r)$.

If $a = \infty$, we understand $B(a,r)$ is with the chordal metric and the lim sup, lim inf have similar interpretations.

Note also that, it follows easily $\lim_{z\to a} f(z)$ exists if and only if $L = l$ $(a \in \partial_\infty G)$.

Theorem 2.9.3 (Maximum Modulus Theorem - Extended version)**.** *Let $G \subset \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ is analytic. Suppose*

$$\limsup_{z\to a} |f(z)| \leq M$$

for some $M > 0$ and all $a \in \partial_\infty G$. Then $|f(z)| \le M$ for all $z \in G$.

Proof. Let
$$H = \{z \in G : |f(z)| > M + \delta\}$$
for a fixed $\delta > 0$. We aim to show that $H = \emptyset$. Since then $|f| \le M$ because $\delta > 0$ is arbitrary. It follows from the elementary fact in real analysis that H is open because $|f|$ is continuous. We next show that H has no intersection with a region near the ∞ and in particular $H \cap \partial_\infty G = \emptyset$, and hence H is a bounded set.

By the hypothesis $\limsup_{z\to a} |f(z)| \le M$ for all $a \in \partial_\infty G$, for the above $\delta > 0$, there exists $r > 0$ such that
$$|f(z)| < M + \delta$$
for all $z \in G \cap B(a, r)$. Hence $\overline{H} \subset G$. This argument works whether G is bounded or unbounded, and $a = \infty$. Thus $H \cap \partial_\infty G = \emptyset$ and hence H is bounded. Therefore $\overline{H}$ is a compact set.

Note that $|f(z)| = M + \delta$ when $z \in \partial H$ since $\overline{H} \subset \{z \in G : |f(z)| \ge M + \delta\}$. Thus either f is constant on H by Theorem 1.7.2 (hence f is constant on G by Identity theorem since H is open and non-empty) or $H = \emptyset$. But if f is constant on G, where $|f| = M + \delta$, then it contradicts the hypothesis that $|f| < M + \delta$ near $\partial_\infty G$. Thus $H = \emptyset$. This completes the proof. □

We shall apply the maximum modulus theorem to characterize certain analytic map of unit disk. We first recall

Theorem 2.9.4 (Schwarz's Lemma)**.** *Let $\Delta = \{z : |z| < 1\}$ be the unit disk. Suppose $f : \Delta \to \mathbb{C}$ is analytic such that $|f(z)| \le 1$ for each $z \in \Delta$, and $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in \Delta$ and $|f'(0)| \le 1$.*

Moreover, $f(z) = e^{i\theta} z$ for a fixed θ whenever $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \ne 0$.

Proof. Define
$$F(z) = \begin{cases} \dfrac{f(z)}{z}, & z \ne 0; \\ f'(0), & z = 0. \end{cases}$$

F is thus analytic on Δ.

Moreover, $|F(z) = \left|\frac{f(z)}{z}\right| \leq \frac{1}{|z|} \to 1$ as $|z| \to 1$. It follows from Theorem 2.9.3 that $|F(z)| \leq 1$.

If $|F(z)| = 1$ for some $z \in \Delta$ (i.e. either $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$), then F is a constant $e^{i\theta}$ for some $\theta \in [0, 2\pi]$ by the maximum modulus theorem 1.7.2 since $|F| \leq 1$ for all $z \in \Delta$. And so $f(z) = e^{i\theta} z$. □

Exercise. Suppose $\phi(z)$ is analytic on $|z| \leq R$, where $|\phi(z)| \leq 1$ and $\phi(0) = 0$. Show that $|\phi(z)| \leq \frac{r}{R}$ on $|z| = r$, where $r < R$.

Proposition 2.9.5. *Suppose $|a| < 1$, then*

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}$$

is a conformal map mapping Δ onto Δ, $\partial\Delta$ to $\partial\Delta$. Moreover, $\varphi_a^{-1} = \varphi_{-a}$, $\varphi_a'(0) = 1 - |a|^2$ and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Proof. Since $|a| < 1$, φ_a is clearly analytic. In fact, φ_a is conformal (Exercise). We only show

$$\begin{aligned} |\varphi_a(e^{i\theta})| &= \left|\frac{e^{i\theta} - a}{1 - \overline{a}e^{i\theta}}\right| \\ &= \left|e^{i\theta} \cdot \frac{e^{i\theta} - a}{e^{-i\theta} - \overline{a}}\right| \\ &= \frac{|e^{i\theta} - a|}{|e^{-i\theta} - \overline{a}|} = 1. \end{aligned}$$

Hence $\varphi_a(\partial\Delta) = \partial\Delta$. The remaining conclusion is left as an exercise. □

Proposition 2.9.6. *Suppose $f : \Delta \to \Delta$ is analytic and $f(a) = \alpha$. Then*

$$|f'(a)| \leq \frac{1 - |\alpha|^2}{1 - |a|^2}. \quad (\textit{max. value of } |f'(a)|)$$

Moreover, equality occurs if and only if $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$, $|c| = 1$.

Remark. We may assume $|\alpha| < 1$. Otherwise f is a constant.

Proof. Define $g = \varphi_\alpha \circ f \circ \varphi_{-a}$, Then $g(\Delta) \subset \Delta$, and $g(0) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = \dfrac{\alpha - \alpha}{1 - \overline{\alpha}\alpha} = 0$. Clearly g is analytic and thus $|g(z)| \le |z|$ and $|g'(0)| \le 1$ by Schwarz's Lemma. But

$$g'(0) = \frac{1 - |a|^2}{1 - |\alpha|^2} f'(a).$$

Thus

$$|f'(a)| \le \frac{1 - |\alpha|^2}{1 - |a|^2}. \tag{2.14}$$

Equality will occur if and only if there exists a c such that $|g'(0)| = |c| = 1$ and $g = cz$. □

We can now prove the converse of Proposition 2.9.5.

Theorem 2.9.7. *Let $f : \Delta \to \Delta$ be an one-to-one analytic function onto Δ. Suppose $f(a) = 0$. Then there is a c such that $|c| = 1$ and*

$$f = c\varphi_a = c\frac{z - a}{1 - \overline{a}z}.$$

Proof. Since f is bijective, we let $g : \Delta \to \Delta$ to be f^{-1}. So $g(f(z)) = z$ for all $z \in \Delta$. We apply (2.14) to both f and g to derive the inequalities:

$$|f'(a)| \le \frac{1}{1 - |a|^2} \quad \text{and} \quad |g'(0)| \le 1 - |a|^2.$$

On the other hand, $1 = g'(0)f'(a)$. Thus, $|f'(a)| = (1 - |a|^2)^{-1}$ since

$$\frac{1}{1 - |a|^2} \le |f'(a)| \le \frac{1}{1 - |a|^2}.$$

Then, since $\varphi_0(z) = z$, Proposition 2.9.6 gives $f = c\varphi_a$ for some c with $|c| = 1$. □

Remark. A simple consequence of the maximum modulus of entire functions is that the function $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ is an increasing function of r, i.e. $M(r_1) \le M(r_2)$ if $r_1 \le r_2$.

2.10 Phragmén-Lindelöf principle

Example 2.10.1. Let $f(z) = \exp(\gamma z^a)$, $\gamma > 0$, $a \geq \frac{1}{2}$, be defined on $\mathbb{C}$.

Note that $|f(z)| = \exp(r^a \gamma \cos(a\theta))$, and $\cos(a\theta) < 0$ if

$$S_n : (2n-1)\frac{\pi}{2a} < \theta < (2n+1)\frac{\pi}{2a}$$

for all odd integers n, $\cos(a\theta) > 0$ for $\theta \in S_n$ and for all even integers; and $|f(z)| = 1$ if $\theta = \frac{\pi}{2a}(2n+1)$ for all integers n. Note that each S_n has an opening $\frac{\pi}{a}$.

We conclude that $|f| \to 0$ (so bounded) on each sector S_n (n odd); and $|f| \to \infty$ on S_n (n even); and f is bounded on the boundary of S_n.

Clearly, $\log M(r, f) = \gamma r^a$ and it is possible for an entire function to be bounded on two rays making angle of $\frac{\pi}{a}$ with each other without being bounded inside the sectors S_n (n even). Phragmén (1863-1937) observed that this example is the *best possible* in 1904.

Theorem 2.10.2 (Phragmén)**.** *Let* $G = \left\{z : |\arg z| < \frac{\pi}{2a},\ a \geq \frac{1}{2}\right\}$ *and* $f : G \to \mathbb{C}$ *is analytic. If* f *is bounded on* ∂G *and*

$$\log M(r, f) = o(r^a),$$

then f *is bounded on* G.

So for each analytic function f on G and bounded on ∂G, either f is bounded on G or

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^a} > 0.$$

We shall prove a more general result than that by Phragmén.

Theorem 2.10.3 (Phragmén-Lindelöf Theorem). *Let G be a simply connected region and $f : G \to \mathbb{C}$ be analytic. Suppose there exists a non-vanishing, bounded analytic function $w(z) : G \to \mathbb{C}$ such that $|w(z)| \leq 1$ on G. Moreover, if $\partial_\infty G = A \cup B$, then*

(i) $\limsup_{z\to a} |f(z)| \leq M$, *for every $a \in A$;*

(ii) $\limsup_{z\to b} |w(z)|^\epsilon |f(z)| \leq M$, *for every $b \in BR$ and $\epsilon > 0$,*

then, $|f(z)| \leq M$ for all $z \in G$.

Proof. Set $F(z) = w(z)^\epsilon f(z)$ for $z \in G$. Since $w \neq 0$ on G and so we can find an analytic branch for $\log w$ and thus $w^\epsilon = \exp(\epsilon \log w)$ is a well-defined analytic function (a branch). It follows from the hypotheses (i) and (ii) that

$$\limsup_{z\to z_0 \in \partial G} |F(z)| \leq M.$$

By the maximum modulus principle (extended version), we deduce immediately that $|F(z)| \leq M$ must hold for all $z \in G$. Thus

$$|f(z)| \leq |w(z)|^{-\epsilon} M, \quad \text{for all } z \in G.$$

Since $\epsilon > 0$ is arbitrary, we may let $\epsilon \to 0$ to obtain

$$|f(z)| \leq M, \quad z \in G$$

as required. □

Theorem 2.10.4 (Phragmén-Lindelöf (1908)). *Let $a \geq 1/2$ and*

$$G = \left\{ z : |\arg z| < \frac{\pi}{2a} \right\}.$$

Suppose $f : G \to \mathbb{C}$ is analytic and $\limsup_{z\to a} |f(z)| \leq M$ for all $a \in \partial_\infty G$, where $M > 0$ is a fixed constant. Suppose further that there exist constants K, $b < a$ such that

$$|f(z)| \leq K \exp(r^b) \quad \text{as } z \to \infty, \qquad z \in G.$$

Then $|f(z)| \leq M$ for each $z \in G$.

Proof. We choose a constant c such that $b < c < a$, and define

$$F(z) = w(z)^{\epsilon} f(z)$$

in G, where $\epsilon > 0$, $w(z) = \exp(-z^c)$. Notice that

$$|w| = \exp(-r^c \cos c\theta) \leq 1.$$

Let $z = re^{i\theta}$, $\theta = \pm\dfrac{\pi}{2a}$. Then $|w(z)| = \exp(-r^c \cos c\theta) \leq 1$.
Hence for $a \in \partial G$,

$$\begin{aligned}\limsup_{z \to a} |F(z)| &= \limsup_{z \to a} |w(z)|^{\epsilon} |f(z)| \\ &\leq \limsup_{z \to a} |f(z)| \leq M.\end{aligned}$$

For $z \in G$,

$$\begin{aligned}|F(z)| &= |w(z)|^{\epsilon} |f(z)| \\ &\leq K \exp[-\epsilon r^c \cos(c\theta) + r^b] \\ &\to 0 < M\end{aligned}$$

when $|z| \to \infty$, since $\cos c\theta > 0$, $\theta \in \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$, $c < a$.

It follows from Theorem 2.10.3 that, $|f(z)| \leq M$ for all $z \in G$. □

We shall consider a *generalization* of Theorem 2.10.4 below. It follows from Example 2.10.1 and the hypothesis of Theorem 2.10.4 that we cannot relax the size of the angle in G or the constant b there. But this is exactly what we try to do.

Theorem 2.10.5 ("Generalisation"**).** *Assuming the hypothesis and notation in Theorem 2.10.4, but f satisfies, instead, for each $\delta > 0$, there exists $K > 0$ such that*

$$|f(z)| \leq K \exp(\delta r^a) \quad (K = K(\delta))$$

uniformly in G. Then $|f(z)| \leq M$ for all $z \in G$.

Proof. Let

$$F(z) = \exp(-\epsilon z^a) f(z)$$

in G, where $\epsilon > 0$ is a fixed constant choosing arbitrarily. We may suppose $0 < \delta < \epsilon$ since $\delta > 0$ is arbitrary. Suppose $z = r \in \mathbb{R}$, then

$$\begin{aligned} |F(z)| &= |\exp(-\epsilon z^a)||f(z)| = \exp[-\epsilon r^a \cos 0]|f(r)| \\ &\leq K \exp[(\delta - \epsilon) r^a] \\ &\to 0 \end{aligned}$$

as $r \to \infty$. Hence $|F(z)| \leq M'$ for all $z > 0$, where $M' = \sup\{|F(z)| : z > 0\}$.

We now apply Theorem 2.10.4 to the sector

$$S_1 : \theta \in \Big(0, \frac{\pi}{2a}\Big) \quad \text{and} \quad S_2 : \theta \in \Big(-\frac{\pi}{2a}, 0\Big).$$

By the hypothesis $|F| \leq M$ on the rays $\theta = \pi/2a$ and $\theta = -\pi/2a$. We conclude that $|F| \leq \max\{M, M'\}$ on S_1 and S_2. We claim that $M' \leq M$. For suppose $M' > M$, then we can find a $z = x_0 \in \mathbb{R}$ such that $|f(x_0)| = M'$. This is a contradiction to maximum modulus principle unless F reduces to a constant, and so $M' \leq M$.

We completes the proof by letting $\epsilon \to 0$ in $|f| \leq \exp(\epsilon r^a) M$. $\square$

Chapter 3

Riemann Mapping Theorem

Let G be an open set in $\mathbb{C}$. We consider families of analytic functions $\{f_n\}$, $f_n : G \to \mathbb{C}$ and ask for condition on $\{f_n\}$ so that we could extract a convergent subsequence $\{f_{n_k}\}$ which converges uniformly in a certain sense. Such consideration is of fundamental importance in complex function theory. As an application, we shall prove the celebrated Riemann mapping theorem at the end of this chapter. We shall develop the theory step by step, first to continuous functions and then to analytic and meromorphic functions. On the other hand, we shall consider functions with values in a general complete metric space Ω although $\Omega = \mathbb{C}$ or $\Omega = \widehat{\mathbb{C}}$ is our primary considerations.

3.1 Metric Space

Definition 3.1.1. Let (Ω, d) to denote a *complete metric space* with the metric d on Ω. Suppose G is an open subset of $\mathbb{C}$, then $C(G, \Omega)$ denotes the *set of all continuous functions* from G to Ω.

In order to develop $C(G, \Omega)$ to have a meaning of *compactness*, we have to clarify several issues, such as how to turn $C(G, \Omega)$ into a metric space, what are the topology on it etc.

Let us first recall some basic facts about point-set topology.

Definition 3.1.2. (i) A metric space S is *complete* if every Cauchy sequence converges;

(ii) A subset X of a metric space S is *compact* if and only if every open covering of X contains a finite subcovering. (Heine-Borel property) (See Ahlfors p.60)

Proposition 3.1.3. *Let X be a compact subset of a metric space. Then X is complete and bounded.*

Proof. Let $\{x_n\}$ be a Cauchy sequence and suppose that $x_n \not\to y$ for any $y \in X$ as $n \to \infty$. Then there exists an $\epsilon > 0$ such that $d(x_n, y) > 2\epsilon$ for infinitely many n. With the same ϵ, there exists n_0 such that $d(x_n, x_m) < \epsilon$ for $n, m > n_0$. We choose a $n > n_0$ such that $d(x_n, y) > 2\epsilon$. Then $2\epsilon < d(x_n, y) \leq d(x_n, x_m) + d(x_m, y) < \epsilon + d(x_m, y)$ for all $m > n_0$. So $d(x_m, y) > \epsilon$ for all $m > n_0$, i.e. all open balls $B(y, \epsilon)$ contains only finitely many x_n.

Let U be the union of open balls which contain only a finite number of x_n. If we suppose $\{x_n\}$ dose not converge, then U is an open covering of X all open balls contains only finitely many x_n by the preceding paragraph, or considering if any one of the open balls contain an infinite number of x_n, then $\{x_n\}$ will converge by the preceding paragraph.

Then, since X is compact, we could find a finite subcovering of the original covering. But this implies $\{x_n\}$ is a finite sequence. A contradiction. Hence x_n must converge.

Fix an $x_0 \in X$. Then $\cup_{r>0} B(x_0, r)$ is am open covering of X. Thus $X \subset B(x_0, r_1) \cup \cdots \cup B(x_0, r_m)$. Let $\tilde{r} = \max_{1\leq i \leq m} r_i$. So for any $x, y \in X$, $d(x, y) \leq d(x, x_0) + d(x_0, y) < 2\tilde{r}$ and thus X is bounded. □

In fact, a compact set is not just bounded, but totally bounded.

Definition 3.1.4. A subset X of a metric space S is *totally bounded* if for every $\epsilon > 0$, X can be covered by *finitely* many balls of radius ϵ.

Theorem 3.1.5. *A metric space is compact if and only if it is complete and totally bounded.*

Proof. It remains to prove a compact set is totally bounded in " $\Longrightarrow$ ". But this is easy, since $\cup_{x\in X} B(x,\epsilon)$ is an open cover of X. We extract a finite subcover $B(x_1,\epsilon)\cup\cdots\cup B(x_m,\epsilon)$ of X by compactness.

" $\Longleftarrow$ " We now assume X to be complete and totally bounded. Suppose X has an open covering U which does not contain any finite subcovering. Let $\epsilon_n = 1/2^n$. We know that X can be covered by finitely many $B(x,\epsilon_1)$, hence there must exist a $B(x_1,\epsilon_1)$ has no finite subcovering otherwise X must have a finite subcovering. But $B(x_1,\epsilon_1)$ is itself totally bounded (why?), hence there exists a ball $B(x_2,\epsilon_2)$ which does not admit a finite subcovering. Continuing the process, we obtain a sequence $\{x_n\}$ with the property that $B(x_n,\epsilon_n)$ has no finite subcovering and $x_{n+1}\in B(x_n,\epsilon_n)$. But then

$$\begin{aligned} d(x_n,x_{n+p}) &\le d(x_n,x_{n+1})+d(x_{n+1},x_{x+2})+\cdots+d(x_{n+p-1},x_{n+p})\\ &< \epsilon_n+\epsilon_{n+1}+\cdots+\epsilon_{n+p-1} < \frac{1}{2^{n-1}}. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and suppose $x_n\to y$. This y must belong to a $B(y,\delta)$ which belongs to an open set in the original cover U. We choose n so large that $d(x_n,y)<\delta/2$ and $\epsilon_n<\delta/2$. But $d(x,y)\le d(x,x_m)+d(x_n,y)<\delta/2+\delta/2$ whenever $d(x,x_n)<\epsilon_n<\delta/2$. That is $B(x_n,\epsilon_n)\subset B(y,\delta)\subset$ an open subset of U. A contradiction since $B(x_n,\epsilon_n)$ has no finite subcovering by construction. □

We state the following results without proofs.

Corollary 3.1.5.1. *A subset of $\mathbb{R}$ or $\mathbb{C}$ is compact is and only if it is closed and bounded.*

Theorem 3.1.6. *A metric space is compact if and only if every infinite sequence has a limit point.*

Corollary 3.1.6.1. *Any infinite sequence in a closed and bounded subset of $\mathbb{R}$ and $\mathbb{C}$ has a convergent subsequence.*

Theorem 3.1.6 can be rephrased as a metric space is compact if and only if every infinite sequence has a convergent subsequence. We called such space to have the *Bolzano-Weierstrass property.*

We shall return to the question asked at the beginning of this chapter namely how to make $C(G, \Omega)$ to have the Bolzano-Weierstrss property. But for $C(G, \Omega)$ we have another name.

Definition 3.1.7. A family $\mathcal{F} \subset C(G, \Omega)$ is *normal* if each infinite sequence in F contains a convergent subsequence converges to a function in $C(G, \Omega)$. (Note that the precise definition is not given at this stage.)

Note that this definition differs to a subset to be sequentially compact (i.e. Theorem 3.1.6) in a metric space, because we do not require the limit of the infinite sequence to be in the subset.

Our first question is how to turn $C(G, \Omega)$ into a metric space. The problem being that G is an open set and even continuous functions may not behave well on an open set. So compact sets are much more suitable for our consideration especially for an infinite sequence. We shall first investigate some fundamental point-set topology result to see how one can *approximate* an open set by compact subsets.

Proposition 3.1.8. *Suppose that G is an open set, then there exists a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$. Moreover, the sequence can be chosen so that*

(i) $K_n \subset \mathit{int}\, K_{n+1}$

(ii) for each compact subset K of G, we can find an n such that $K \subset K_n$;

(iii) every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus G$.

Proof. Let $A \subset X$ and $x \in X$, recall that the *distance* from x to A is defined by

$$d(x, A) = \inf\{d(z, a) : a \in A\}$$

, where (X, d) is any metric space.

One way to construct the compact subset K_n is to let K_n consist of all points in G at distance $\leq n$ from the origin, *and* at distance $\geq 1/n$ from the boundary ∂G. That is, we define

$$K_n = \{z \in G : |z| \leq n\} \cap \{z \in G : d(z, \mathbb{C} \setminus G) \geq 1/n\}$$

which is bounded; and being the intersection of two closed sets must itself be closed. The interior $int\, K_n$ is just $\{z \in G : |z| < n\} \cap \{z \in G : d(z, \mathbb{C} \setminus G) > 1/n\}$. Hence $int\, K_{n+1} \supset K_n$ and (i) is satisfied. It is also easy to see from the definition of K_n that $G = \cup_1^\infty K_n$.

But since also $K_{n+1} \supset int\, K_{n+1}$, we get $G = \cup_1^\infty int\, K_n$ as well. Suppose now K is a compact subset of G. $G = \cup_1^\infty int\, K_n$ implies that $\{int\, K_n\}$ forms an open cover of G and also of K. But K is compact so we can find a finite subcovering $\cup_1^N int\, K_n$ of K. Since $\cup_1^N int\, K_n \subset int\, K_N \subset K_N$, there exists an N such that $K \subset K_N$.

To prove part (iii), we need to show every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus G$. Since $K_n \subset G$ for each n, we have $\widehat{\mathbb{C}} \setminus G \subset \widehat{\mathbb{C}} \setminus K_n$. It follows that the *unbounded* component of $\widehat{\mathbb{C}} \setminus G$ must be a subset of the unbounded component of $\widehat{\mathbb{C}} \setminus K_n$ for each n. It also follows from the definition of K_n that the unbounded component of $\widehat{\mathbb{C}} \setminus K_n$ must contain $\{z : |z| > n\}$ as a subset. So for any *bounded* component D (open) of $\widehat{\mathbb{C}} \setminus K_n$, it must contain a point z such that $d(z, \mathbb{C} \setminus G) < 1/n$. By definition we can therefore find a $w \in \mathbb{C} \setminus G$ such that $|w - z| < 1/n$. But then $z \in B(w, 1/n) \subset \widehat{\mathbb{C}} \setminus K_n$. Since disks are connected and z is in the component D of $\widehat{\mathbb{C}} \setminus K_n$, $B(w, 1/n) \subset D$. If D_1 is the component of $\widehat{\mathbb{C}} \setminus G$ that contains w, then it follows that $D_1 \subset D$. □

The sequence of compact sets K_n such that $\cup K_n = G$, $K_n \subset K_{n+1}$ is called an *exhaustion* of G by compact sets.

Metric Space $C(G, \Omega)$

Suppose (S, d) is a metric space then it is easy to show that

$$d'(s,t) = \frac{d(s,t)}{1 + d(s,t)} \quad (s, t \in S)$$

is also a metric on S, and hence (S, d') is another metric space. (Verify that $d'(s,t) \leq d'(s,q) + d'(q,t)$ and $d'(s,t) = 0 \iff s = t$.)

It is also not difficult to check that d and d' induce the same topology on S i.e. a subset T is open in (S,d) if and only if it is open in (S,d'); a sequence is a Cauchy sequence in (S,d) if and only if it is a Cauchy sequence in (S,d'), etc.

Let G be an open set in $\mathbb{C}$ and according to Proposition 3.1.8, there is an exhaustion of G by the compact set $\{K_n\}$, $K_n \subset int\, K_{n+1}$, $G = \cup_1^\infty K_n$. Suppose $f, g \in C(G, \Omega)$, and we recall that $C(G,\Omega)$ denotes the set of all continuous functions $f : G \to \Omega$. We define

$$\rho_n(f,g) = \sup\{d(f(z), g(z)) : z \in K_n\}.$$

It is easy to see that ρ_n is a metric on $C(K_n, \Omega)$ for each n since (Ω, d) is a metric space. We further define

$$\begin{aligned} \rho(f,g) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \\ &\leq \sum_{n+1}^{\infty} \frac{1}{2^n} = 1 \end{aligned}$$

since $\rho_n(f,g)/(1+\rho_n(f,g)) \leq 1$. By the above discussion ρ satisfies the triangle inequality, $\rho(f,g) = \rho(g,f)$. Finally suppose $\rho(f,g) = 0$. Then $\rho_n(f,g) = 0$ and $f = g$ on K_n. But $G = \cup K_n$. So $f = g$ identically on G. So ρ is a metric on $C(G,\Omega)$ and $(C(G,\Omega), \rho)$ is a metric space. (We shall see later that $(C(G,\Omega),\rho)$ is in fact a complete metric space.)

If $f_m \to f$ in $C(G,\Omega)$ with sequence to ρ, then $f_m \to f$ uniformly on each compact subset K_n of G. (See later if this is unclear to you at this point.)

Since the construction of the metric space $(C(G,\Omega),\rho)$ depends on a particular exhaustion $\{K_n\}$, we naturally ask will $\{K_n\}$ affects the topology on $(C(G,\Omega),\rho)$ i.e. if O is open with respect to $\{K_n\}$, would O be still open with respect to another exhaustion? To do so, we require the following characterization of open sets in $(C(G,\Omega),\rho)$ in terms of the metric d on Ω.

Proposition 3.1.9. *Let ρ be the above metric defined on $C = C(G, \Omega)$.*

(i) For every $\epsilon > 0$, there exist a $\delta > 0$ and a compact set $K \subset G$ such that for $f, g \in C$, $\sup\{d(f(z), g(z) : z \in K\} < \delta$ implies $\rho(f, g) < \epsilon$.

(ii) Conversely, if we are given a $\delta > 0$ and *a compact set $K \subset G$, there exists an $\epsilon > 0$ such that for $f, g \in C$, $\rho(f, g) < \epsilon$ implies* $\sup\{d(f(z), g(z) : z \in K\} < \delta$.

Proof. (i) Let $\epsilon > 0$ be given, we choose an integer p so large such that $\sum_{p+1}^{\infty} 1/2^n < \epsilon/2$. Let $\delta > 0$ be chosen so small such that for $0 < t < \delta$, we have $t/(t+1) < \epsilon/2$. Recall that $G = \cup K_n$, now let $K = K_p$, and consider those f and g such that $\sup\{d(f(z), g(z)); z \in K\} < \delta$. But $\rho_k(f, g) \leq \rho_p(f, g)$ for $1 \leq k \leq p$. Hence

$$\begin{aligned}\rho(f, g) = \sum_{1}^{\infty} \frac{\rho_k(f, g)}{2^k(1 + \rho_k(f, g))} &= \left(\sum_{1}^{p} + \sum_{p+1}^{\infty}\right) \frac{\rho_k(f, g)}{2^k(1 + \rho_k(f, g))} \\ &\leq \sum_{1}^{p} \frac{1}{2^k} \cdot \frac{\epsilon}{2} + \sum_{p+1}^{\infty} \frac{1}{2^k} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

as required.

(ii) Suppose now a $\delta > 0$ and a compact set $K \subset G$ is given. Suppose $\cup K_n = G$ is an exhaustion of G by compact set. Then there exists an integer p such that $K \subset K_p$. Choose $\epsilon > 0$ so small such that $\dfrac{2^p\epsilon}{1 - 2^p\epsilon} < \delta.$

Suppose $\rho(f, g) < \epsilon$, then

$$\frac{\rho_p(f, g)}{2^p(1 + \rho_p(f, g)} < \epsilon,$$

i.e.

$$\rho_p(f, g) < \frac{2^p\epsilon}{1 - 2^p\epsilon} < \delta.$$

Thus $\sup\{d(f(z), g(z)) : z \in K\} \leq \rho_p(f, g) < \delta$ as required. $\square$

What is an *open ball* in (C, ρ)?
Ans: $B(f, \epsilon) = \{g : \rho(g, f) < \epsilon\}$.

What about an *open set* in (C, ρ)?
Ans: Since open set is an union of open balls, or for each f in the open set, there exists an $\epsilon > 0$ such that $B(f, \epsilon)$ is a proper subset of the open set.

We immediately obtain:

Proposition 3.1.10. *A set $U \subset (C, \rho)$ is open if and only if for each $f \in U$, there exist a compact set $K \subset G$ and a $\delta > 0$ such that*

$$U \supset \{g : d(f(z), g(z)) < \delta : z \in K\}.$$

Proposition 3.1.10 clearly indicates that any open set U of (C, ρ) is independent of the particular exhaustion $\{K_n\}$ used to define ρ_n and hence ρ. This answers the question raised before Proposition 3.1.9.

Here we again answer a claim made before Proposition 3.1.9.

Proposition 3.1.11. *Let $\{f_n\}$ be an infinite sequence in $(C(G, \Omega), \rho)$. Then $f_n \to f \in (C(G, \Omega), \rho)$ if and only if $\{f_n(z)\}$ converges to $f(z)$ uniformly on every compact subset of G.*

Proof. " $\Longrightarrow$ " Let $K \subset G$ be an arbitrary compact set. By (ii) of Proposition 3.1.8, there exists a compact set K_N in the exhaustion $\cup K_n = G$ so that $K \subset K_N \subset K_n$ for all $n \geq N$. Thus $\rho_N(f_m, f) \to 0$ as $m \to \infty$ since

$$\frac{\rho_N(f_m, f)}{2^N(1 + \rho_N(f_m, f))} \leq \sum_1^\infty \frac{\rho_N(f_m, f)}{2^N(1 + \rho_N(f_m, f))} = \rho(f_m, f) \to 0$$

as $m \to \infty$. But

$$\sup\{d(f_m(z), f(z)) : z \in K\} \leq \sup\{d(f_m(z), f(z)) : z \in K_N\} \to 0$$

as $m \to \infty$ by Proposition 3.1.9(ii). Hence $f_m \to f$ on any compact set $K \subset G$.

The converse is left as an exercise. □

So far we have not used the assumption at the beginning that Ω is a complete metric space.

Theorem 3.1.12. $(C(G, \Omega), \rho)$ *is a complete metric space.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $(C(G, \Omega), \rho)$. That is , given $\epsilon > 0$, there exists a $N > 0$ such that $\rho(f_n, f_m) < \epsilon$ whenever $n, m > N$.

By Proposition 3.1.9(ii), given any compact set $K \subset G$ and $\delta > 0$, we have

$$\sup\{d(f_n(z), f_m(z) : z \in K\} < \delta \tag{3.1}$$

whenever $n, m > N$. That is, $\{f_n(z)\}$ is a Cauchy sequence in $\mathbb{C}$. Thus $f_n(z)$ must converge to a complex number $f(z)$, say. This is true for every $z \in K$. So we obtain a function by $f : K \to \mathbb{C}, z \mapsto f(z)$.

We need to verify that $f_n \to f$ with respect to ρ and that $f \in C(G, \Omega)$. Let z be an arbitrary element of K, then there exists an $m_0 = m_0(z)$ such that $d(f_m(z), f(z)) < \delta$ for $m > m_0$.

Let $n > N$ and $z \in K$, we have

$$d(f_n(z), f(z)) \le d(f_n(z), f_m(z)) + d(f_m(z), f(z)) \le \delta + \delta = 2\delta \tag{3.2}$$

by choosing $m > m_0$ sufficiently large. It follows from (3.1) that (3.2) holds uniformly for all $z \in K$ and $n > N$. That is, $f_n \to f$ uniformly on every compact subset K of G. Proposition 3.1.10 implies that $\rho(f_n, f) \to 0$ as $n \to \infty$. Moreover since $f_n \to f$ uniformly on K, f must be continuous. Since K is arbitrary, f must be continuous on G by Proposition 3.1.8, i.e. $f \in C(G, \Omega)$. □

Recall that a family $\mathcal{F} \subset C(G, \Omega)$ is normal if every infinite sequence has a subsequence which converges to a function in $C(G, \Omega)$. Note that the limit is not required to be a member of $\mathcal{F}$. This and Theorem 3.1.6 imply that

Proposition 3.1.13. *A family $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if $\overline{\mathcal{F}}$ is compact (or $\mathcal{F}$ is relatively compact in $C(G, \Omega)$).*

We now relate the concepts of normality and total boundedness. We recall, from Theorem 3.1.5 that, a subset is compact if and only if it is complete and totally bounded. Hence Proposition 3.1.13 can be rephrased as: $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if $\overline{\mathcal{F}}$ is complete and totally bounded. $\mathcal{F}$ being a subset of $\overline{\mathcal{F}}$ is also totally bounded, i.e. given $\epsilon > 0$, $\mathcal{F} \subset \cup_1^N B(f_i, \epsilon)$ for some $\{f_1, \ldots, f_N\}$ of $\mathcal{F}$. So for every $\epsilon > 0$, there exist $f_1, \ldots, f_N \in \mathcal{F}$ such that for every $f \in \mathcal{F}$, there exist an i such that $\rho(f, f_i) < \epsilon$.

We now state this in terms of the original metric d.

Exercise.

Let $S = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R},\ \text{only finitely many } x_i \neq 0\}$. Then (S, d) is a metric space, where $d(x, y) = \max\{|x_i - y_i|\}$. Is (S, d) complete? Show that the $\delta-$neighbourhoods are not totally bounded.

Theorem 3.1.14. *A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if for every compact set $K \subset G$ and $\delta > 0$, there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that for each $f \in \mathcal{F}$, there exists an i among $\{1, \ldots, n\}$ with*

$$\sup\{d(f(z), f_i(z)) : z \in K\} < \delta. \tag{3.3}$$

Proof. Suppose $\mathcal{F}$ is normal; hence $\overline{\mathcal{F}}$ is compact and thus totally bounded. So for each $\epsilon > 0$, there exist $f_1, \ldots, f_n$ among $\mathcal{F}$ such that $\mathcal{F} \subset \cup_1^n B(f_i, \epsilon)$.

Let $K \subset G$ be compact and $\delta > 0$ be given. According to Proposition 3.1.9(ii), we may choose $\epsilon > 0$ such that for each $f \in B(f_i, \epsilon)$, we have

$$\sup\{d(f(z), f_i(z)) : z \in K\} < \delta.$$

Conversely, suppose $\mathcal{F}$ has the property (3.3), then it is clear that $\overline{\mathcal{F}}$ also has this property (3.3). By Proposition 3.1.13, it is equivalent to show that $\overline{\mathcal{F}}$ is a compact subset of (C, ρ) in order to show that $\mathcal{F}$ is normal. But $\overline{\mathcal{F}}$ is compact if and only if it is complete and totally bounded. Since $\overline{\mathcal{F}}$ satisfies (3.3), $\overline{\mathcal{F}}$ is totally bounded by Proposition 3.1.9(i). But $\overline{\mathcal{F}}$ is a closed subset of the complete metric space (C, ρ), so it must be complete also. This proves that $\mathcal{F}$ is normal. □

We have essentially established the *theory part* of Normal family. However, it is still too general to be applicable. For example, one main result is by Montel: A family of analytic functions is normal if and only if the family is locally bounded. We shall define the term *locally bounded* precisely later. It essentially means each f in the family is bounded on every ball. To make the connection, we still need to establish several *links*, some of them are very important on their own.

3.2 Arzela-Ascoli Theorem

Definition 3.2.1. A set $\mathcal{F} \subset C(G, \Omega)$ is *equicontinuous at a point* $z_0 \in G$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every $f \in \mathcal{F}$.

Similarly, $\mathcal{F}$ is *equicontinuous over a set* $E \subset G$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $|z - z'| < \delta$, $d(f(z), f(z')) < \epsilon$ whenever $z, z' \in E$ and for every $f \in \mathcal{F}$.

Remark. If $\mathcal{F} = \{f\}$, then $\mathcal{F}$ is equicontinuous at z_0 means just f is continous at z_0. And $\mathcal{F} = \{f\}$ is equicontinous over a set $E \subset G$ if f is uniformly continuous over E.

Lemma 3.2.2 (Lebesgue's Covering Lemma)**.** *Let (X, d) be a compact metric space. If $\mathcal{G}$ is an open covering of X, then there is an $\epsilon > 0$ such that for each $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x, \epsilon) \subset G$.*

Proof. Since X is compact, Theorem 3.1.6 implies that every infinite sequence has a convergent subsequence. Let $\mathcal{G}$ be an open cover of X, suppose on the contrary that there is no such $\epsilon > 0$ can be found. In particular, for every integer n there is a point $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any member G of $\mathcal{G}$. But $\{x_n\}$ must have a subsequence $\{x_{n_k}\}$ converging to $x_0 \in X$, say. There must be a $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Choose $\epsilon > 0$ such that $B(x_0, \epsilon) \subset G_0$. Let $N > 0$ such that $d(x_0, x_{n_k}) < \epsilon/2$ for all $n_k > N$. We further choose n_k such that $n_k \geq \max\{N, 2/\epsilon\}$, $y \in B(x_{n_k}, 1/n_k)$. Then $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \epsilon/2 + \epsilon/2 = \epsilon$. That is $B(x_{n_k}, 1/n_k) \subset B(x_0, \epsilon) \subset G_0 \in \mathcal{G}$. A contradiction. □

Remark. The $\epsilon > 0$ in the above lemma is known as *Lebesgue's number.*

Proposition 3.2.3. *Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Then $\mathcal{F}$ is equicontinuous over each compact subset of G.*

Proof. Let $K \subset G$ be a compact set and fix $\epsilon > 0$. $\mathcal{F}$ is equicontinuous at each point w of K means that there exists a $\delta_w > 0$ such that $d(f(w), f(w')) < \epsilon/2$, for all $f \in \mathcal{F}$ and $|w - w'| < \delta_w$.

The set $\{B(w, \delta_w) : w \in K\}$ forms an open cover of K. By Lebesgue's Covering Lemma, there exists a $\delta > 0$ such that for each $z \in K$, $B(z, \delta)$ is contained in one of these $B(w, \delta_w)$. So if $z' \in B(z, \delta)$, then $d(f(z), f(z')) \leq d(f(z), f(w)) + d(f(w), f(z')) < \epsilon/2 + \epsilon/2 = \epsilon$ for all $f \in F$ whenever $z' \in B(z, \delta)$. Hence $\mathcal{F}$ is equicontinuous over K. □

Theorem 3.2.4 (Arzela-Ascoli Theorem)**.** *A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if*

(i) $\mathcal{F}$ is equicontinuous at each point of G;

(ii) for each $z \in G$, $\overline{\{f(z) : f \in \mathcal{F}\}}$ is compact in Ω.

We shall postpone the proof of Arzela-Ascoli Theorem and give an application first. (Full detail will be given later.)

Theorem 3.2.5 (Montel's Theorem)**.** *Let $H(G)$ be a subset of $C(G, \Omega)$ of all analytic functions $f : G \to \Omega = \mathbb{C}$. (Note that $H(G)$ is complete.) Then $F \subset H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.*

In order to prove the Arzela-Ascoli Theorem, we need the following lemma.

Lemma 3.2.6 (Cantor Diagonalization Process). *Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$, and let $X = \prod_1^\infty X_n$ be their Cartesian product. Let $\xi = (x_n)$, $\eta = (y_n) \in X$. Then*

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))}$$

defines a metric on X ((X, d) is a metric space). Let

$$\xi^k = (x_n^k)_{k=1}^\infty = (x_1^k, x_2^k, x_3^k, \ldots) \in X,$$

then $\xi^k \to \xi = (x_n)$ say, in (X, d) if and only if $x_n^k \to x_n \in X_n$ for each n as $k \to \infty$.

Moreover (X, d) is compact if (X_n, d) is compact for each n.

Proof. It is left to the reader to verify that (X, d) is a metric space.

" $\Longrightarrow$ " Suppose first that $\xi^k \to \xi$ in (X, d), i.e. $d(\xi^k, \xi) \to 0$ as $k \to \infty$. Then, for each $n \in \mathbb{N}$, $d_n(x_n^k, x_n) \to 0$ as $k \to \infty$ since

$$\lim_{k\to\infty} \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq \lim_{k\to\infty} d(\xi^k, \xi) 2^n = 0.$$

" $\Longleftarrow$ " Suppose now that $d_n(x_n^k, x_n) \to 0$ for each $n \in \mathbb{N}$ as $k \to \infty$.

Given $\epsilon > 0$, we choose l so large that $\sum_{n=l+1}^\infty 1/2^n < \epsilon/2$, and choose a $\delta > 0$ so small that $\dfrac{t}{1+t} < \dfrac{\epsilon}{2}$ if $t < \delta$. Since $d_n(x_n^k, x_n) \to 0$ as $k \to \infty$, there exists a $K > 0$ such that $d_n(x_n^k, x_n) < \delta$ if $k > K$ for $1 \leq n \leq l$. Hence

$$\begin{aligned} d(\xi^k, \xi) &= \left(\sum_1^l + \sum_{l+1}^\infty \right) \frac{d_n(x_n^k, x_n)}{2^n(1 + d_n(x_n^k, x_n))} \\ &< \sum_1^l \frac{1}{2^n} \cdot \frac{\epsilon}{2} + \sum_{l+1}^\infty \frac{1}{2^n} < \epsilon \end{aligned}$$

by the choice of l and k above. Hence $d(\xi^k, \xi) \to 0$ as $k \to \infty$. This proves the first part of the lemma.

Suppose now that (X_n, d_n) is compact for each $n \in \mathbb{N}$. By Theorem 3.1.6 it suffices to prove that every infinite sequence contains a convergent subsequence. We now come to describe the famous *Cantor diagonalization process*. Let $\xi^k = (x_n^k) = (x_1^k, x_2^k, x_3^k, \ldots)$, $k = 1, 2, 3, \ldots$, be a sequence in (X, d) where each $x_n^k \in (X_n, d_n)$

Since X_1 is assumed to be compact, so $(x_1^k)_1^\infty$ has a convergent subsequence converges to a point x_1 say, in X_1 (by Theorem 3.1.6). So there is a subset of $\mathbb{N}$ denoted by $\mathbb{N}_1$ such that $k \in \mathbb{N}_1$. Similarly since X_2 is compact, we can find a subset of $\mathbb{N}_1$ denoted by $\mathbb{N}_2$ such that $x_2^k \to x_2 \in X_2$ as $k \to \infty, k \in \mathbb{N}_2$. It is to be noted that $x_1^k \to x_1$ and $x_2^k \to x_2$ as $k \to \infty, k \in \mathbb{N}_2$. By the same method we may repeat the above procedure for $X_3, X_4, \ldots$ and obtain $\mathbb{N}_2 \supset \mathbb{N}_3 \supset \mathbb{N}_4 \supset \mathbb{N}_5 \supset \cdots$.

We now let k_j be the $j-$th element in $\mathbb{N}_j$, then

$$\xi^{k_j} = (x_1^{k_j}, x_2^{k_j}, x_3^{k_j}, \ldots)$$

converges to $\xi = (x_n) = (x_1, x_2, x_3, \ldots)$ as $k_j \to \infty$ with j. To see this, we note that $\lim_{k_j\to\infty} x_n^{k_j} = x_n$ for each n, since $k_j \in \mathbb{N}_j \subset \mathbb{N}_n$ when $j \geq n$. This completes the proof. □

Now we are ready to prove the Arzela-Ascoli Theorem (Theorem 3.2.4).

Proof of Arzela-Ascoli Theorem. " $\Longrightarrow$ " Let us first assume that $\mathcal{F}$ is normal. We deal with (ii) first. So fix a $z \in G$ and define a map $F : C(G, \Omega) \to \Omega$ by $f \mapsto f(z)$. We aim to prove that F is a continuous mapping. Proposition 3.1.9(ii) implies that given $f, g \in C(G, \Omega)$ and $\epsilon > 0$, we can find a $\delta > 0$ such that

$$d(f(z), g(z)) < \epsilon \quad \text{whenever} \quad \rho(f, g) < \delta. \quad (K = \{z\})$$

The statement is equivalent to

$$d(F(f), F(g)) < \epsilon \quad \text{whenever} \quad \rho(f, g) < \delta.$$

That is, F is a continuous mapping from $C(G, \Omega)$ to Ω. Since $\mathcal{F}$ is normal, and so $\overline{\mathcal{F}}$ is compact, it follows $F(\overline{\mathcal{F}})$ is also compact in Ω.

Since this argument works for each $z \in G$, it completes the argument.

We now show that $\mathcal{F}$ is equicontinuous at each point z_0 of G. Fix $z_0 \in G$, and let $\epsilon > 0$ be given. We choose $R > 0$ such that $\overline{B(z_0, R)} \subset G$. Let $K = \overline{B(z_0, R)}$ which is a compact set. According to Theorem 3.1.14, there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that for each $f \in \mathcal{F}$, there exists a $k \in \{1, \ldots, n\}$ with

$$\sup\{d(f(z), f_k(z)) : z \in \overline{B(z_0, R)} = K\} < \frac{\epsilon}{3}.$$

We now make use of the fact that f_k is continuous at z_0. That is, there exists a $0 < \delta < R$ such that $|z - z_0| < \delta$ implies

$$d(f_k(z), f_k(z_0)) < \frac{\epsilon}{3}$$

for $1 \le k \le n$. Therefore given $\epsilon > 0$, $f \in \mathcal{F}$, there exists a $\delta > 0$ (with a suitable k) such that $|z - z_0| < \delta$ implies

$$\begin{aligned} d(f(z), f(z_0)) &\le d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

" $\Longleftarrow$ " We now prove the converse. So suppose (i) and (ii) of the theorem hold. Let $\{z_n\}$ be an rational enumeration of G (i.e. z_n has rational real and imaginary parts, $z_n \in G$). We define

$$X_n = \overline{\{f(z_n) : f \in \mathcal{F}\}} \subset \Omega$$

for every n. By (ii) of the hypothesis (X_n, d) is a compact metric space. Hence Lemma 3.2.6 implies $X = \prod_1^\infty X_n$, with the metric as defined in Lemma 3.2.6, is again a compact metric space .

For each $f \in \mathcal{F}$ we define a sequence

$$\tilde{f} = (f(z_1), f(z_2), f(z_3), \ldots) \in X.$$

Suppose $\{f_k\}$ is an infinite sequence in $\mathcal{F}$, we shall prove $f_k \to f \in C(G, \Omega)$ by proving that $\{f_k\}$ is a Cauchy sequence in the $C(G, \Omega)$. But $C(G, \Omega)$ is complete and hence $\mathcal{F}$ must be normal.

As for $\widetilde{f}$, we define

$$\widetilde{f_k} = (f_k(z_1), f_k(z_2), \ldots)$$

which is an infinite sequence in the compact metric space X. By Theorem 3.1.6 $\{\widetilde{f_k}\}$ has a convergent subsequence which we still denote by $\{\widetilde{f_k}\}$. Suppose $\lim_{k\to\infty} f_k(z_n) = w_n$, Lemme 3.2.6 implies $\lim_{k\to\infty} \widetilde{f_k} = \xi = (w_n)$.

So our strategy is to show given $\epsilon > 0$, K is an arbitrary compact subset, there exists a $J > 0$ such that

$$d(f(_k(z), f_j(z)) < \epsilon \quad \text{whenver } k, j > J$$

and for $z \in K$. Then by Proposition 3.1.9(i), $\{f_k\}$ will be a Cauchy sequence in $C(G, \Omega)$.

Since K is compact, let $R = \text{dist}(K, \partial G) > 0$, and

$$K_1 = \left\{ z \in G : d(z, K) \leq \frac{R}{2} \right\}.$$

So K_1 is again compact and $K \subset int\, K_1 \subset K_1 \subset G$.

We clearly have the values of f_k at z_n when k is large, $f_k(z_n) \sim w_n$ (k sufficiently large). We use the hypothesis that $\mathcal{F}$ is equicontinuous over K to gain control of $f_k(z)$ when z is close to one of z_n. Since $\mathcal{F}$ is equicontinuous at each point of G, it is equicontinuous over K_1. That is, with the $\epsilon > 0$ given above, we can find a $\delta > 0$ such that $\delta < \dfrac{R}{2}$ and

$$d(f(z), f(z')) < \frac{\epsilon}{3}$$

for all $f \in \mathcal{F}$ whenever $|z - z'| < \delta$ and z, $z' \in K_1$. Let $D = \{z_n\} \cap K_1 = \{\xi_i\}$. Then the open sets $\{B(\xi_i, \delta) : \xi_i \in D\}$ is an open cover of K. (See Figure 3.1)

But K is compact, so we can find a subcovering of disks with centres $\xi_1, \xi_2, \ldots, \xi_n \in D$.

Note that $\lim_{k\to\infty} f_k(\xi_i)$ exists for each i, hence there exists a $J > 0$ such that for $j, k > J$, $d(f_k(\xi_i), f_j(\xi_i)) < \dfrac{\epsilon}{3}$ for each of $i = 1, \ldots, n$.

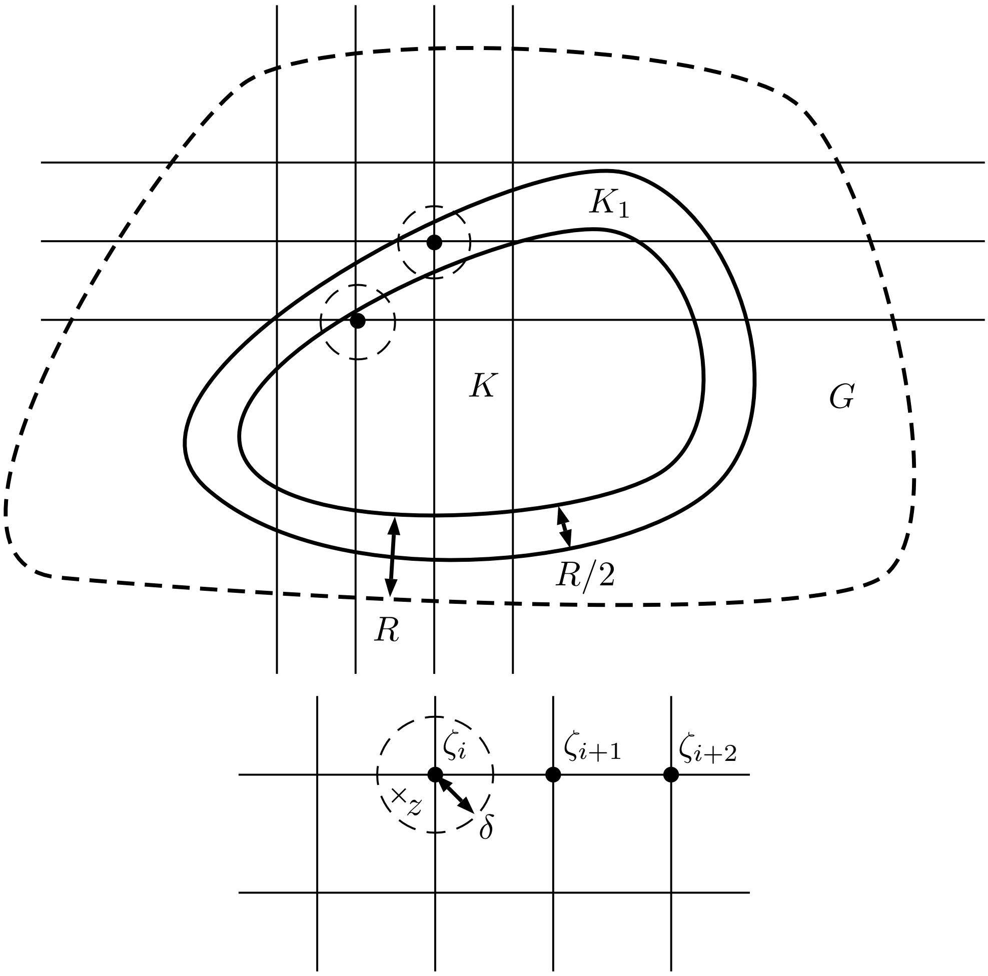

Figure 3.1: $\{B(\xi_i, \delta) : \xi_i \in D\}$

Now let z be an arbitrary point in K, $z \in B(\xi_i, \delta)$ for some i, so

$$\begin{aligned} d(f_k(z), f_j(z)) &\leq d(f_k(z), f_k(\xi_i)) + d(f_k(\xi_i), f_j(\xi_i)) + d(f_j(\xi_i), f_j(z)) \\ &< \underbrace{\frac{\epsilon}{3}}_{\text{equicontinuous}} + \underbrace{\frac{\epsilon}{3}}_{\text{convergence}} + \underbrace{\frac{\epsilon}{3}}_{\text{equicontinuous}} = \epsilon \end{aligned}$$

provided $j, k > J$. This completes the proof. □

3.3 Normal Family of Analytic Functions

Let G be an open subset of $\mathbb{C}$ and let $H(G)$ be a subset of $C(G,\mathbb{C})$ consisting of analytic functions $f : G \to \mathbb{C}$. Thus almost all basic properties of $C(G,\Omega)$ are carried over to $H(G)$. However, it is not clear that if $H(G)$ is closed (and hence complete).

Theorem 3.3.1. *Suppose $\{f_n\}$ is a sequence in $H(G)$ and $f \in C(G,\Omega)$ such that $f_n \to f$. Then $f \in H(G)$, and $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$.*

Proof. Let T be a triangle contained inside a disk $D \subset G$. Since T is a compact set, $\{f_n\}$ converges to f uniformly over T. Hence $\oint_T f = \lim \oint_T f_n = 0$ by Cauchy's Theorem. But this is true for every T, Morera's Theorem implies that f must be analytic on every disk $D \subset G$. That is, f is analytic on G.

To show $f_n^{(k)} \to f^{(k)}$, this follows from Cauchy's integral formula. Let $a \in G$. Then there exists $R > r$ such that $B(a,r) \subset B(a,R) \subset G$. Let $\gamma = \partial B(a,R)$ then Cauchy's integral formula gives, for $z \in B(a,r)$,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\,dw.$$

Let $M_n = \max\{|f_n(w) - f(w)| : w \in \gamma\}$. Then $M_n \to 0$ as $n \to \infty$ since $f_n \to f$ in $C(G,\Omega)$. Thus

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &\leq \frac{k!}{2\pi} M_n \int_0^{2\pi} \frac{1}{(R-r)^{k+1}} R\,d\theta \\ &= \frac{k! M_n R}{(R-r)^{k+1}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $f_n^{(k)} \to f^{(k)}$ uniformly on $B(a,r)$. Suppose K is an arbitrary compact set of G. Then we can find $a_1, \ldots, a_m$ such that $K \subset \cup_1^m B(a_i, r)$. So $f_n^{(k)} \to f^{(k)}$ uniformly on K and thus $\rho(f_n^{(k)}, f^{(k)}) \to 0$ in $H(G)$ by Proposition 3.1.11. □

Corollary 3.3.1.1. *(i) $H(G)$ is a complete metric space;*

(ii) *If each $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to f, then*

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note that both Theorem 3.3.1 and Corollary 3.3.1.1 have no analogues in real variable theory. Can you think of some examples?

Here is again an unusual theorem.

Theorem 3.3.2 (Hurwitz's Theorem)**.** *Let G be a region and $f_n : G \to \mathbb{C}$ are in $H(G)$. Suppose $f_n \to f \not\equiv 0$, $\overline{B(a, R)} \subset G$ and $f(z) \neq 0$ on $|z - a| = R$, then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a, R)$.*

Proof. Let us recall *Rouché's Theorem*: (see Conway p.125) Suppose f and g are analytic in a neighborhood of $\overline{B(a, R)}$ and have no zeros on $|z - a| = R$. Suppose further that

$$|f(z) + g(z)| < |f(z) + |g(z)|$$

for all $|z - a| = R$, then f and g have the same number of zeros with due count of multiplicities of multiple zeros.

Since $f(z) \neq 0$ on $|z - a| = R$, therefore

$$\delta = \inf\{|f(z)| : |z - a| = R\} > 0.$$

The hypothesis $f_n \to f$ uniformly on $|z - a| = R$ implies there is an N such that $f_n \neq 0$ for all $n \geq N$. But

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)|$$

for all n sufficiently large. We conclude the theorem by applying Rouché's theorem. □

Corollary 3.3.2.1. *Suppose G is a region and $\{f_n\} \subset H(G)$, $f_n \to f$ in $H(G)$. Suppose $f_n(z) \neq 0$ for each $z \in G$ and n, then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in G$.*

Definition 3.3.3. A family $\mathcal{F} \subset H(G)$ is *locally bounded* if each $a \in G$, there is a $M > 0$ and an $r > 0$ such that for all $f \in \mathcal{F}$,

$$|f(z)| \le M, \quad \text{for all } z \in B(a, r).$$

We immediately deduce

Proposition 3.3.4. *A family $\mathcal{F} \subset H(G)$ is* locally bounded *if and only if for each compact set $K \subset G$ there is a constant M such that*

$$|f(z)| \le M, \quad \textit{for all } f \in \mathcal{F} \textit{ and } z \in K.$$

Theorem 3.3.5 (Montel's Theorem)**.** *A family $\mathcal{F} \subset H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.*

Proof. " $\Longrightarrow$ " Suppose $\mathcal{F}$ is normal and not locally bounded. By Proposition 3.3.4, there exists a compact set $K \subset G$ and $f \in \mathcal{F}$ such that $\sup\{|f(z)| : z \in K\} = \infty$. So we can find a sequence $\{f_n\} \subset \mathcal{F}$ such that $\sup\{|f_n(z)| : z \in K\} \ge n$. But $\mathcal{F}$ is normal, so there exist a subsequence $f_{n_k} \to f$ uniformly on any compact subsets. That is $\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$.

Since $f \in H(G)$ and $|f| \le M$, $z \in K$ for some $M > 0$. But

$$\begin{aligned} n_k &\le \sup\{|f_{n_k}(z)| : z \in K\} \\ &\le \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + \sup\{|f(z)| : z \in K\} \\ &\to 0 + M \quad \text{as} \quad k \to \infty \end{aligned}$$

A contradiction.

" $\Longleftarrow$ " Suppose now that $\mathcal{F}$ is locally bounded. Then the set $\overline{\{f(z) : f \in \mathcal{F}\}}$ is clearly compact, and it remains to show $\mathcal{F}$ is equicontinuous at each point of G. Let $a \in G$ and $\epsilon > 0$ be given. Tt follows from the hypothesis that there exists an $M > 0$ and $r > 0$ such that for all $f \in \mathcal{F}$, $|f(z)| \le M$ for $z \in \overline{B(a, r)}$. Now choose a z in $|z-a| < \dfrac{r}{2}$ $(z \in B(a, r/2))$. Put $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$. Then we have, for $w \in \gamma$, $|w - z| \ge |w - a| - |a - z| > \dfrac{r}{2}$. An application of Cauchy's integral formula on γ gives

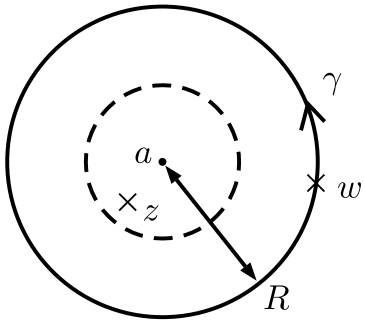

Figure 3.2: $z \in B(a, r/2)$

$$
\begin{aligned}
|f(z) - f(a)| &\leq \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(z-a)}{(w-a)(w-z)}\,dw\right| \\
&\leq \frac{1}{2\pi} 2\pi \frac{M|z-a|}{|re^{it}|\frac{r}{2}} |ire^{it}| = \frac{2M}{r}|z-a| < \epsilon \quad \text{(independent of } f\text{)}
\end{aligned}
$$

provided we choose $\delta < \min\left\{\frac{r}{2}, \frac{r}{2M}\epsilon\right\}$. Hence given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for all $f \in \mathcal{F}$ and $z \in B(a, \delta)$. □

Corollary 3.3.5.1. *$\mathcal{F} \subset H(G)$ is compact if and only if $\mathcal{F}$ is closed and locally bounded.*

Example 3.3.6. Let S be the normalized class of one-to-one conformal mapping on the unit disk with Taylor's expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

It is well-known that

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad \text{for all } |z| < 1 \text{ and } f \in S.$$

Montel's theorem implies that S is a normal family.

Theorem 3.3.7 (Another theorem of Montel)**.** *Let G be a region and $F \subset H(G)$. Suppose each $f \in \mathcal{F}$ omits same two fixed values $a, b \in \mathbb{C}$ in their range. Then $\mathcal{F}$ is normal.*

The above theorem is called as *Fundamental normality test.*

Remark (Bieberbach conjecture)**.** $|a_n| \leq n$, for all $n \geq 2$ and $f \in S$. Proved by de Branges in 1984.

3.4 Riemann Mapping Theorem

Definition 3.4.1. Two regions G_1 and G_2 in $\mathbb{C}$ are said to be *conformally equivalent* if there exists an one-to-one analytic map f with $f(G_1) = G_2$.

We note that Louville's theorem implies that $\mathbb{C}$ is not equivalent to the unit disk Δ.

Theorem 3.4.2 (Riemann Mapping Theorem)**.** *Let $G \subset \mathbb{C}$ be a simply connected region where its complement contains at least one point. Let $a \in G$. Then there is a unique one-to-one analytic mapping $f : G \to \mathbb{C}$ that satisfies $f(G) = \Delta = \{z : |z| < 1\}$ and $f(a) = 0$, $f'(a) > 0$.*

Suppose f and g are Riemann mappings for G_1 and G_2 respectively with $f(G_1) = \Delta$, $g(G_2) = \Delta$. Then $g^{-1} \circ f : G_1 \to G_2$ is an one-to-one analytic map such that $(g^{-1} \circ f)(G_1) = G_2$.

It is clear to see that conformally equivalent is an equivalence relation mapping all simply connected regions where their complements are non-empty.

Proof of Riemann Mapping Theorem. Let G be a region as assumed in the theorem. We shall divide the proof into five stages. Let $a \in G$, we define the family

$$\mathcal{F} = \{f \in H(G) : f \text{ one-to-one}, \ f(G) \subset \Delta, \ f(a) = 0, \ f'(a) > 0\}.$$

The theorem will be proved if we can find a $f \in \mathcal{F}$ such that $f(G) = \Delta$.

(A) ($\mathcal{F}$ *is non-empty*). Let $b \in \mathbb{C} \setminus G$ is non-empty by the hypothesis. Since G is simply connected, Theorem 1.10.13 asserts that we can find an analytic function g with

$$g(z) = \sqrt{z-b} = \exp\left(\frac{1}{2}\log(z-b)\right), \quad g(z)^2 = z - b.$$

It is easily observed that g is one-to-one analytic function.

Then the open mapping theorem (Theorem 1.11.4) asserts that there is a real number $r > 0$ with $B(g(a), r) \subset g(G)$. We next show $B(-g(a).r) \cap g(G) = \emptyset$. For suppose there exists a $z \in G$ with $g(z) \in B(-g(a), r)$, then

$$|g(z) - (-g(a))| < r.$$

This inequality can be written as

$$| - g(z) - g(a)| < r.$$

In other words, $-g(z) \in B(g(a), r)$. Hence there exists a $w \in G$ such that $g(w) = -g(z)$, squaring both sides yields $w - b = g(w)^2 = g(z)^2 = z - b$. So $w = z$, and $2g(z) = 0$. A contradiction. Hence $B(-g(a), r) \cap g(G) = \emptyset$.

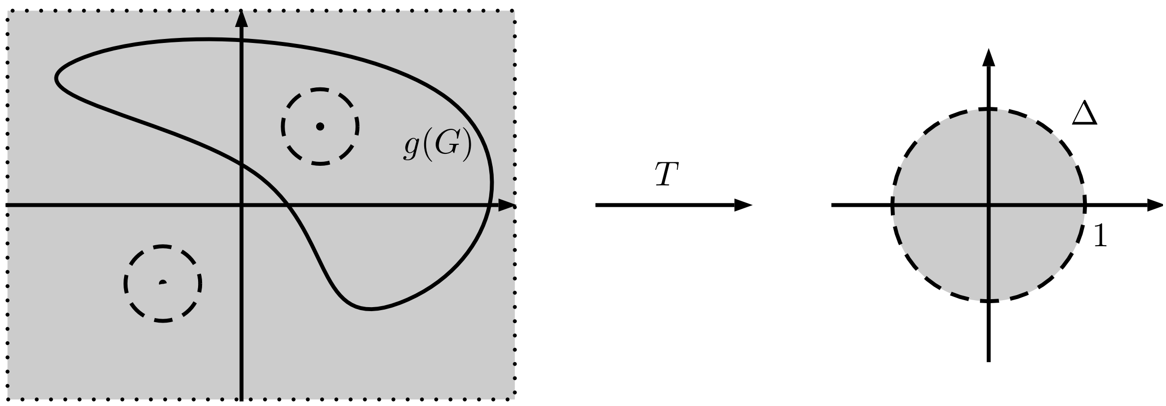

Figure 3.3: $T \circ g : G \to \Delta$

For any three points fixed on $\partial B(-g(a), r)$, we can always find a unique Möbius mapping $T(z) = \dfrac{az+b}{cz+d}(: \mathbb{C} \to \mathbb{C})$ such that $T(\partial B(-g(a), r)) = \partial\Delta$ and $T(\mathbb{C} \setminus \overline{B(-g(a), r)}) = \Delta$. Hence $T \circ g : G \to \Delta$. It remains to make $T \circ g$ a member of $\mathcal{F}$. But this is easy. Suppose $T \circ g(a) = \alpha$, then we define $\varphi_\alpha = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ which is an automorphism with $\varphi_\alpha(\alpha) = 0$. Hence $(\varphi_\alpha \circ T \circ g)(G) \subset \Delta$ with $(\varphi_\alpha \circ T \circ g)(a) = 0$.

Since each of φ_α, T and g is conformal, so is $\varphi_\alpha \circ T \circ g$. That is, $(\varphi_\alpha \circ T \circ g)'(z) \neq 0$ for all $z \in G$. We finally choose a suitable θ, so that $e^{i\theta}(\varphi_\alpha \circ T \circ g) \in \mathcal{F}$. Hence $\mathcal{F}$ is non-empty.

(B) ($\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$). Note that the zero function 0 is not conformal. Let $\{f_n\}$ be a sequence in $\mathcal{F}$. Suppose $f_n \to f$. We show either $f \in \mathcal{F}$ (not identically zero) or $f \equiv 0$. We first deduce that $f(a) = 0$ and $f'(a) \geq 0$ since the convergence is uniform on every compact subsets of G.

Let $z_1, z_2 \in G$. We choose an $r > 0$ so small that $z_1 \notin \overline{B(z_2, r)}$. Then $f_n(z) - f_n(z_1) \neq 0$ on $\overline{B(z_2, r)}$ since $f_n \in \mathcal{F}$ and so one-to-one. According to Corollary 3.3.2.1, we have

$$f_n(z) - f_n(z_1) \to f(z) - f(z_1) = \begin{cases} \neq 0, & \text{for all } z \in \overline{B(z_2, r)}; \\ \equiv 0, & \text{for all } z \in \overline{B(z_2, r)}. \end{cases}$$

If $f(z) \equiv f(z_1)$ for all $z \in \overline{B(z_2, r)}$, then $f(z) \equiv 0$ on G since $f(a) = 0$. If, however, $f(z) \neq f(z_1)$ for all $z \in \overline{B(z_2, r)}$, this means $f(z_2) \neq f(z_1)$ whenever $z_1 \neq z_2$. So f is one-to-one on G. But this implies $f'(z) \neq 0$ for each $z \in G$, and in particular $f'(a) > 0$. Hence $f \in \mathcal{F}$ as required.

(C) (*Existence of the largest* $f'(a) > 0$). Note that (C) and (D) below are related. Consider the mapping $H(G) \to \mathbb{C}$ given by $f \mapsto f'(a)$ (a is already fixed in G). By Theorem 3.3.1 the mapping $f \to$

$f'(a)$ is continuous. But $\mathcal{F}$ is locally bounded (since $|f| < 1$ for each $f \in \mathcal{F}$) and so normal. That is, $\overline{\mathcal{F}}$ is compact by Proposition 3.1.13. The image of $\overline{\mathcal{F}}$ under the above continuous mapping must also be compact in $\mathbb{C}$. Hence there exists a $f \in \overline{\mathcal{F}}$ such that $f'(a) \geq g'(a) > 0$ for all $g \in \mathcal{F}$. But $\mathcal{F} \neq \emptyset$ by (A) so there exists a non-constant $f \in F$ such that $f'(a) \geq g'(a) > 0$ for *all* $g \in \mathcal{F}$.

(D) (*The f found in (C) has $f(G) = \Delta$*). We suppose that there exists a $w \in \Delta$ such that $f(z) \neq w$ for all $z \in G$. Then the function

$$\frac{f - w}{1 - \overline{w} f} \neq 0$$

for all $z \in G$. We may define an analytic branch $h : G \to \mathbb{C}$ by

$$(h(z))^2 = \frac{f(z) - w}{1 - \overline{w} f(z)}.$$

Let

$$k(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}.$$

It is not difficult to observe that $h(G) \subset \Delta$ and $k(G) \subset \Delta$. We also have $k(a) = 0$ and $k'(z) \neq 0$. In fact, $k \in \mathcal{F}$ since

$$\begin{aligned} k'(a) &= \frac{|h'(a)|}{h'(a)} h'(a) \frac{1 - |h(a)|^2}{(1 - |h(a)|^2)^2} \\ &= \frac{|h'(a)|}{1 - |h(a)|^2} > 0. \end{aligned}$$

On the other hand, $|h(a)|^2 = \left| \dfrac{f(a) - w}{1 - \overline{w} f(a)} \right| = \left| \dfrac{0 - w}{1 - 0} \right| = |w|.$

Notice that

$$2h(z)h'(z) = \frac{d}{dz}(h(z))^2 = \frac{f'(z)(1 - |w|^2)}{(1 - \overline{w} f(z))^2}.$$

Thus

$$2h(a)h'(a) = f'(a)(1 - |w|^2).$$

Finally,

$$k'(a) = \frac{|h'(a)|}{1-|h(a)|^2} = \frac{\dfrac{f'(a|(1-|w|^2)}{2|h(a)|}}{1-|h(a)|^2}$$
$$= f'(a)\left(\frac{1+|w|}{2\sqrt{|w|}}\right)$$
$$> f'(a).$$

A contradiction. This completes the proof of (D).

(E) (*Uniqueness of* f). Suppose g also satisfies (A)-(D), then $f \circ g^{-1} : \Delta \to \Delta$ is an one-to-one, onto analytic map. Notice that $f \circ g^{-1}(0) = f(a) = 0$. So Theorem 2.9.7 shows that there is a constant $c = e^{i\theta}$ and $f \circ g^{-1}(z) = cz$ for all $z \in \Delta$. That is $f(z) = cg(z)$ for all $z \in G$ which gives $0 < f'(a) = cg'(a)$. But $g'(a) > 0$, so $c = 1$ and $f(z) = g(z)$.

Remark. The simply connectedness implies the existence of analytic square root function which is all we need to prove the conclusion.

□

Corollary 3.4.2.1. *Among the simply connected regions, there are only two equivalence classes; one consisting of* $\mathbb{C}$ *alone and the other containing proper simply connected regions.*

3.5 Boundary Correspondence of Conformal Mappings

Suppose f is a conformal mapping from the unit disc Δ to a simply connected domain D. We are concerned with under what circumstance that we could extend the f to the boundary $|z| = 1$.

Lemma 3.5.1. *Let $f : \Delta \to \mathbb{C}$ be continuous, $f(\Delta) = D$. Suppose $\lim_{z\to\xi} f(z)$ exists for every ξ with $|\xi| = 1$. Then the function $\tilde{f} : \Delta \to \overline{\mathbb{C}}$ defined by*

$$\tilde{f}(z) = \begin{cases} f(z), & |z| < 1, \\ \lim_{z\to\xi} f(z), & |\xi| = 1, \end{cases}$$

is the unique continuous extension of f to $|z| \leq 1$. Moreover, $f(\bar{\Delta}) = \bar{D}$.

The lemma provides a way to define a possible meaning of a continuous extension of f to $|z| = 1$. Interested reader can consult Palka's book [8, Chap. XI] or Ahlfors' [1].

Definition 3.5.2. A plane domain/region G is *finitely connected along its boundary* if corresponding to each point z of ∂G and each $r > 0$, there exists an $s \in (0, r)$ such that $G \cap B(z, s)$ intersects at most finitely many components of the open set $G \cap B(z, r)$.

Theorem 3.5.3 (Väisälä & Näkki)**.** *Let $f : \Delta \to \mathbb{C}$ be conformal. The f can be extended to a continuous mapping $\tilde{f}$ of $\overline{\Delta}$ onto $\overline{f(\Delta)}$ if and only if $f(\Delta)$ is finitely connected along its boundary.*

Definition 3.5.4. A plane domain/region G is *locally connected along its boundary* if corresponding to each point z of ∂G and each $r > 0$, there exists an $s \in (0, r)$ such that $G \cap B(z, s)$ intersects exactly one component of $G \cap B(z, r)$.

Theorem 3.5.5. *Let $f : \Delta \to \mathbb{C}$ be conformal. Then f can be extended to a homeomorphism $\tilde{f}$ of $\overline{f(\Delta)}$ if and only if $f(\Delta)$ is locally connected along its boundary.*

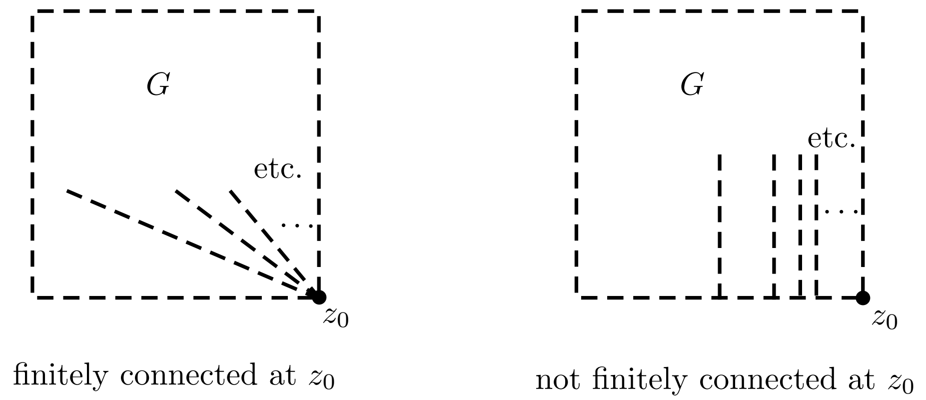

Figure 3.4: Finitely connectedness along different boundaries

Definition 3.5.6. A set J of points in $\mathbb{C}$ is called a *Jordan curve* if J is the boundary of some simple closed path. (J is compact and hence bounded.)

Theorem 3.5.7 (Jordan Curve Theorem, Jordan 1887)**.** *The complement of a Jordan curve J has exactly two components, each having J as its boundary. One of these components is a bounded set (the* inside *of J), while the other is unbounded (the* outside *of J).*

Definition 3.5.8. A domain/region $G \subset \mathbb{C}$ with the property that ∂G is a Jordan curve is called a *Jordan domain*.

Theorem 3.5.9 (Caratheodory-Osgood Theorem)**.** *A conformal mapping f of Δ onto a domain D can be extended to a homeomorphism of $\overline{\Delta}$ onto $\overline{D}$ if and only if D is a Jordan domain.*

3.6 Space of Meromorphic Functions

Definition 3.6.1. Let $M(G) \subset C(G, \widehat{\mathbb{C}})$ denote the space of meromorphic functions on the region G.

Theorem 3.6.2. *Let $\{f_n\} \subset M(G)$, $f_n \to f$ in $C(G, \widehat{\mathbb{C}})$. Then either f is meromorphic or $f \equiv \infty$. If each $\{f_n\}$ is analytic or $f \equiv \infty$.*

Corollary 3.6.2.1. *$M(G) \cup \{\infty\}$ is a complete metric space. (w.r.t. spherical metric)*

Corollary 3.6.2.2. *$H(G) \cup \{\infty\}$ is closed in $C(G, \widehat{\mathbb{C}})$.*

Example 3.6.3. $f_n(z) = n(z^2 - n)$ is analytic on $\mathbb{C}$ for each n. The $f_n \to \infty$ uniformly on each compact subset of $\mathbb{C}$. While $\{f_n'(z)\} = \{2nz\}$ is not a normal family, since $f_n'(0) = 0$ and $f_n'(z) \to \infty$ for $z \neq 0$. So $\mathcal{F}$ is normal $\not\Longrightarrow \mathbb{F}'$ is normal.

Definition 3.6.4. $\rho(f)(z) = \dfrac{2|f'(z)|}{1 + |f(z)|^2}$ is called the *spherical derivative of f*. It is defined even at the poles of f.

Recall that the chordal distance under the stereographic projection is given by

$$\begin{aligned} d(f(z_1), f(z_2)) &= \frac{2|f(z_1) - f(z_2)|}{\sqrt{(1 + |f(z_1)|^2)(1 + |f(z_2)|^2)}} \\ &\sim \frac{2|f'(z_1)|dz}{1 + |f(z_1)|^2} \quad \text{as } z_2 \to z_1. \end{aligned}$$

Let γ be the curve in $\mathbb{C}$. The length of $f(\gamma)$ under the stereographic projection on the Riemann sphere is given by

$$\int_\gamma \rho(f)(z)\,|dz|.$$

Theorem 3.6.5. *$\mathcal{F} \subset M(G)$ is normal in $C(G, \widehat{\mathbb{C}})$ if and only if $\rho(f)(z)$ is locally bounded on $\mathcal{F}$.*

3.7 Schwarz's reflection principle

Let $G \subset \mathbb{C}$ be a region, and $\bar{G} = \{\bar{z} : z \in G\}$. Clearly if a region G is symmetrical with respect to $\mathbb{R}$, then $\bar{G} = G$.

Theorem 3.7.1. *Suppose $\bar{G} = G$. We denote $G_+ = \{z \in G : \Im z > 0\}$, $G_- = \{z \in G : \Im z < 0\}$ and $G_0 = G \cap \mathbb{R}$. Suppose $f : G_+ \cup G_0 \to \mathbb{C}$ is continuous, analytic on G_+ such that f is* real *on G_0. Then*

$$g(z) := \begin{cases} f(z) & z \in G_+ \\ \overline{f(\bar{z})} & z \in G_0 \cup G_- \end{cases} \tag{3.4}$$

is analytic on G.

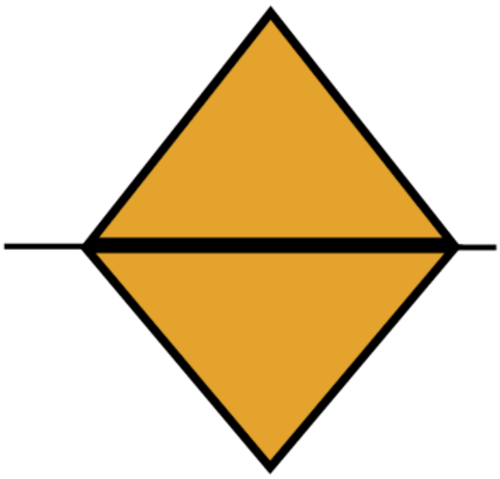

Figure 3.5: Schwarz's relfection along the $\mathbb{R}$

Remark. We note that if f is only defined on G_+ and continuous and real on G_0, then we can use the above g to extend f across to G_- by reflection. By the identity theorem applied to $\mathbb{R}$, so that such an extension is unique.

Proof. It is clear that g is analytic on G_+ and G_-. It remains to consider if g is analytic on G_0. That is, if g is analytic in a neighbourhood $B(x_0, , r)$, where x_0 real and for every $x_0 \in G_0$ and a corresponding $r > 0$. We could achieve this by proving for each triangle T within $B(x_0, r)$ the integral $\int_T g\,dz = 0$. Then g is analytic in $B(x_0, r)$ by Morera's theorem. Thus, if the triangle T lies entirely in G_+ with no intersection with G_0, then $\int_T f = 0$ since f is analytic there. Similarly if T lies entirely in G_-. So we assume that $T \cap G_0 \neq \emptyset$.

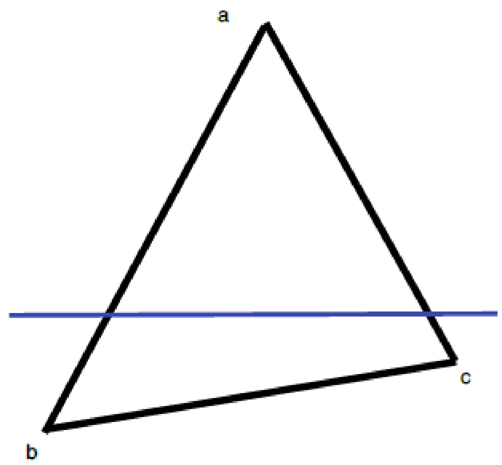

Figure 3.6: One triangle and one quadrilaterial

In general, either $T \cap G_0$ is a single point or it is a line segment. The former consideration obviously gives $\int_T f = \int_T g = 0$. The latter means that the G_0 deivdes the T into two pieces. Without loss of generality, we may assume that $G_+ \cup G_0$ contains the triangle $T' = [a, b, c, a]$ part of T and $[a, b]$ lies on G_0, leaving the quadrilateral part in $G_- \cup G_0$.

Notice that $g = f$ is uniformly continues on T' since T' is a compact set. That is, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $z, z' \in T'$, and $|z - z'| < \delta$, then

$$|f(z) - f(z')| < \varepsilon.$$

We construct a sub-triangle $T'' = [\alpha, \beta, c, \alpha]$ of T' such that one of

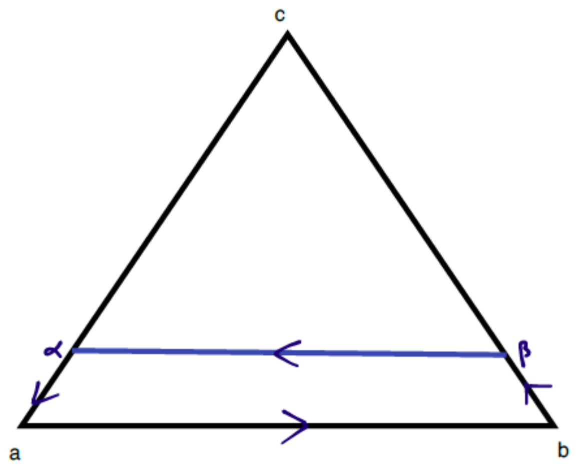

Figure 3.7: Integration along the quadrilaterial

the sides $[\alpha, \beta]$ is parallel and close to $[a, b]$ and hence to the $\mathbb{R}$. We may parametrise the horizontal line segments $[a, b]$ and $[\beta, \alpha]$ by

$$(1-t)a + tb, \quad (1-t)\alpha + t\beta, \qquad (0 \le t \le 1).$$

So now with the given $\varepsilon > 0$, we choose $\delta > 0$ so that

$$|\alpha - a| < \delta, \quad |\beta - b| < \delta \qquad (0 \le t \le 1),$$

hence

$$\begin{aligned} |(1-t)\alpha + t\beta - ((1-t)a + tb)| &\le (1-t)|\alpha - a| + t|\beta - b| \\ &\le \delta(1 - t + t) \\ &= \delta. \end{aligned}$$

This implies

$$|f[(1-t)\alpha + t\beta] - f[(1-t)a + tb)]| < \varepsilon, \qquad (0 \le t \le 1).$$

Thus

$$\begin{aligned}
\Big|\int_{[a,\,b]} f - \int_{[\alpha,\,\beta]} f\Big| \\
&= \Big|(b-a)\int_0^1 f[(1-t)a+tb) - (\beta-\alpha)\int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\le |b-a|\Big|\int_0^1 f[(1-t)a+tb) - \int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\quad + |(b-a)-(\beta-\alpha)|\Big|\int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\le |b-a|\,\varepsilon + |(b-a)-(\beta-\alpha)|\,M \\
&\le \varepsilon\ell(T') + |(b-a)-(\beta-\alpha)|\,M \\
&\le \varepsilon\ell(T') + 2\delta\,M
\end{aligned}$$

where $\ell(T')$ stands for the length of the parameter of T', and $M = \max\{|f(z)| : z \in T'\}$. The estimates of the remaining integrals are easy:

$$\Big|\int_{[a,\,\alpha]} f\Big| \le |\alpha-a|M \le M\delta, \quad \Big|\int_{[b,\,\beta]} f\Big| \le |\beta-b|M \le M\delta.$$

We finally deduce

$$\begin{aligned}
\Big|\int_{T'} f\Big| &= \Big|\int_{T''} f + \int_{[a,\,b,\,\beta,\,\alpha,\,a]} f\Big| \\
&= \Big|\int_{[a,\,b,\,\beta,\,\alpha,\,a]} f\Big| \\
&= \Big|\int_{[a,\,b]} f - \int_{[\alpha,\,\beta]}\Big| + \Big|\int_{[a,\,\alpha]} f\Big| + \Big|\int_{[b,\,\beta]} f\Big| \\
&\le \varepsilon\,\ell(T') + 4\delta\,M \\
&\le \varepsilon\,(\ell(T') + 4\,M)
\end{aligned}$$

since we may choose $\delta < \varepsilon$. This shows that $\int_{T'} f = 0$. We conclude that f is analytic in $B(x_0, r)$. Hence g is analytic on G. □

The above is called Schwarz's[1] reflection principle. We can map the above upper half-plane onto a circle and the real-axis $\mathbb{R}$ to $|z-a| = r$.

[1] H. A. Schwarz (1843-1921): advisor Karl Weierstrass

Theorem 3.7.2 (Schwarz reflection principle: second version)**.** *Let G_1 denote a simply-connected domain interior to $C_a := \{z : |z-a| = r\}$ with an arc γ on C_a such that every point of* $\text{int}(\gamma)$ *has a semi-circular neighbourhood in $B(a, r) \cap \gamma$. Let $f : G_1 \to \mathbb{C}$ be analytic and continuous on $G_1 \cup \gamma$. Suppose $f(\gamma) = \Gamma$ consists of an arc of the circle $C_b := \{w : |w-b| = R\}$. Then we can extend f to the region G_2, obtained by reflecting G_1 with respect to C_a, mapping every $z \in G_1$ to*

$$z^* = a + \frac{r^2}{\bar{z} - \bar{a}}$$

being the symmetric (inverse) point of z in G_1, and

$$f(z^*) = b + \frac{R^2}{\overline{f(z)} - \bar{b}},$$

in G_2 so that the new function is analytic in $G = G_1 \cup \gamma \cup G_2$.

Proof. Let $z \in G_1$. Then we recall that the symmetric point z^* with respect to the circle C_a is given by

$$z^* = a + \frac{r}{\bar{z} - \bar{a}}.$$

Let M_{C_a} be the Möbius transformation that maps the circle onto $\mathbb{R}$ with the notaton $z \mapsto Z$. We also denote the inverse point of $w = f(z)$ with respect to the circle$|w-b| = R$ to be

$$w^* = b + \frac{R}{\overline{f(z)} - \bar{b}}.$$

We also denote the Möbius transformation that maps the circle

$|w-b|=R$ onto $\mathbb{R}$ by M_{C_b} with the notaton $w \mapsto W$. Then we have

$$\begin{aligned} f(z^*) &= f \circ M_{C_a}(Z^*) \\ &= F(Z^*) = F(\bar{Z}) \\ &= \overline{F(\bar{Z})} \\ &= \overline{W} \ (= W^*) \\ &= M_{C_b}(w^*) \\ &= M_{C_b}(f(z)^*) \\ &= b + \frac{R^2}{\overline{f(z)} - \bar{b}}, \end{aligned}$$

where $F = f \circ M_{C_a}$. □

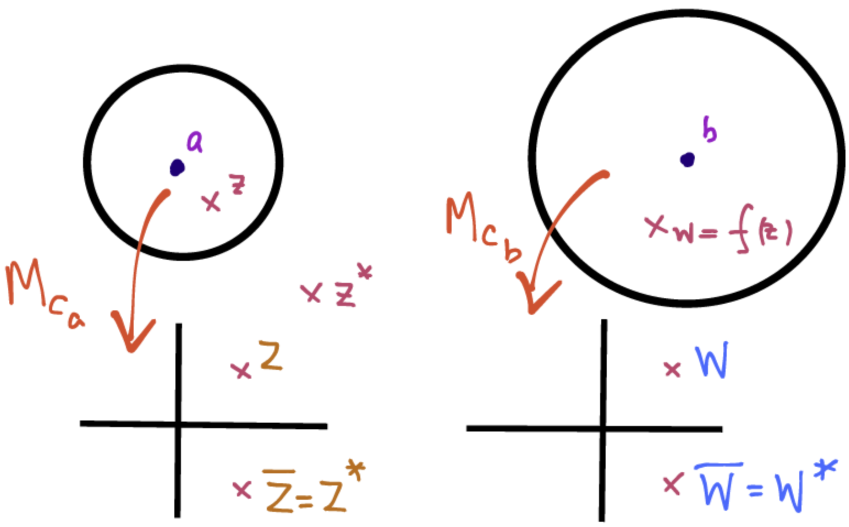

Figure 3.8: Schwarz reflection with respect to circles

One can achieve a more general reflection below.

Theorem 3.7.3. *Let G_1 and G_2 be two simply-connected domains such that*

1. $G_1 \cap G_2 = \emptyset$;

2. $\bar{G}_1 \cap \bar{G}_2 = \gamma$ *where* γ *is a smooth curve such that every interior point* $int(\gamma)$ *of* γ *has a neighbourhood lying entirely inside* $G := G_1 \cup int(\gamma) \cup G_2$.

Let $f_j(z)$ *be analytic in* G_j, *continuous in* $G_j \cup \gamma$, $j = 1, 2$ *such that for every point* $\xi \in \gamma$

$$\lim_{D_1 \ni z \to \xi} f_1(z) = h(\xi) = \lim_{D_2 \ni z \to \xi} f_2(z)$$

for some complex-valued function $h : \gamma \to \mathbb{C}$. *Then there exists an analytic function* f *in* G *such that* $f(z) = f_j(z)$ *for each* $z \in G_j$, $j = 1, 2$.

3.8 Schwarz-Christoffel formulae

The Riemann mapping theorem that we discussed is an existence result. It is rather difficult to construct explicit formulae that actually realise the theorem for even reasonable shape simple-connected regions. But a given simply connected can be approximated by polygons, so it becomes of interest to find explicit formulae for conformal of polygons.

Theorem 3.8.1 (Schwarz (1869), Christoffel (1867))**.** *Let* f *be a one-one conformal mapping that maps the upper half-plane* $\mathbb{H}^+$ *onto the interior of the a polygon* $D = [w_1, w_2, \cdots w_n]$ *with the interior angles*

$$0 < \alpha_k \pi := (1 - \nu_k)\pi < 2\pi,$$

at each of the vertices w_k, $k = 1, \cdots n$. *Suppose* $-\infty < a_1 < a_2 < \cdots < a_n < \infty$ *are real numbers on* $\mathbb{R}$ *such that* $f(a_k) = w_k$, $k = 1, \cdots n$. *Then* f *is given by*

$$\begin{aligned} f(z) &= \alpha \int_0^z \frac{dz}{(z-a_1)^{1-\alpha_1}(z-a_2)^{1-\alpha_2} \cdots (z-a_n)^{1-\alpha_n}} + \beta \\ &= \alpha \int_0^z \frac{dz}{(z-a_1)^{\mu_1}(z-a_2)^{\mu_2} \cdots (z-a_n)^{\mu_n}} + \beta \end{aligned} \tag{3.5}$$

where α, β *are two integration constants, where the* ν_k, $k = 1, \cdots, n$ *are the corresponding exterior angles.*

We recall that from elementary geometry that if the above polygon D is convex, that is, $0 < \nu_k < 1$, then

$$\sum_{k=1}^{n} \nu\pi_k = 2\pi.$$

Proof. Since the boundary of the proposed polygon D is certainly a Jordan curve, we immediately deduce from Theorem 3.5.9 that there is a conformal mapping f from the upper half-plane $\mathbb{H}^+$ onto the D such that f can be extended continuously to the real-axis $\mathbb{R}$ and $f(\mathbb{R}) = \partial D$. Let us label

$$f(a_k) = w_k, \quad k = 1, \cdots, n$$

$w_{n+1} = w_1$ that are the vertices of the polygon D. Let us denote $f(a_k,\, a_{k+1}) = L_k,\ k = 1, \cdots n$. Then we can apply Schwarz's reflection principle (Theorem 3.7.2) to a chosen $\mathbb{H}^+ \cup (a_k,\, a_{k+1})$ for some $k \in \{1, \cdots, n\}$ and reflect along $(a_k,\, a_{k+1})$ to continue f to the lower half-plane $\mathbb{H}^-$. But this corresponds to a reflection image D' obtained from D after a reflection of D along its side L_k. In fact, the $D' = f(\mathbb{H}^-)$. where we have reused the notation for the extension of f onto the domain $\mathbb{H}^+ \cup (a_k,\, a_{k+1}) \cup \mathbb{H}^-$. But the Riemann mapping theorem again asserts that there is a one-one conformal mapping $\hat{f}$ that maps $\mathbb{H}^-$ onto D'. So we may apply the Schwarz reflection principle (Theorem 3.7.2) again to reflect $\mathbb{H}^-$ along one of the other intervals $(a_{k+1},\, a_{k+2})$[2] say, to the upper half-plane $\mathbb{H}^+$. This again corresponds to the reflection of D' along its side L_{k+1} to a symmetrical region. The resulting image, which we denote by D'' is of identical shape as D where we started off, but located in a different position. The Riemann mapping theorem again implies that there is a $\tilde{f}$ that maps the upper half-plane $\mathbb{H}^+$ onto the D''. Since we can superimpose the$D$ to D'' by a translation and a rotation, so we have

$$\tilde{f}(z) = Af(z) + B \tag{3.6}$$

in $\mathbb{H}^+$ for some constants A, B[3].

[2] Any other side will do.

[3] In fact, $A = e^{i\theta_k}$.

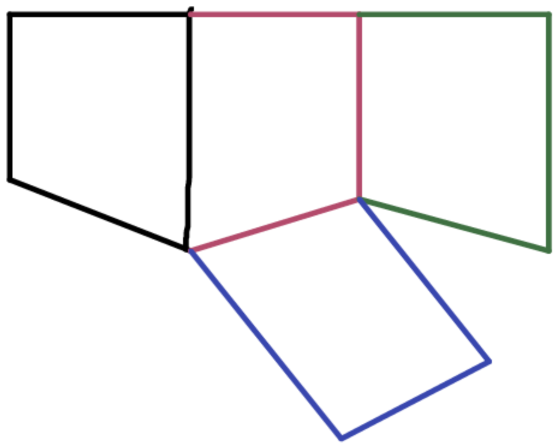

Figure 3.9: Even number of reflections

We deduce

$$\tilde{f}'(z) = Af'(z) \neq 0$$

throughout the $\mathbb{H}^+$ since f is conformal there. Moreover,

$$g(z) := \frac{\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{f''(z)}{f'(z)} \tag{3.7}$$

in $\mathbb{H}^+$. This shows that the function g is analytic in $\mathbb{H}^+$. A similar consideration leads to a similar conclusion that g is analytic in $\mathbb{H}^-$, and hence on

$$\mathbb{H}^+ \cup_{k=1}^{n} (a_k,\, a_{k+1}) \cup \mathbb{H}^-$$

by the Schwarz reflection principle. Hence g is analytic on $\mathbb{C}$ except perhaps at $a_k,\ k = 1, \cdots n$. Let us investigate what happens at these a_k. Let us consider the behaviour of f when z changing from the line segment $(a_{k-1},\, a_k)$ to $(a_k,\, a_{k+1})$. We have

$$f(z) = f(a_k) + (z - a_k)^{\alpha_k} h(z)$$

where h is analytic in a neighbourhood at $z = a_k$ and $h(a_k) \neq 0$ (imagine that z lies on a line segment slight above the $\mathbb{R}$. Thus $f(z) -$

$f(a_k)$ changes an angle $\alpha_k\pi$ from L_{k-1} to L_k when z "passes through" a_k.

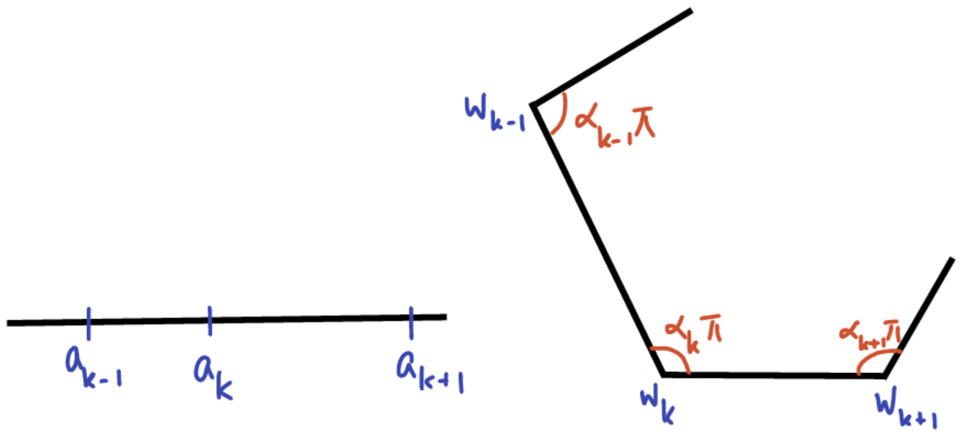

Figure 3.10: "Opening" an angle

Hence

$$\begin{aligned} f'(z) &= \alpha_k(z-a_k)^{\alpha_k-1}h(z) + (z-a_k)^{\alpha_k}h'(z) \\ &= (z-a_k)^{\alpha_k-1}\Big[\alpha_k h(z) + (z-a_k)h'(z)\Big] \\ &:= (z-a_k)^{\alpha_k-1}\phi(z), \end{aligned} \tag{3.8}$$

where $\phi(z)$ is analytic at a_k and $\phi(a_k)\neq 0$. Thus,

$$\frac{f''(z)}{f'(z)} = \frac{\alpha_k-1}{z-a_k} + \frac{\phi'(z)}{\phi(z)}.$$

This shows that the function g defined above is analytic in $\mathbb{C}$ except at the a_k, $k=1,\cdots,n$ where it has a residue α_k-1 at each simple pole a_k. Thus the function

$$\frac{f''(z)}{f'(z)} - \sum_{k=1}^{n}\frac{\alpha_k-1}{z-a_k}$$

is an entire function, and in fact including $z = \infty$. To see this we note that f and its analytic continuation are bounded at ∞, that is, we have the Laurent expansion at ∞:

$$f(z) = f(\infty) + O\Big(\frac{1}{z^m}\Big), \quad z \to \infty$$

for some integer $m \geq 1$. We deduce that g has a simple pole at ∞. This shows that

$$\frac{f''(z)}{f'(z)} - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k} \equiv 0$$

by Liouville's theorem. The above formula implies that

$$f'(z) = \alpha \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1}.$$

integrating the above formula from 0 to z yields the desired formula. $\square$

Remark. We can continue the above reflection along one $(a_k,\, a_{k+1})$ from $\mathbb{H}^+$ to $\mathbb{H}^-$ and then from $\mathbb{H}^-$ to $\mathbb{H}^+$ via another interval $(a_j,\, a_{j+1})$ any number of times for different k and j in the above construction. The upshoot is that evey time we complete a cycle we end up with a different function valued at the same point in the upper half-plane and similarly in the lower half-plane. This suggests that we should consider that these different values from different "reflected values" to be different branches of an analytic function $w = F(z)$ defined on $\mathbb{C} \setminus \cup_{k=1}^{n} (a_k,\, a_{k+1})$. The above proof shows that the $g = f''/f'(z)$ so constructed is independent of the branches chosen. In fact, we have shown that it is globally defined in $\hat{\mathbb{C}}$.

Remark. The reader may have noticed that we did not discuss the actual locations of the real numbers $-\infty < a_1 < a_2 < \cdots < a_n < \infty$ and the constants α, β in the Schwarz-Christoffel formula above. This turns out to be a difficult unsolved problems. However, we can still prescribe a_1, a_2, a_n to w_1, w_2, w_n say after a suitably chosen Möbius transformation. However, given a polygon with more than three vertices, it becomes a non-trivial problem to determine the other points $a_4, \cdots, a_n$ on the real axis. This is partly due to the fact that the Schwarz-Christoffel formula only precribes the angles α_k, but not the length of $(a_k,\, a_{k+1})$ (recall that conformal map does not preserve lenghts in general). The remaining unknowns are $a_4,\, \cdots, a_n$ real numbers and two complex numbers α and β. We deduce from the formula (3.5) that when $z = x > a_n$, then

$$\arg f'(x) = \arg \alpha,$$

and the line segment $(a_n,\, a_1)$ (via $\hat{\mathbb{C}}$) corresponds to the side $L_n = [w_n,\, w_1]$ of the polygon D. But $\arg f'(x) = \arg \alpha$ corresponds to the angle that L_n makes with the real-axis $\mathbb{R}$. This shows that $\arg \alpha$ is known. On the other hand, putting $z = a_1$ in (3.5) yields $f(a_1) = \beta$. This implies $\beta = w_1$ is therefore also known. We are left with $n - 2$ real unknown constants

$$a_4,\, \cdots, a_n,\; |\alpha|$$

to be determined. On the other hand, we have a further $n-2$ equations

$$\ell([w_k,\, w_{k+1}]) = |\alpha| \int_{a_k}^{a_{k+1}} \Big| \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} \Big|\, |dz|$$

$k = 4, \cdots, n$ (with $a_{n+1} = a_n$) that can be used to compute the $a_4,\, \cdots, a_n,\; |\alpha|$. But the determination is generally difficult if not impossible.

Example 3.8.2. Find a conformal mapping from the upper half-plane onto an equilateral triangle of side lenght ℓ.

That is the three angles of the triangle are all equal to $\alpha_k\pi = \pi/3,\; k = 1,\, 2,\, 3$. According to the last remark, the Schwarz-Christoffel formula completely determine the $a_j,\, w_j = f(a_j),\; k = 1,\, 2,\, 3$. So let us choose

$$a_1 = -1,\, a_2 = 0,\, a_3 = 1.$$

Then the SC-formula (3.5) yields

$$w = f(z) = \alpha \int_0^z \frac{dt}{(t-(-1))^{1-1/3} t^{1-1/3} (t-1)^{1-1/3}} + \beta,$$

Without loss of generality, we may choose $f(a_2) = f(0) = 0$. Hence $\beta = 0$. Moreover, we have

$$\ell = \left|\alpha \int_0^1 \frac{dt}{\sqrt[3]{t^2(t^2-1)^2}}\right|,$$

implying that

$$\alpha = \frac{\ell}{\displaystyle\int_0^1 \frac{dt}{\sqrt[3]{t^2(1-t^2)^2}}}.$$

Hence

$$f(z) = \ell\,\frac{\displaystyle\int_0^z \frac{dt}{\sqrt[3]{t^2(t^2-1)^2}}}{\displaystyle\int_0^1 \frac{dt}{\sqrt[3]{t^2(1-t^2)^2}}}$$

is the desired mapping.

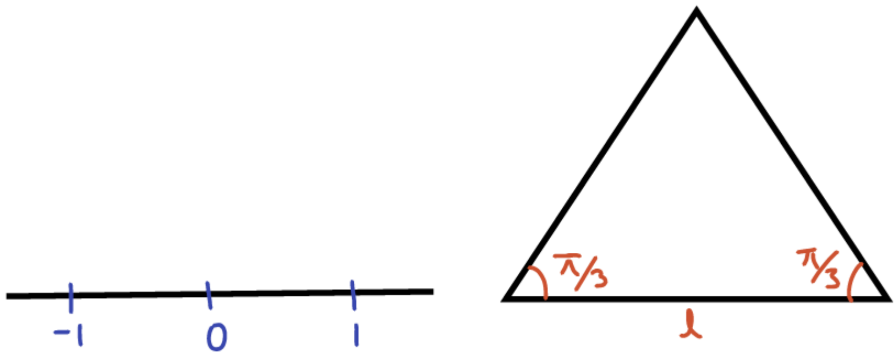

Figure 3.11: Schwarz equaliterial triangle

Exercise 3.8.1. Replace the above equilateral triangle with an isosceles right trangle with $\alpha_2 = \frac{1}{2}$, $\alpha_1 = \alpha_3 = \frac{1}{4}$, with the length of the hypotenuse ℓ.

Example 3.8.3. Construct a one-one conformal map from the upper half-plane $\mathbb{H}^+$ to a rectangle with coordinates $[-K,\, K,\, K+iK',\, -K+iK']$ for some $K>0$.

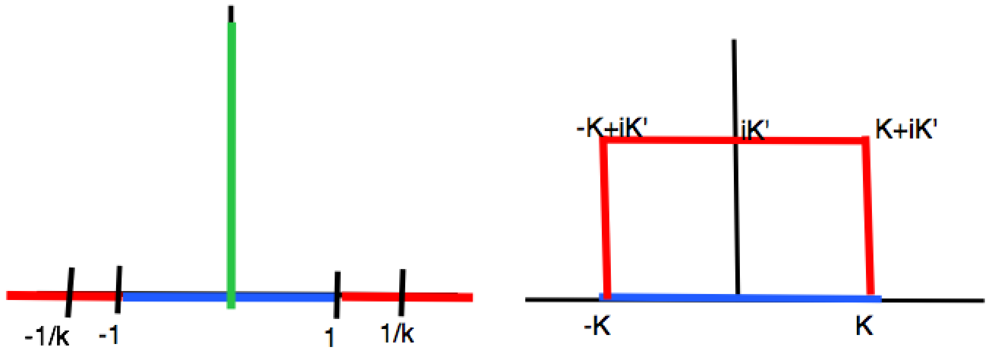

Figure 3.12: Elliptic function of the 1st kind

We recall that a slight variation of Riemann mapping theorem allows us to assert tht there is a one-one conformal mapping from the first quadrant of the $z-$plane to the rectangle with vertices $[0,\, K,\, K+iK',\, iK']$ such that the points 0, 1 and ∞ in the $z-$plane are mapped onto the points $0,\, K,\, iK$ respectively. So we have the following correspondences:

$$[0,\,1]\mapsto[0,\,K],\quad [1,\,\infty)\mapsto[K,\,K+iK']\cup[K+iK',\,iK'].$$

So there is a $0<k<1$ so that the point $z=1/k>1$ is mapped onto the point $K+iK'$. This also implies that the positive imaginary axis $\{z=iy: y>0\}$ is being mapped onto the line segment $[0,\, iK']$.

So we obtain the desired mapping $\mathbb{H}^+\to[-K,\,K,\,K+iK',\,-K+iK']$ after reflecting the Riemann mapping obtained above with respect to the imaginary axis, so that the real-axis $\mathbb{R}$ is mapped onto $[-K,\,K,\,K+iK',\,-K+iK']$, and the points $-1/k, -1,\, 1,\, 1/k$ are mapped onto the points $-K+iK',\, -K,\, K,\, K+iK'$ respectively. The explicit formula is therefore given by

$$\begin{aligned} f(z) &= \alpha \int_0^z \Big(z+\frac{1}{k}\Big)^{\frac{1}{2}-1}(z-1)^{\frac{1}{2}-1}(z+1)^{\frac{1}{2}-1}\Big(z-\frac{1}{k}\Big)^{\frac{1}{2}-1} + \beta \\ &= \alpha' \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + \beta \end{aligned}$$

Let $z = 0$ in the variable above. Then clearly $\beta = 0$. We choose the branch of square root above in accord to positive value when z lies in $(0, 1)$. But $f(1) = K$. So

$$K = \alpha' \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

This allows us to determine the constant $\alpha' > 0$ provided we know the value of k. Moreover, since $f(\frac{1}{k}) = K + iK'$, so

$$\begin{aligned} K + iK' &= \alpha' \int_0^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &= \alpha' \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &\quad + \alpha' i \int_1^{1/k} \frac{dz}{\sqrt{(z^2-1)(1-k^2z^2)}} \end{aligned}$$

since there is a change of $\arg(1-z)$, amongst all the factors of $(1-z^2)(1-k^2z^2)$, by $-\pi$. It follows that

$$K' = \alpha' \int_1^{1/k} \frac{dz}{\sqrt{(z^2-1)(1-k^2z^2)}}.$$

Let

$$z = \frac{1}{\sqrt{1-k'^2t^2}}$$

in the above integration, where $k'^2 = 1-k^2$ and $0 < k' < 1$. It is routine to check that the above substitution yields

$$K' = \alpha' \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

We therefore deduce the relationship:

$$\frac{K'}{2K} = \frac{\displaystyle\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}}{\displaystyle 2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}}. \tag{3.9}$$

We see that both the numerator and denominator have similar integrands. As k increases from 0 to 1, the integral

$$\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

increases from

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2} \quad \text{to} \quad \int_0^1 \frac{dz}{1-z^2} = +\infty.$$

That is the interval $(0, 1)$ is being mapped onto $[\frac{\pi}{2}, +\infty)$. While k increases from 0 to 1, its complementary value k' decreases from 1 to 0. So the numerator

$$2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}$$

behaves in a similar behaviour but in the opposite direction, namely, it decreases monotonically from $+\infty$ to π. We deduce that the ratio $K'/2K$, decreases monotonically, as a function of k, from $+\infty$ to 0. So there is a unique $0 < k < 1$ such that (3.9) holds for a given K and K'. This allows us to compute an approximate (and hopefully to know exactly) value of k, and hence α'.

Definition 3.8.4. The above integral where $\alpha' = 1$,

$$K(k) = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

is called the (Legendre form) of **complete elliptic integral of the first kind**.

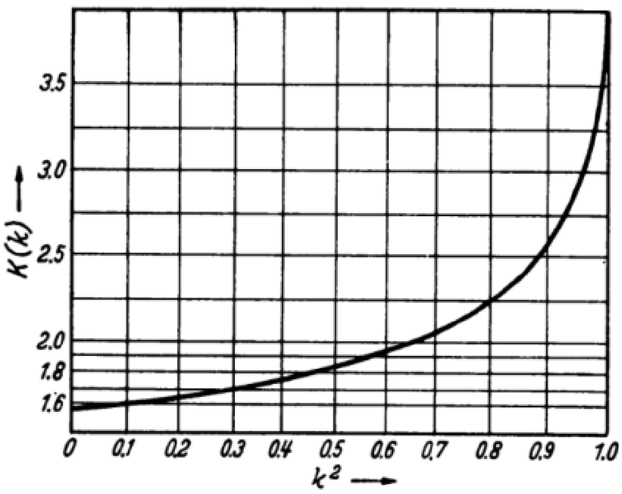

Figure 3.13: Modulus of an elliptic integral: Byrd and Friedman, p. 17

Theorem 3.8.5 (Schwarz-Christoffel: second version)**.** *Let f be a one-one conformal mapping that maps the upper half-plane $\mathbb{H}^+$ onto the interior of the a polygon $D = [w_1, w_2, \cdots w_n]$ with the interior angles*

$$0 < \alpha_k\pi := (1 - \nu_k)\pi < 2\pi,$$

at each of the given vertex w_k, $k = 1, \cdots n$. Suppose the corresponding points $-\infty < a_1 < a_2 < \cdots < a_{n-1} < \infty$ are real numbers on $\mathbb{R}$ such that $f(a_k) = w_k$, $k = 1, \cdots n-1$, and $a_n = \infty$, $f(\infty) = w_n$. Then f is given by

$$f(z) = \alpha \int_0^z \frac{dz}{(z - a_1)^{1-\alpha_1}(z - a_2)^{1-\alpha_2} \cdots (z - a_{n-1})^{1-\alpha_{n-1}}} + \beta \quad (3.10)$$

where α, β are two integration constants.

Proof. The transformation

$$z = a - \frac{1}{\zeta} \qquad (\text{i.e., } \zeta = -1/(z - a)), \quad a < a_1$$

transforms the upper half-plane $\mathbb{H}^+$ onto itself such that the $a_1 < \cdots < a_{n-1}$ are mapped onto $b_1 < \cdots < b_{n-1}$ and $a_n = \infty$ to $b_n = 0$. Hence we may apply (3.5) to

$$F(\zeta) = f\Big(a - \frac{1}{\zeta}\Big)$$

and this yields

$$\begin{aligned} F(\zeta) &= \alpha' \int_0^\zeta \frac{d\zeta}{(\zeta - b_1)^{1-\alpha_1} \cdots (\zeta - b_{n-1})^{1-\alpha_{n-1}} \zeta^{\alpha_n - 1}} + \beta' \\ &= \alpha' \int_0^\zeta \prod_{k=1}^{n-1} (\zeta - b_k)^{\alpha_k - 1} \zeta^{\alpha_n - 1}\, d\zeta + \beta'. \end{aligned}$$

Hence

$$\begin{aligned} f(z) = F(\zeta) &= \alpha' \int_{z_0}^z \prod_{k=1}^{n-1} \Big(\frac{-1}{z-a} + \frac{1}{a_k - a}\Big)^{\alpha_k - 1} \Big(\frac{-1}{z-a}\Big)^{\alpha_n - 1} \zeta^2\, dz + \beta' \\ &= \alpha' \int_{z_0}^z \prod_{k=1}^{n-1} \Big(\frac{a_k - z}{(z-a)(a_k - a)}\Big)^{\alpha_k - 1} \Big(\frac{-1}{z-a}\Big)^{\alpha_n - 1} \frac{dz}{(z-a)^2} + \beta' \\ &= \alpha'' \int_{z_0}^z \prod_{k=1}^{n-1} (z - a_k)^{\alpha_k - 1} \frac{1}{(z-a)^{\sum \alpha_k - n + 2}}\, dz + \beta' \\ &= \alpha'' \int_0^z \prod_{k=1}^{n-1} (z - a_k)^{\alpha_k - 1}\, dz + \beta'' \end{aligned}$$

since $\sum_{k=1}^n \alpha_k = n - 2$. □

Example 3.8.6. Let us apply the above formula to obtain an equilateral triangle of side length ℓ. That is, we may assume the three points on the real axis to be

$$a_1 = 0,\ a_2 = 1,\ a_3 = \infty.$$

Then $\alpha_k = \pi/3$, $k = 1, 2, 3$. The formula (3.10) yields

$$f(z) = \alpha \int_0^z \frac{dz}{(z-1)^{\frac{2}{3}}\, z^{\frac{2}{3}}} + \beta.$$

The side length ℓ can be expressed as integration of arc-lenght:

$$\begin{aligned}\ell &= |\alpha|\int_0^1 |f'(z)|\,|dz| \\ &= |\alpha|\int_0^1 |z^{\frac{1}{3}-1}(z-1)^{\frac{1}{3}-1}||dz| \\ &= |\alpha|\int_0^1 t^{\frac{1}{3}-1}(t-1)^{\frac{1}{3}-1}\,dt \\ &= |\alpha|\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3}+\frac{1}{3})} = |\alpha|\frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})},\end{aligned}$$

where $\Gamma(z)$ denotes the Euler-Gamma function (see later) and it is known that

$$B(\alpha,\,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

provided that $\Re\alpha > -1$ and $\Re\beta > -1$.

Example 3.8.7. In general, if we consider the image of 0, 1, ∞ to be the general triangle ABC with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ with side lengths a, b, c respectively, then we have the Schwarz-Christoffel map to be

$$f(z) = \int_0^z z^{\alpha-1}(1-z)^{\beta-1}\,dz$$

where we have chosen $C_1 = 1$ and C_2 so that $f(0) = 0$. Then we can compute the side length of, say,

$$c = \int_0^1 |f'(z)|\,|dz| = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

But since $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin\pi x}$, so

$$c = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1-\gamma)} = \frac{1}{\pi}\sin(\gamma\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)$$

since $\alpha+\beta+\gamma = 1$. Similarly, the side lengths of the other two sides are given by

$$a = \frac{1}{\pi}\sin(\alpha\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)$$

and

$$b = \frac{1}{\pi}\sin(\beta\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma).$$

Example 3.8.8. Apply a SC-formula to show that the conformal mapping f that maps $\mathbb{H}^+$ onto the half vertical strip:

$$-\frac{\pi}{2} < \Re(w = f(z)) < \frac{\pi}{2}; \quad \Im(w) > 0.$$

such that $-1 \mapsto -\frac{\pi}{2}$, $1 \mapsto \frac{\pi}{2}$, $\infty \mapsto \infty$ is given by

$$f(z) = \int_0^z \frac{dz}{\sqrt{1-z^2}} = \sin^{-1} z.$$

Exercise 3.8.2. Show that the formula

$$f(z) = \int_0^z \frac{dz}{\sqrt{z(1-z^2)}}$$

maps the upper half-plane $\mathbb{H}^+$ onto the interior of a square of side length

$$\frac{1}{2\sqrt{2\pi}}\Gamma\Big(\frac{1}{4}\Big)^2.$$

Exercise 3.8.3. Given a polygon D with vertices $w_1, \cdots, w_n$ and interior angles α_k $k = 1, \cdots, n$, has one of its angles, $\alpha_2 = 0$, say. See the figure below. Derive a Schwarz-Christoffel formula mapping the upper half-plane to this polygon. (Hint: Consider the polygon with $n+1$ sides constructed from that of the original polygon with a line segment drawn from new vertices w_{21} and w_{22} each on the parallel sides of D with $\alpha_2 = 0$ and perpendicular to the parallel sides. Use the Schwarz-Christoffel formula of this polygon to approximate the desired mapping).

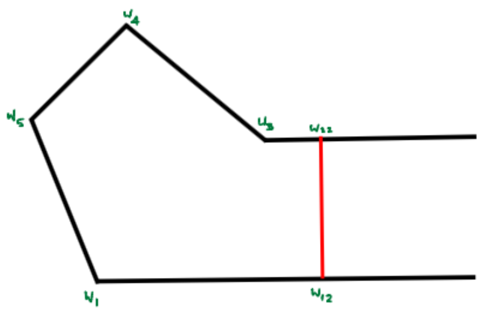

Figure 3.14: The second angle is 0

Chapter 4

Entire Functions

4.1 Infinite Products

Definition 4.1.1. Let $\{z_n\}$ be a sequence of complex numbers, and we write $p_n = \prod_1^n z_i$ to denote the nth-partial product of $\{z_n\}$. If $p_n \to p$ as $n \to \infty$. We say the infinite product exists and denote the limit by $p = \lim p_n = \prod_1^\infty z_i$. If p_n does not tend to a finite number or $p_n \to 0$, then we say $\prod_1^\infty z_i$ diverges.

Example 4.1.2. Determine the convergence of $(1+1)(1-\dfrac{1}{2})(1+\dfrac{1}{3})\cdots$.

Solution. Define

$$p_n = \begin{cases} (1+1)(1-\frac{1}{2})\,(1+\frac{1}{3})\,\cdots(1-\frac{1}{n}), & n \text{ even;} \\ (1+1)(1-\frac{1}{2})\,(1+\frac{1}{3})\,\cdots(1+\frac{1}{n}), & n \text{ odd.} \end{cases}$$
$$= \begin{cases} (1+1)\dfrac{1}{2}\dfrac{4}{3}\dfrac{3}{4}\cdots\dfrac{n}{n-1}\dfrac{n-1}{n} = 1, & n \text{ even;} \\ (1+1)\dfrac{1}{2}\dfrac{4}{3}\dfrac{3}{4},\cdots\dfrac{n-1}{n-2}\dfrac{n-2}{n-1}\dfrac{n+1}{n} = 1+\dfrac{1}{n}, & n \text{ odd.} \end{cases}$$

Hence $p_n \to 1$ as $n \to \infty$, and we conclude that

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots = 1.$$

Note that we also have

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots = \prod_{n=1}^{\infty}\left(1-\frac{(-1)^n}{n}\right).$$

□

From Example 4.1.2, we see that the last number $1-\frac{(-1)^n}{n}$ in the partial product p_n tends to one as $n \to \infty$. This is true in general. For suppose $p_n \to p$, then $z_N = \frac{\prod_1^N z_i}{\prod_1^{N-1} z_i} \to \frac{p}{p} = 1$ as $N \to \infty$. In view of this observation, it will be more convenient for us to consider infinite product of the form $\prod_1^\infty (1+a_n)$ where $a_n \to 0$ as $n \to \infty$ if the infinite product converges. We now prove a fundamental convergence criterion.

Theorem 4.1.3. *The infinite product $\prod_1^\infty (1+a_n)$ is convergent if and only if given $\epsilon > 0$, there exists an $N > 0$ such that*

$$|(1+a_{n+1})\cdots(1+a_m)-1| < \epsilon$$

for all $m > n \geq N$.

Proof. Suppose $\prod_1^\infty (1+a_i) = p$. Let p_n be the nth-partial product of $\prod_1^\infty (1+a_i)$, then $\{p_n\}$ is a Cauchy sequence in $\mathbb{C}$. That is, given $\epsilon > 0$, there exists an N such that $|p_n| > \frac{|p|}{2}$ and

$$|p_n - p_m| < \epsilon\frac{|p|}{2}$$

for all $m > n \geq N$. Thus

$$\begin{aligned} |(1+a_{n+1})\cdots(1+a_m)-1| &= |p_n|\left|\frac{p_m}{p_n}-1\right|\frac{1}{|p_n|} \\ &= |p_m - p_n|\frac{1}{|p_n|} \\ &< \epsilon\frac{|p|}{2}\frac{2}{|p|} = \epsilon \end{aligned}$$

for all $m > n \geq N$, as required.

Conversely, suppose given $1 > \epsilon > 0$, there exists an N such that for $m > n \geq N$ we have

$$\left|\frac{p_m}{p_n} - 1\right| < \epsilon.$$

Let $p_k' = \dfrac{p_k}{p_N}$ for all $k \geq N$ and N fixed, then

$$1 - \epsilon < |p_k'| < 1 + \epsilon < 2.$$

Notice that the assumption is equivalent to

$$\left|\frac{p_m'}{p_n'} - 1\right| < \epsilon.$$

That is

$$|p_m' - p_n'| < \epsilon |p_n'| < 2\epsilon$$

for all $m > n \geq N$. Hence $\{p_m'\}$ is a Cauchy sequence. So $\{p_n\}$ is also a Cauchy sequence and thus converges in $\mathbb{C}$. □

If all the a_n are positive. Then we have

Proposition 4.1.4. *Suppose all the $a_n > 0$. Then $\prod(1 + a_n)$ converges if and only if $\sum a_n$ converges.*

Proof. Suppose $\prod(1 + a_n)$ converges. By

$$a_1 + \cdots + a_n \leq (1 + a_1) \cdots (1 + a_n)$$

we conclude immediately that $\sum a_n < \infty$.

Conversely, since $1 + a < e^a$ for all $a > 0$, hence $(1 + a_1) \cdots (1 + a_n) < e^{a_1 + \cdots + a_n}$. Thus $\prod(1 + a_i)$ converges. □

Definition 4.1.5. The infinite product $\prod(1 + a_n)$ is said to be *absolutely convergent* if the product $\prod(1 + |a_n|)$ converges.

We recall from Example 4.1.2 that the infinite product

$$(1+1)(1-\frac{1}{2})(1+\frac{1}{3})\cdots$$

is not absolutely convergent. The converse, however, is definitely true.

Theorem 4.1.6. *If* $\prod(1+|a_n|)$ *converges, then* $\prod(1+a_n)$ *converges.*

Proof. Method I: The result follows immediately from the observation that

$$|(1+a_{n+1})\cdots(1+a_m)-1| \leq (1+|a_{n+1}|)\cdots(1+|a_m|)-1,$$

and by Theorem 4.1.3.

Method II: Let $p_n = \prod_1^n(1+a_i)$ and $P_n = \prod_1^n(1+|a_i|)$. Then

$$\begin{aligned} p_n - p_{n-1} &= (1+a_1)\cdots(1+a_{n-1})a_n, \\ P_n - P_{n-1} &= (1+|a_1|)\cdots(1+|a_{n-1}|)|a_n|, \end{aligned}$$

and

$$|p_n - p_{n-1}| \leq P_n - P_{n-1}.$$

Since $P_n \to \prod_1^\infty(1+|a_i|)$, we have $\sum_2^n(P_i - P_{i-1}) = P_n - P_1$ converges. But then $\sum_2^\infty(p_i - p_{i-1})$ converges absolutely by the above inequality. Hence the limit $\prod(1+a_n)$ exists. □

Theorem 4.1.7. *A product* $\prod(1+a_n)$ *is absolutely convergent if and only if* $\sum a_n$ *converges absolutely.*

Proof. If $\prod_1^\infty(1+|a_n|)$ converges then $\sum|a_n|$ must converges by Proposition 4.1.4. The converse also follows from Proposition 4.1.4. □

We deduce immediately that

Proposition 4.1.8. $\prod_1^\infty(1+a_n)$ *converges if* $\sum_1^\infty a_n$ *converges absolutely.*

We next turn to the study whether the statement "if $\prod(1+a_n) = p$, then $\sum \log(1+a_n) = \log p$" holds? Here $\log p$ is the principal logarithm.

Proposition 4.1.9. *If* $\sum \log(1+a_n)$ *converges, then* $\prod(1+a_n)$ *converges. If* $\prod(1+a_n)$ *converges, then* $\sum \log(1+a_n)$ *converges to a branch of* $\log(\prod(1+a_n))$.

Proof. Let $s_n = \sum_1^n \log(1+a_i)$ then the hypothesis implies that $s_n \to \sum_1^\infty \log(1+a_i) = s$, say, as $n \to \infty$. That is,

$$\prod_1^n (1+a_i) = e^{s_n} \to e^s, \quad n \to \infty$$

i.e.

$$\prod_1^\infty (1+a_i) = e^s.$$

Suppose now $p = \prod_1^\infty (1+a_i)$ converges. Let $p_n = \prod_1^n (1+a_i)$. Then we must have $\log \frac{p_n}{p} \to 0$ as $n \to \infty$. We decompose it as $\log \frac{p_n}{p} = S_n - \log p + h_n(2\pi i)$. Then

$$\log \frac{p_{n+1}}{p} - \log \frac{p_n}{p} = \log(1+a_{n+1}) + (h_{n+1} - h_n)2\pi i.$$

But the left side tends to zero as $n \to \infty$. Also $\log(1+a_{n+1}) \to 0$ as $n \to \infty$. Thus $h_{n+1} - h_n = 0$ for all n sufficiently large. Let it be h. Then

$$s_n - \log p + h(2\pi i) = \log \frac{p_n}{p} \to 0.$$

That is $s_n \to S = \log p - h(2\pi i)$ answering the question raised before the proposition. □

Finally, we give a criterion for the absolutely convergent product $\prod(1+a_i)$ in terms of $\sum \log(1+a_i)$.

Theorem 4.1.10. $\prod_1^\infty (1+a_i)$ *converges absolutely if and only if* $\sum_1^\infty \log(1+a_i)$ *converges absolutely.*

Proof. The result follows immediately from Theorem 4.1.7 and the limit

$$\lim_{z\to 0}\frac{\log(1+z)}{z}=1.$$

It suffices to show $\sum|\log(1+a_i)|$ and $\sum|a_i|$ converges and diverges together. The details is left to the reader. □

Example 4.1.11. 1. $\prod_1^\infty\left(1+\frac{1}{n^\alpha}\right)$ converges whenever $\alpha>1$.

2. $\prod_1^\infty\left(1-\frac{2}{n(n+1)}\right)=\frac{1}{3}$.

3. $\prod_1^\infty\left(1+\frac{x}{n}\right)=\begin{cases}+\infty, & x>0\\ 0, & x<0.\end{cases}$

4. $\prod(1+z^n)$ is absolutely convergent for every $|z|<1$.

5. If $\sum a_n$ converges absolutely, then $\prod(1+a_nz)$ converges absolutely for every z. For example $\prod\left(1+\frac{z}{n^2}\right)$ converges absolutely.

6. If $\sum a_n$ and $\sum|a_n|^2$ are convergent, then $\prod(1+a_n)$ is convergent (Hint: $\log(1+a_n)=a_n+O(|a_n|^2)$).

7. Suppose $a_{2n-1}=\frac{-1}{\sqrt{n+1}}$, $a_{2n}=\frac{1}{\sqrt{n+1}}+\frac{1}{n+1}+\frac{1}{(n+1)\sqrt{n+1}}$. Then $\prod(1+a_n)$ converges, but $\sum a_n$ and $\sum a_n^2$ both diverge.

8. If a_n is real and $\sum a_n$ is convergent, then the product $\prod(1+a_n)$ converges or diverges to zero according to $\sum a_n^2$ converges or diverges respectively.

4.2 Infinite Product of Functions

It is not difficult to see that the main results from the previous section can be generalized to infinite product of functions.

Let G be a region in $\mathbb{C}$, and $\{f_n\}$ be a sequence of analytic functions defined on G.

Theorem 4.2.1. *Let $\{f_n\} \subset H(G)$ and $\sum_1^\infty |f_n|$ converges uniformly on every compact subsets of G. Then the infinite product $\prod(1 + f_n(z))$ converges uniformly to an analytic function f on G , i.e. $\prod_1^\infty(1+f_n) = f \in H(G)$.*

Moreover, f has a zero at those, and only those points of G at which at least one of the factors is equal to zero. The order of such a zero is finite and is equal to the sum of the orders to which those factors vanish there.

Proof. Let K be any compact subset of G. Since $\sum |f_n|$ converges uniformly on K, there exists a $M > 0$ such that $\sum_1^\infty |f_n(z)| < M$ for all $z \in K$. Thus, for any $n \in \mathbb{N}$, we have

$$(1 + |f_1(z)|) \cdots (1 + |f_n(z)|) \leq e^{|f_1(z)|+\cdots+|f_n(z)|} < e^M$$

for all $z \in K$. Set $P_n(z) = \prod_1^n(1 + |f_i(z)|)$. Then

$$\begin{aligned} P_n(z) - P_{n-1}(z) &= (1 + |f_1(z)|) \cdots (1 + |f_{n-1}(z)|)|f_n(z)| \\ &< e^M|f_n(z)| \end{aligned}$$

for all $n \geq 2$ and all $z \in K$. Hence

$$\begin{aligned} \left|\prod_1^n(1 + f_i(z)) - (1 + f_1(z))\right| &\leq \sum_{i=2}^n(P_i(z) - P_{i-1}(z)) \\ &< e^M \sum_{i=2}^n |f_i(z)| < e^{2M} \end{aligned}$$

for all $z \in K$. So we deduce that $\prod_1^\infty(1 + f_i(z)) - (1 + f_1(z))$ converges uniformly on K. But $H(G)$ is complete, so $\prod_1^\infty(1+f_i(z))-(1+f_1(z))$ is analytic and hence $\prod_1^\infty(1 + f_i(z))$ is analytic on K. But K is arbitrary, so $\prod_1^\infty(1 + f_i(z))$ is analytic on G.

Since $\sum |f_n(z)| < \infty$ for each $z \in K$, there exists an $N \in \mathbb{N}$ such that $\sum_n^\infty |f_i(z)| < \dfrac{1}{2}$ for all $n > N$. Suppose now that $z \in K$ and

$f(z) = 0$. It follows that $\Pi_{N+1}^{\infty}(1 + f_i(z)) \neq 0$ on K and hence the order of the zero is equal to the sum of orders of those factors (i.e. $\Pi_1^N(1 + f_i(z))$ vanishes there). $\square$

Remark. It is clear from the proof of Theorem 4.1.10 that $\sum |f_i(z)|$ and $\sum |\log(1 + f_i(z))|$ converge and diverge together. So we could rephrase Theorem 4.2.1 such that the hypothesis $\sum |f_i| < \infty$ is replaced by $\sum |\log(1 + f_i)| < \infty$. It turns out that both conditions are useful in applications.

4.3 Weierstrass Factorization Theorem

Suppose f is entire and non-vanishing. Then we can write f as e^g where g is an entire function (see Theorem 1.10.13). If f has only a finite number of zeros (can be repeated) $z_1, z_2, \ldots, z_n$, say, then $\dfrac{f}{(z - z_1)\cdots(z - z_n)}$ is entire and non-vanishing. Thus we have $f(z) = \Pi_{i=1}^n(z - z_i)e^g$. A natural question is for an representation for f as above when f has an infinite number of zeros. We can also view the above question as an interpolation problem: Given $z_1, z_2, \ldots, z_n, \ldots$ and $w_1, w_2, \ldots, w_n, \ldots$, find an entire function f such that $f(z_i) = w_i$ for $i = 1, 2, 3, \ldots$. If $w_i = 0$ for $i = 1, 2, \ldots$, then our question become a special case of the interpolation problem.

Thus a natural guess of an answer of the interpolation is the *function*

$$f(z) = z^m e^{g(z)} \prod_1^{\infty}\left(1 - \frac{z}{z_i}\right).$$

But it is unclear of whether such a *function* exists since the infinite product may diverge. According to Proposition 4.1.8, $\displaystyle\prod_1^{\infty}\left(1 - \frac{z}{z_i}\right)$ converges if $\displaystyle\sum\left|\frac{z}{z_i}\right| = |z|\sum\left|\frac{1}{z_i}\right|$ is convergent for every z. Thus if $z_n = n^2$, then $\displaystyle\sum\left|\frac{1}{z_n}\right| = \sum\frac{1}{n^2} < \infty$ and so f has the above factorized form.

Weierstrass was able to construct a convergent-producing factor called *primary factor* so that a factorization of f always exist regardless of the given sequence $\{z_n\}$.

Definition 4.3.1. Let $p \geq 0$ be an integer. We define the *Weierstrass primary factor* by

$$E_p(z) = E(z,p) = \begin{cases} (1-z)\exp\left(z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots + \dfrac{z^p}{p}\right), & p \geq 1; \\ 1-z, & p = 0. \end{cases}$$

Theorem 4.3.2 (Weierstrass Factorization Theorem)**.** *Let $\{a_n\}$ be a sequence of complex numbers with $\lim a_n = \infty$. Then there exists an entire function f with $f(a_n) = 0$ for all n and f has a zero at $z = 0$ of order $m \geq 0$. In fact, f is given by*

$$\begin{aligned} f(z) &= z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big) \\ &= z^m e^{g(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{a_n}\Big) \exp\Big[\frac{z}{a_n} + \cdots + \frac{1}{p_n}\Big(\frac{z}{a_n}\Big)^{p_n}\Big], \end{aligned}$$

where $g(z)$ is an entire function and $\{p_n\}$ is any non-negative integer sequence for which

$$\sum_{1}^{\infty} \Big(\frac{r}{|a_n|}\Big)^{p_n+1} < \infty$$

for each $r > 0$.

Remark. (i) $p_n = n - 1$ always satisfy the hypothesis. The idea is to choose $\{p_n\}$ as simple as possible.

(ii) The above *factorization* has already taken care of the multiplicity of $\{a_n\}$.

Lemma 4.3.3. *Let p be a non-negative integer. Then*

(i) $|E_p(z) - 1| \leq |z|^{p+1}$ *if* $|z| \leq 1$*;*

(ii) $|\log E_p(z)| < \frac{k}{k-1}|z|^{p+1}$ *if* $|z| < \frac{1}{k}$ *and* $k > 1$;

(iii) $|E_p(z) - 1| < 6|z|^{p+1}$, *if* $|z| < \frac{1}{2}$.

Proof. (i) We expand E_p into a power series:

$$E_p(z) = 1 + \sum_1^\infty a_k z^k$$

where all the a_k are real. Differentiating both sides yields

$$E_p'(z) = \sum_1^\infty k a_k z^{k-1}. \tag{4.1}$$

But the left side is equal to

$$\begin{aligned} E_p'(z) &= [(1-z)(1+z+z^2+\cdots+z^{p-1}-1]\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\ &= [(1-z^p)-1]\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right). \end{aligned} \tag{4.2}$$

By comparing the coefficients of (4.1) and (4.2), we deduce $a_1 = \cdots = a_p = 0$ and $a_k \le 0$ for the rest of k. Thus for $|z| \le 1$,

$$\begin{aligned} |E_p(z) - 1| &= \left|\sum_{p+1}^\infty a_k z^k\right| = |z^{p+1}|\left|\sum_0^\infty a_{p+k+1} z^k\right| \\ &\le |z|^{p+1}\sum_0^\infty |a_{p+k+1}| = -|z|^{p+1}\sum_0^\infty a_{p+k+1} = |z|^{p+1} \end{aligned}$$

since $0 = E_p(1) = 1 + \sum_{p+1}^\infty a_k$ and so $\sum |a_k| = -\sum a_k = 1$.

(ii) Since

$$\begin{aligned}
|\log E_p(z)| &= \left|\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right| \\
&= \left|\left(-z - \frac{z^2}{2} - \cdots\right) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right| \\
&= \left|-\frac{1}{p+1}z^{p+1} - \frac{1}{p+2}z^{p+2} - \cdots\right| \\
&\le |z|^{p+1}\left(\frac{1}{p+1} + \frac{1}{p+2}|z| + \frac{1}{p+3}|z|^2 + \cdots\right) \\
&\le |z|^{p+1}(1 + |z| + |z|^2 + \cdots) \\
&< |z|^{p+1}\left(1 + \frac{1}{k} + \frac{1}{k^2} + \cdots\right) = \frac{k}{k-1}|z|^{p+1}.
\end{aligned}$$

(iii) By the definition of $E_p(z)$,

$$\begin{aligned}
|E_p(z) - 1| &= \left|(1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) - 1\right| \\
&\le \exp\left(\frac{|z|^{p+1}}{p+1} + \frac{|u|^{p+2}}{p+2} + \cdots\right) - 1 \\
&\quad \left(\text{by } 1 - z = \exp\left(-z - \frac{z^2}{2} - \cdots\right)\right) \\
&\le \exp[|z|^{p+1}(1 + |z| + |z|^2 + \cdots)] - 1 \\
&= \exp\left(|z|^{p+1}\frac{1}{1-|z|}\right) - 1 \\
&< \exp(2|z|^{p+1}) - 1 \\
&\le 2|z|^{p+1}\exp(2|z|^{p+1}) \quad \because e^x - 1 \le xe^x \text{ for } x \ge 0 \\
&< 2|z|^{p+1}e^1 < 6|z|^{p+1}.
\end{aligned}$$

□

Now we can prove Theorem 4.3.2.

Proof of Theorem 4.3.2. Let $\{a_n\}$ be the given sequence of complex numbers such that $a_n \to \infty$ as $n \to +\infty$. Thus given any $|z| = r$, we can find an $N > 0$ such that $|a_n| > 2r$, or $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ for $|z| < r$. Thus Lemma 4.3.3(iii) gives

$$\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| < 6\left|\frac{z}{a_n}\right|^{p_n+1} < \left(\frac{r}{|a_n|}\right)^{p_n+1}$$

for $n > N$ and $|z| < r$. It follows from the hypothesis that the sum $\sum(E_{p_n}(z/a_n) - 1)$ converges uniformly and absolutely on any compact subset of $B(0, r)$. Theorem 4.2.1 implies that the infinite product $\prod_1^\infty E_{p_n}(z/a_n)$ converges to an analytic functions in $B(0, r)$. But r is arbitrary, so it is actually an entire function.

Suppose f is an entire function with zeros given by $\{a_n\}$, then $f/\prod_1^\infty E_{p_n}(z/a_n)$ is zero-free. Hence we can find an entire function g such that

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

where $m \geq 0$ is an integer.

It is easy to see that we can always find the sequence $\{p_n\}$ by choosing $p_n = n - 1$. Since $\sum\left|\frac{r}{a_n}\right|^{p_n+1} < \sum\left(\frac{1}{2}\right)^n < +\infty$ for each r. This completes the proof of the theorem.

Alternatively, we can prove the theorem by applying Lemma 4.3.3(ii). We choose $k > 1$ and N so large that $|a_n| > kr$ for $n > N$ and $|z| < r$. Thus

$$\left|\log E_{p_n}\left(\frac{z}{a_n}\right)\right| < \frac{k}{k-1}\left|\frac{z}{a_n}\right|^{p_n+1} < \frac{k}{k-1}\left(\frac{1}{k}\right)^{p_n+1}.$$

Choose $p_n = n - 1$ again implies $\sum|\log E_{p_n}(z/a_n)|$ converges uniformly. The discussion in the remark after Theorem 4.2.1 shows that $\prod_1^\infty E_{p_n}(z/a_n)$ converges to an entire function. You may fill in the details as an exercise. □

Remark. Note that some authors will phrase Theorem 4.3.2 as: *Let* $\{a_n\}$ *be a sequence of complex numbers with* $\lim a_n = \infty$, *then there exists* $\{p_n\}$ *such that the following* f *is an entire function*

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

where g *is an entire function.*

This is because we can always obtain the estimate, as in the proof,

$$\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| < 6\left(\frac{1}{2}\right)^{p_n+1}.$$

Hence any increasing non-negative integer sequence $\{p_n\}$ will make $\sum |E_{p_n}(z/a_n) - 1|$ converges uniformly.

Proposition 4.3.4. *Suppose* G *is an open set and* $\{f_n\} \subset H(G)$ *such that* $f = \prod f_n$ *converges in* $H(G)$. *Then*

(a) $f' = \sum_{k=1}^\infty \left(f_k' \prod_{n\neq k} f_n\right)$

(b) $\dfrac{f'}{f} = \sum_{k=1}^\infty \dfrac{f_k'}{f_k}$

on any compact subset $K \subset G$ *provided* $f \neq 0$ *on* K. *(See Conway p.174)*

Proof. (*Sketch*) For (a), Consider $F_k' = \sum_{i=1}^k \left(f_i' \prod_{n\neq i} f_n\right) = \left(\prod_1^k f_i\right)'$. By Theorem 3.3.1, since we have $F_k \to f$, then $\left|f' - \sum_{i=1}^k \left(f_i' \prod_{n\neq i} f_n\right)\right|$ converges in $H(G)$ and $f' = \sum_{i=1}^\infty \left(f_i' \prod_{n\neq i} f_n\right)$ as required.

For (b), let K be an arbitrary compact set. Hence $|f| > a > 0$ for all $z \in K$. Then

$$\left|\frac{f'}{f} - \frac{F_k'}{F_k}\right| = \left|\frac{f'F_k - fF_k'}{fF_k}\right| \to 0 \quad \text{as } k \to \infty$$

since $F_k' \to f'$ and $F_k \to f$ in $H(G)$. □

4.4 Factorization of Sine Function

We define

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots .$$

Since this series is convergent uniformly and absolutely on any closed disk centred at the origin, we could rearrange the terns so that

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It follows that each zero of $\sin(\pi z)$ is simple. In fact, the zeros are real and equal to $0, \pm 1, \pm 2, \ldots, \pm n, \ldots$. Let a_k be the non-zero zeros. Then

$$\sum_{k=1}^{\infty} \left(\frac{r}{|a_k|}\right)^2 = \sum_{\substack{-\infty \\ (n\neq 0)}}^{\infty} \left(\frac{r}{n}\right)^2 = r^2 \sum_{\substack{-\infty \\ (n\neq 0)}}^{\infty} \frac{1}{n^2}$$

always converge for each $r > 0$ by choosing $\{p_n\} = \{1\}$. It follows from the Weierstrass factorization theorem (Theorem 4.3.2) that

$$\begin{aligned} \sin \pi z &= z e^{g(z)} \prod_{-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \\ &= z e^{g(z)} \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \end{aligned}$$

for some entire function $g(z)$. We deduce from Proposition 4.3.4 that

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z) + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2}$$

converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

We now need a standard contour integration result which can be found p.122 in Conway :

$$\pi \cot \pi z = \frac{1}{z} + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2} \quad \text{for } z \in \mathbb{Z}.$$

Hence g is identically a constant. In fact $g(z) = \log \pi$ because $\dfrac{\sin \pi z}{\pi z} \to 1$ as $z \to 0$. We finally obtain

$$\sin \pi z = \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right).$$

4.5 Introduction to Gamma Function

We shall only introduce the definition of Gamma function and leave its more difficult asymptotic expansion to a later chapter when time allows. To begin with, let us consider the following entire function defined by

$$G(z) = \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

The infinite product G converges to an entire function in $H(\mathbb{C})$ with only negative zeros $-1, -2, -1, \ldots$. Similarly the function $G(-z)$ has similar properties except the zeros are $1, 2, 3, \ldots$. It is readily seen that $\pi z G(z) G(-z) = \sin \pi z$.

Consider now $G(z-1)$ which has the same zeros as $G(z)$ plus a new zero at the origin. Hence there exists an entire function $\gamma(z)$ such that

$$G(z-1) = z e^{\gamma(z)} G(z).$$

We shall determine $\gamma(z)$. To do so, we take the logarithmic derivative on both sides:

$$\sum_1^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right).$$

Rewrite

$$\begin{aligned}\sum_1^\infty\left(\frac{1}{z-1+n}-\frac{1}{n}\right)&=\frac{1}{z}-1+\sum_2^\infty\left(\frac{1}{z-1+n}-\frac{1}{n}\right)\\&=\frac{1}{z}-1+\sum_1^\infty\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\&=\frac{1}{z}-1+\sum_1^\infty\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_1^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&=\frac{1}{z}-1+\sum_1^\infty\left(\frac{1}{z+n}-\frac{1}{n}\right)+1=\frac{1}{z}+\sum_1^\infty\left(\frac{1}{z+n}-\frac{1}{n}\right).\end{aligned}$$

This implies that $\gamma'(z)=0$ and $\gamma(z)=\gamma$ is a constant. Putting $z=1$ into $G(z-1)=e^\gamma zG(z)$ gives $1=G(0)=e^\gamma G(1)$. That is

$$e^{-\gamma}=G(1)=\prod_1^\infty\left(1+\frac{1}{n}\right)e^{-1/n}.$$

The nth-partial product is

$$(n+1)e^{-(1+1/2+1/2+\cdots+1/n)}$$

and this implies

$$\begin{aligned}\gamma&=\lim\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log(n+1)\right)\\&=\lim\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n-\log\left(1+\frac{1}{n}\right)\right)\\&=\lim\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)-\lim\log\left(1+\frac{1}{n}\right)\\&=\lim\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).\end{aligned}$$

The number $\gamma(\approx 0.57722)$ is called *Euler's constant* whose numerical value is still unknown. In fact, it is still undecided whether γ is rational or irrational.

Using $H(z)=e^{\gamma z}G(z)$ on $G(z-1)=e^\gamma zG(z)$ gives us a new relation:

$$H(z-1)=zH(z).$$

A further change of notation $\Gamma(z) = \dfrac{1}{zH(z)}$ gives us the right order: $\Gamma(z-1) = \dfrac{\Gamma(z)}{z-1}$ or

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for } z \neq -1, -2, \ldots.$$

Of course

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_1^\infty \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

is now an infinite product of meromorphic functions. The convergence can easily be justified by considering compact sets K in $\mathbb{C} \setminus \{-1, -2, \ldots\}$. $\Gamma(z)$ is call *(Euler's) gamma function.* Clearly $\Gamma(1) = 1$ and we deduce from the functional equation above that $\Gamma(2) = \Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2!, \ldots, \Gamma(n) = (n-1)!$. Thus the gamma function can be considered as a generalization of the factorial. Also

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

which gives $\Gamma(1/2) = \sqrt{\pi}$.

One can show that

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \quad \text{(Mellin transform)}$$

We shall extend Weierstrass factorization theorem to an arbitrary region.

Theorem 4.5.1. *Let G be a region and $\{a_j\} \subset G$ is a sequence of points without a limit point in G. Then there exists an analytic function $f : G \to \mathbb{C}$ such that $f(a_j) = 0$ and f has no other zeros in G.*

Proof. We first show that it is possible to simplify the problem by considering G unbounded and $\lim_{z\to\infty} f(z) = 1$. More precisely, we consider G such that $\{z : |z| > R\} \subset G$ and $|a_j| < R$ for all j.

For suppose this is true, then given any region G_1 and an arbitrary sequence $\{\alpha_j\}$ such that $\{\alpha_j\}$ does not have a limit point in G_1. We choose $a \in G_1$, $r > 0$ such that $\overline{B}(a,r) \subset G_1$ and $\alpha_j \notin B(a,r)$ for all j. Let $T(z) = \dfrac{1}{z-a}$, then $G := T(G_1) \setminus \{\infty\}$ is such that $\{z : |z| > R\} \subset G$ and $|a_j| = |1/(\alpha_j - a)| < R$ for all j and some $R > 0$.

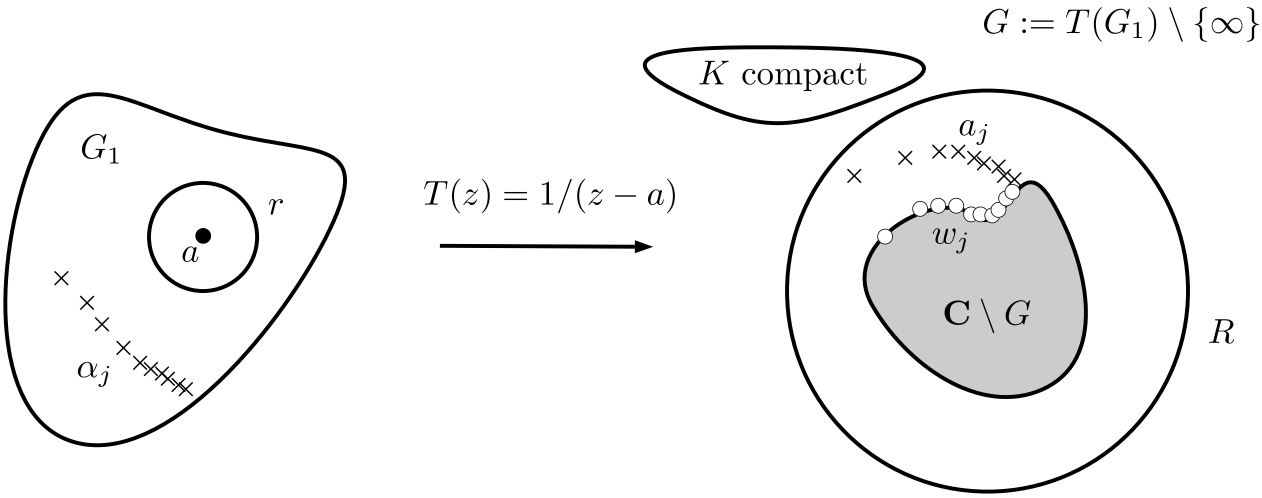

Figure 4.1: $\mathbb{C} \setminus G$

Since $\lim_{z\to\infty} f = 1$, $f(T(z))$ has a removable singularity at a and f has zeros at precisely $a_j = \dfrac{1}{\alpha_j - a}$ for all j. Then according to the definition, there exists an analytic function g on G_1 such that $g = f(T(z))$ on $G_1 \setminus \{a\}$. Clearly g has the zeros precisely on $\{\alpha_j\}$. It remains to prove the special case mentioned above.

Since G is open, so for each a_n we can find $w_n \in \mathbb{C} \setminus G$ such that

$$|w_n - a_n| = d(a_n, \mathbb{C} \setminus G)$$

and

$$\lim_{n\to\infty} |w_n - a_n| = 0$$

We aim to show that the infinite product $\prod E_n\left(\dfrac{a_n - w_n}{z - w_n}\right)$ converges in $H(G)$. So let K be any compact subset of G, and hence $d(K, \mathbb{C}\setminus G) > 0$.

Then, for $z \in K$,

$$\left|\frac{a_n - w_n}{z - w_n}\right| \le \frac{|a_n - w_n|}{d(w_n, K)} \le \frac{|a_n - w_n|}{d(K, \mathbb{C} \setminus G)}.$$

Hence given δ, $0 < \delta < 1$, there exists an N such that

$$\left|\frac{a_n - w_n}{z - w_n}\right| < \delta$$

for all $n > N$ and all $z \in K$. Thus Lemma 4.3.3(i) implies that

$$\sum_{N+1}^{\infty} \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| < \sum_{N+1}^{\infty} \delta^{n+1}.$$

That is, $\sum_1^\infty \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right|$ converges uniformly, and Theorem 4.2.1 implies that $f := \prod_1^\infty E_n\left(\frac{a_n - w_n}{z - w_n}\right)$ converges to an analytic function in $H(G)$.

The only remaining fact to verify is that $\lim_{z\to\infty} f(z) = 1$. Given $\epsilon > 0$, let $R_1 > R$ so that if $|z| > R_1$, $|a_n| < R$, we have

$$\left|\frac{a_n - w_n}{z - w_n}\right| \le \frac{2R}{R_1 - R}.$$

In particular, we can choose R_1 sufficiently large such that $\frac{2R}{R_1 - R} < \delta$ for any $0 < \delta < 1$. Thus by Lemma 4.3.3(i) again,

$$\left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| \le \left(\frac{2R}{R_1 - R}\right)^{n+1} < \delta^{n+1}$$

for all $|z| > R_1 > R$. Recall that $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$. Thus we may choose R_1 so large that when $|z| > R_1$, there exists $C > 0$ such that

$$\begin{aligned}\left|\sum_1^\infty \log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right| &\le \sum_1^\infty \left|\log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right| \\ &\le C\sum_1^\infty \left|E_n\left(\frac{a_n - w_n}{z - w_n}\right) - 1\right| \\ &\le C\sum_1^\infty \delta^{n+1} \\ &= C\frac{\delta^2}{1-\delta}.\end{aligned}$$

Thus by choose δ sufficiently small and hence R_1 sufficiently large that,

$$\begin{aligned}|f(z) - 1| &= \left|\exp\left(\sum \log E_n\left(\frac{a_n - w_n}{z - w_n}\right)\right) - 1\right| \\ &< \epsilon\end{aligned}$$

for all $|z| > R_1$. This completes the proof. □

4.6 Jensen's Formula

We shall derive a useful formula called Jensen's formula. It is a special case of the more general Poisson-Jensen formula. Jensen's formula will be used again in later sections.

Theorem 4.6.1. *Let f be analytic on a region containing $\overline{B(0,r)}$ and that $a_1, \ldots, a_n$ are the zeros of f in $B(0,r)$. Suppose in addition that $f(z) \neq 0$ on $|z| = r$ and $f(0) \neq 0$, then*

$$\log|f(0)| = -\sum_{k=1}^n \log\frac{r}{|a_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Alternatively, Jensen's formula can be written as

$$|f(0)|\prod_1^n \frac{r}{|a_k|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta\right).$$

Proof. We first prove Jensen's formula when f is non-vanishing on $\overline{B(0,r)}$. Hence we may find an analytic branch of $\log f(z)$. We then have, by Cauchy's integral formula

$$\log f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\log f(\zeta)}{\zeta - z}\, d\zeta$$

where $\gamma = \partial B(0,r)$ and $\zeta = re^{i\theta}$. Thus

$$\log f(0) = \frac{1}{2\pi}\int_0^{2\pi} \log f(re^{i\theta})\, d\theta,$$

and we obtain the Jensen formula by taking the real parts on both sides.

We next consider f to have a finite number of zeros in $B(0,r)$. Let $b \in \Delta = \{z : |z| < 1\}$, then it is known that the map $\dfrac{z-b}{z-\bar{b}z}$ is an automorphism of Δ with $|z| = 1$ being mapped to $|z| = 1$. Based on this automorphism, it is not difficult to check that

$$\frac{r(z-a_k)}{r^2 - \overline{a_k}z}$$

maps $B(0,r)$ onto $B(0,1)$ in an one-to-one manner with $|z| = r$ mapped to $|z| = 1$ and $a_k \mapsto 0$. So the function defined by

$$F(z) = \frac{f(z)}{\prod_1^n \dfrac{r(z-a_k)}{r^2 - \overline{a_k}z}} = f(z)\prod_1^n \frac{r^2 - \overline{a_k}z}{r(z-a_k)}$$

is non-vanishing on $B(0,r)$, and $|F(z)| = |f(z)|$ on $|z| = r$.

We now apply the result in the first part to $F(z)$ to obtain

$$\begin{aligned}\log|F(0)| &= \frac{1}{2\pi}\int_0^2 \pi \log|F(re^{i\theta})|\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.\end{aligned}$$

But $\log|F(0)| = \log|f(0)| + \sum_1^n \log\dfrac{r}{|a_k|}$. This completes the proof. □

Remark. (i) If f has a finite number of poles $b_1, \ldots, b_m$ except at the origin, then the Jensen formula becomes

$$\log|f(0)| = -\sum_1^n \frac{r}{|a_k|} + \sum_1^m \frac{r}{|b_v|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

(ii) The Jensen formula in Theorem 4.6.1 still holds even if there are finite number of zeros on $|z| = r$. It suffices to show that f has only a simple zero $a = re^{i\varphi}$ on $|z| = r$. Let us recall that the function $F(z)$ defined in the proof of Theorem 4.6.1. Now the function $\dfrac{F(z)}{z-a}$ is zero-free on $\overline{B(0,r)}$ and hence

$$\begin{aligned}\log\left|\frac{F(0)}{0-a}\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{F(re^{i\theta})}{re^{i\theta}-a}\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log(r|1-e^{i(\theta-\varphi)}|)\, d\theta.\end{aligned}$$

Hence

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{i(\theta-\varphi)}|\, d\theta.$$

The above equation will become the Jensen formula if the second integral on the right hand side vanishes. This will be done in the next lemma.

Lemma 4.6.2.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{i\theta}|\, d\theta = 0.$$

Proof. Consider the simply connected region $\Omega = \{z : \Re(z) < +1\}$. Hence we may define an analytic branch $\log(1-z)$ in Ω since $1-z \neq 0$. In particular, the branch is unique if we choose $\log(1-0) = 0$. Notice that $\Re(1-z) > 0$, so we have

$$\Re(\log(1-z)) = \log|1-z| \quad \text{and} \quad |\arg(1-z)| < \frac{\pi}{2}.$$

We then consider two paths:

$$\Gamma(t) = e^{it},\ \delta \le t \le 2\pi - \delta \quad \text{and} \quad \gamma(t) = 1 + \rho e^{it},\ \text{joining } e^{-i\delta} \text{ to } e^{i\delta}$$

We apply Cauchy's integral formula to $\log(1-z)$ to obtain

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} \log|1-e^{i\theta}|\, d\theta = \Re\left[\frac{1}{2\pi i}\int_{\Gamma} \log(1-z)\frac{dz}{z}\right] = \Re\left[\frac{1}{2\pi i}\int_{-\gamma} \log(1-z)\frac{dz}{z}\right].$$

But on $\gamma(t)$,

$$\begin{aligned}\frac{\log(1-z)}{z} &= \log(-\rho e^{it})(1 - \rho e^{it} + \frac{(\rho e^{it})^2}{2!} - \cdots)\\ &= \log(-\rho e^{it})(1 + O(\rho))\\ &= -\log\frac{1}{\rho}(1 + O(\rho)) + i(\text{imaginary part})(1 + O(\rho)).\end{aligned}$$

Hence

$$\left|\Re\left(\frac{1}{2\pi i}\int_{-\gamma}\frac{\log(1-z)}{z}\, dz\right)\right| \le C\delta \log\frac{1}{\delta} \to 0$$

as $\delta \to 0$; thus proving the lemma. □

We shall study Weierstrass factorization type problem for the unit disc in this section (briefly).

Definition 4.6.3. Let $\Delta = \{z : |z| < 1\}$. Then we define a subset of $H(\Delta)$ as

$$H^\infty = H^\infty(\Delta)$$

where $\sup\{|f(z)| : z \in \Delta\} < +\infty$ for all $f \in H^\infty$, i.e. *the set of all bounded analytic functions on* Δ.

Definition 4.6.4. We also define

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}\frac{|\alpha_n|}{\alpha_n} \tag{4.3}$$

on Δ, which is called *Blaschke product* provided the infinite product converges. Here the sequence $\{\alpha_n\}$ consisting of complex numbers in the unit disc.

Thus the natural questions is under what condition on $\{\alpha_n\}$ will (4.3) converges. We shall give an answer to this question in the following theorem. In fact we shall give a characterization to the existence of Blaschke product.

Theorem 4.6.5. *Let $\{\alpha_n\}$ be a sequence in Δ without limit points. Then (4.3) converges uniformly to an analytic function if and only if $\sum_1^\infty (1-|\alpha_n|) < +\infty$.*

Lemma 4.6.6. *Suppose $0 \le a_n < 1$. Then $\prod_1^\infty (1-a_n) > 0$ exists if and only if $\sum_1^\infty a_n < \infty$.*

Proof. Since $\sum a_n < \infty$, there exists N such that $\sum_{N+1}^\infty a_n < 1/2$. Note that

$$\begin{aligned}(1-a_{N+1})(1-a_{N+2}) &\ge 1-a_{N+1}-a_{N+2}\\ &\cdots\cdots\\ (1-a_{N+1})\cdots(1-a_{N+k}) &\ge 1-a_{N+1}-\cdots-a_{N+k} \quad \text{for all } k\\ &> 1/2.\end{aligned}$$

Hence $p_n = (1-a_1)\cdots(1-a_n)$ is monotonic decresing and bounded below by a positive number. Thus $\prod_1^\infty (1-a_n) > 0$ exists.

Conversely, suppose $\prod_1^\infty (1-a_n) = p > 0$. Then

$$0 < p \le p_n = \prod_1^\infty (1-a_k) \le \exp\left(-\sum_1^n a_k\right)$$

and if we assume $\sum_1^\infty a_k = +\infty$ then $\exp\left(-\sum_1^\infty a_k\right) \to 0$. A contradiction. □

Proof of Theorem 4.6.5. Suppose $\sum_1^\infty (1-|\alpha_n|) < \infty$. According to Theorem 4.2.1, it is sufficient to show

$$\sum_1^\infty \left| 1 - \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}\right| < \infty.$$

Notice that

$$1 - \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n} = \frac{\alpha_n + |\alpha_n|z}{(1-\overline{\alpha_n}z)\alpha_n}(1-|\alpha_n|).$$

For each $|z| \leq r < 1$, we have

$$\left|\frac{(1-|\alpha_n|)(\alpha_n+|\alpha_n|z)}{(1-\overline{\alpha_n}z)\alpha_n}\right| \leq \frac{1+r}{1-r}(1-|\alpha_n|)$$

since $|1-\overline{\alpha_n}z| \geq 1-|\overline{\alpha_n}|r \geq 1-r$. Hence (4.3) converges if $\sum_1^\infty(1-|\alpha_n|) < \infty$.

Suppose now the Blaschke product converges. Then $|B(z)| < 1$ for all $z \in \Delta$. We may assume $B(0) \neq 0$, since the factor z^k does not affect the convergence of $\sum(1-|\alpha_n|)$. By Jensen's formula we have for $r < 1$ and n zeros in $|z| < r$,

$$|B(0)| \prod_1^n \frac{r}{|\alpha_n|} = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log|B(re^{i\theta})|\, d\theta\right].$$

But $|B(z)| < 1$ for all $z \in \Delta$. Hence the right hand side of the Jensen's formula is bounded by a constant $C > 0$ for all $0 < r < 1$. Thus

$$\prod_1^\infty |\alpha_n| \geq C^{-1}|B(0)| > 0$$

as $r \to 1$. Lemma 4.6.6 implies $\sum(1-|\alpha_n|)$ must converges. □

4.7 Hadamard's Factorization Theorem

We have applied Weierstrass factorization theorem to obtain an infinite product representation of $\sin \pi z$:

$$\begin{aligned}\sin \pi z &= z e^{g(z)} \prod_{-\infty}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} \\ &= \pi z \prod_1^\infty \left(1-\frac{z^2}{n^2}\right).\end{aligned}$$

It is perhaps difficult to see at the beginning that $g(z)$ reduces to a constant $\log \pi$ and thus $2z\prod_1^\infty\left(1-\dfrac{z^2}{n^2}\right)$ behaves as $e^{i\pi z} - e^{-i\pi z}$

(which is the original definition) in the above representation for $\sin \pi z$. A question comes into our minds immediately is that how will the growth of e^g and $\prod_1^\infty E_{p_n}(z/a_n)$ relate to the growth of the function f. We shall study this question when g is taken as a polynomial and $p_n = p$ for all n in this section. This line of research has dominated the development of function theory of one complex variable for the past seventy years. This area of research is related to subharmonic functions ($\log |f(z)|$ is harmonic away from the zeros of f; see next chapter for harmonic functions) and potential theory. Most easier problems have been solved, with the remaining open problems exceedingly difficult.

Let $\{a_n\}$ be a sequence of numbers in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ and that there exists a non-negative integer p such that $\sum \dfrac{1}{|a_n|^{p+1}} < \infty$. Then according to Weierstrass factorization theorem that

$$\prod_1^\infty E_p\left(\frac{z}{a_n}\right)$$

converges to an entire function on $\mathbb{C}$.

Definition 4.7.1. If the integer p described above is chosen so that $\sum 1/|a_n|^p = +\infty$ and $\sum 1/|a_n|^{p+1} < +\infty$, then the integer is called the *genus* of $\{a_n\}$, and the infinite product is said to be *canonical (standard).* We also call p the genus of the canonical product.

Example 4.7.2. The infinite product

$$\sin \pi z = \pi z \prod_{-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right)$$

has genus one, since $\sum_1^\infty \dfrac{1}{n} = +\infty$ and $\sum_1^\infty \dfrac{1}{n^2} < +\infty$.

So if a function has a representation

$$f(z) = z^m e^{g(z)} \prod_1^\infty E_p\left(\frac{z}{a_n}\right), \quad p = \text{ genus}$$

where g is a polynomial, then the growth is determined by that of $\Pi_1^\infty E_p(z/a_n)$ and e^g. Note that the above infinite product representation of f is unique if p is chosen to be the genus. We now define the *genus of* f to be $\mu = \max\{\deg g, p\}$. We next show the genus of f determines its growth.

Theorem 4.7.3. *Let f be an entire function of genus μ (refered to its Weierstrass factorization). Then given $\alpha > 0$, there exists $R > 0$ such that*

$$|f(z)| \le \exp(\alpha |z|^{\mu+1})$$

for $|z| > R$.

We first obtain a result for canonical product.

Theorem 4.7.4. *Let $P(z)$ be a canonical product with genus p. Then given $\alpha > 0$, there exists $R > 0$ such that*

$$|P(z)| < \exp(\alpha |z|^{p+1})$$

for $|z| > R$.

Proof. We need some elementary estimates for the primary factors. Since

$$|E_p(z)| \le (1 + |z|) \exp\left(|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^p}{p}\right),$$

thus

$$\log |E_p(z)| \le \log(1 + |z|) + |z| + \cdots + \frac{|z|^p}{p}.$$

Thus given any $A > 0$, there exists $R > 0$ such that

$$\log |E_p(z)| < A|z|^{p+1} \quad \text{for } |z| > R. \tag{4.4}$$

We also recall from Lemma 4.3.3(ii) that for $k > 1$,

$$\log |E_p(z)| \le |\log E_p(z)| \le \frac{k}{k-1} |z|^{p+1}, \quad \text{for } k < 1, \quad |z| < 1/k.$$

Without loss of generality, we may assume $\frac{1}{k} < R$. For $\frac{1}{k} \le |z| \le R$, the function

$$\frac{\log |E_p(z)|}{|z|^{p+1}}$$

is easily seen to be continuous there except when $z = 1$ where $\log |E_p(z)| = -\infty$. In any case an absolute upper bound exists. Thus there exists $B > 0$ such that

$$\log |E_p(z)| \le B|z|^{p+1}$$

for $\frac{1}{k} \le |z| \le R$. Let $M = \max\{A, B, \frac{k}{k-1}\}$, we have

$$\log |E_p(z)| \le M|z|^{p+1}$$

for all $z \in \mathbb{C}$.

Since $\sum_1^\infty \frac{1}{|a_n|^{p+1}} < \infty$, we choose N so large that

$$\sum_{N+1}^{\infty} \frac{1}{|a_n|^{p+1}} < \frac{\alpha}{4M}.$$

Thus

$$\begin{aligned} \sum_{N+1}^{\infty} \log \left| E_p\left(\frac{z}{a_n}\right) \right| &\le M|z|^{p+1} \sum_{N+1}^{\infty} \frac{1}{|a_n|^{p+1}} \\ &\le \frac{\alpha |z|^{p+1}}{4}. \end{aligned} \tag{4.5}$$

To estimate $\sum_1^N \log |E_p(z/a_n)|$, we note that $|z/a_n|$ are large for $1 \le n \le N$, hence we may assume the constant $A > 0$ in (4.4) is chosen such that $A = \frac{\alpha}{4N} \min_{1\le i\le N} |a_i|^{p+1}$ for $|z| > R_1 > R$, say. Thus

$$\begin{aligned} \sum_1^N \log |E_p(z/a_n)| &< \frac{\alpha}{4N}|z|^{p+1} \left(\sum_1^N \frac{1}{|a_n|^{p+1}} \right) \min_{1\le i\le N} |a_i|^{p+1} \\ &\le \frac{\alpha}{4N}|z|^{p+1} \left(N \max_{1\le i\le N} \frac{1}{|a_i|^{p+1}} \right) \min_{1\le i\le N} |a_i|^{p+1} \\ &\le \frac{\alpha}{4}|z|^{p+1} \end{aligned} \tag{4.6}$$

for $|z| > R_1 > R$. Combining (4.5) and (4.6) yields

$$\log|P(z)| = \sum_1^\infty \log|E_p(z/a_n)| < \alpha|z|^{p+1}$$

for all $|z| > R_1$. This completes the proof. □

Proof of Theorem 4.7.3. It is now easy to complete the proof of Theorem 4.7.3. For $\deg g < \mu + 1$, so $|z|^m e^{|g|}/\exp(\alpha|z|^{\mu+1}) \to 0$ as $|z| \to \infty$. But $p + 1 \le \mu + 1$. The required estimate follows from Theorem 4.7.4. □

Example 4.7.5. If $\sum 1/|a_n|^2 < \infty$ and $\sum 1/|a_n| = +\infty$, then

$$f = \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

has genus 1. So $|f| < \exp(\alpha|z|^2)$. It also follows that $\sin \pi z$ has genus 1.

Suppose $\sum_1^\infty 1/|a_n| < \infty$. Then $\prod \left(1 - \dfrac{z}{a_n}\right)$ has genus zero. Hence

$$f = e^z \prod \left(1 - \frac{z}{a_n}\right)$$

also has genus 1.

The above theorems show that we can know the growth of f provided we know the function g and the growth (genus) of the zeros of f. We shall study the converse problem in what follows. Namely, what can we say about g and the zeros of f if we know the growth of f.

Definition 4.7.6. Let $S(r)$ be a positive and monotonic increasing function of $r > 0$. The *order* λ of $S(r)$ is defined to be

$$\limsup_{r \to +\infty} \frac{\log S(r)}{\log r}.$$

We say $S(r)$ has infinite order if no finite λ can be found.

Remark. The above definition is equivalent to: given any $\epsilon > 0$, there exists a $r_0 > 0$ such that

(i) $S(r) < r^{\lambda+\epsilon}$ for $r > r_0$, and

(ii) $S(r) > r^{\lambda-\epsilon}$ holds for infinitely many $r > r_0$.

Example 4.7.7. The order r and $r(\log r)^3$ where $s \neq 0$ are both equal to 1. The order of e^r is infinite.

Definition 4.7.8. Let f be an entire function and $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$. Then the *order of f* is defined to be the real number:

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r,\ f)}{\log r}.$$

Example 4.7.9. (i) $\lambda(e^z) = 1$.

(ii) $\lambda(e^{p(z)}) = n$, where $p(z)$ is a polynomial of degree n.

(iii) $\lambda(\exp(e^z)) = \infty$.

Definition 4.7.10. Let $n(r)$ be the *number of zeros of f in $|z| < r$* (counted according to multiplicities).

Proposition 4.7.11. *The order of $n(r)$ does not exceed that of f, i.e. $\lambda(n(r)) \leq \lambda(f)$.*

Proof. We may assume $f(0) \neq 0$. Given $\epsilon > 0$, there exists $r_0 > 0$ such that

$$\log M(r, f) < r^{\lambda(f)+\epsilon} \quad \text{for } r > r_0.$$

Putting $2r$ into Jensen's formula yields

$$\begin{aligned} 2^{n(r)}|f(0)| \leq |f(0)| \prod_1^{n(r)} \frac{2r}{|a_n|} \leq |f(0)| \prod_1^{n(2r)} \frac{2r}{|a_n|} &= \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|\, d\theta\right) \\ &\leq M(2r, f). \end{aligned}$$

Hence

$$\log|f(0)| + n(r)\log 2 \le \log M(2r, f)$$
$$< 2^{\lambda+\epsilon} r^{\lambda+\epsilon} \quad \text{for } r > r_0.$$

Thus, proving the proposition. □

Proposition 4.7.12. *Suppose* $\lambda(f) = \lambda < +\infty$ *and* $\{r_i\}$ *are the moduli of the zeros of* f*. Then the series* $\sum r_n^{-\alpha} < +\infty$ *whenever* $\alpha > \lambda$.

Proof. Let $\lambda < \beta < \alpha$. It follows from Proposition 4.7.11 that $n(r) < Ar^{\beta}$ for $r > r_0$, say. Suppose $r = r_n$, then $n = n(r_n) < Ar_n^{\beta}$. Hence $r_n^{-\beta} < An^{-1}$, and so $r_n^{-\alpha} < An^{-\alpha/\beta}$. Thus $\sum r_n^{-\alpha} < A\sum n^{-\alpha/\beta} < +\infty$ since $\alpha/\beta > 1$. □

Definition 4.7.13. The real number

$$v = \inf\left\{\alpha : \sum_{1}^{\infty} \frac{1}{r_n^{\alpha}} < +\infty\right\}$$

is called the *exponent of convergence* of the sequence $\{r_n\}$.

If $\{a_n\}$ is a sequence of the zeros of $P(z)$ and $|a_n| = r_n$, then

$$p \le v < p + 1$$

where p is the genus of $\{a_n\}$. Also, it was proved in Theorem 4.7.4 that $|P(z)| < \exp(\alpha|z|^{p+1})$. We can prove a more precise result.

Theorem 4.7.14. *The order of a canonical product is equal to the exponent of convergence of its zeros.*

Proof. See exercise/homework. □

It follows that if v is the exponent of convergence of $P(z)$, then

$$|P(z)| < \exp(\alpha|z|^{v+\epsilon})$$

for all $|z|$ sufficiently large. It also follows that

$$\text{genus } p \le \text{order of a canonical product} < p + 1.$$

Note also that $p = [v]$, $p \le v \le \lambda$ (for an entire function).

Lemma 4.7.15. *Let f be an entire function of order $\lambda < +\infty$ and $f(0) = 1$. Suppose $\{a_i\}$ are the zeros of f and an integer $p > \lambda - 1$. Then*

$$\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$$

for $z \neq a_1, a_2, \ldots$.

The proof of this lemma will be given after the Poisson-Jensen formula.

Theorem 4.7.16 (Hadamard's Factorization Theorem)**.** *Let f be an entire function of order $\lambda < +\infty$, and suppose $\{a_i\}$ are the zeros of f where $f(0) = 1$. Then*

$$f(z) = e^{g(z)} P(z)$$

where g is a polynomial of degree $\leq \lambda$, and $P(z)$ is the canonical product form from the zeros of f.

Proof. Let p be an integer such that $p \leq \lambda < p + 1$. Since the order of f, $\lambda(f) < \infty$, Proposition 4.7.12 implies that the zeros $a_1, a_2, \ldots$ of f satisfy $\sum \frac{1}{|a_n|^{p+1}} < +\infty$ since $p + 1 > \lambda$. Let $P(z)$ be the canonical product forms from the zeros of f, and v be its exponent of convergence of zeros.

Weierstrass factorization theorem implies that there exists an entire function $g(z)$ such that $f(z) = e^g P(z)$.

It remains to show that g is a polynomial. But it is easy to check that

$$\frac{d^p}{dz^p}\left(\left[E_p\left(\frac{z}{a_n}\right)\right]' / E_p\left(\frac{z}{a_n}\right)\right) = -p! \frac{1}{(a_n - z)^{p+1}}.$$

Combining this and by using Lemma 4.7.15, we obtain

$$g^{(p+1)}(z) - p! \sum_{1}^{\infty} \frac{1}{(a_n - z)^{p+1}} = \frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p! \sum_{1}^{\infty} \frac{1}{(a_n - z)^{p+1}}.$$

Hence g must be a polynomial of degree at most p. □

4.8 Poisson-Jensen Formula

Theorem 4.8.1. *Let f be analytic on $\overline{B(0,r)}$ and let $a_1,\dots,a_n$ be the zeros of $f(z)$ in $B(0,r)$. Suppose $f(z)\neq 0$ for $z\in B(0,r)$, then*

$$\begin{aligned}\log|f(z)| &= -\sum_{k=1}^{n}\log\left|\frac{r^2-a_n z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\Re\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta\\ &= -\sum_{k=1}^{n}\log\left|\frac{r^2-a_n z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-\rho^2}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}\log|f(re^{i\theta})|\,d\theta\end{aligned}$$

where $z=\rho e^{i\theta}$.

Proof. We need to quote the quote following result from Chapter 6: Suppose g is *analytic* on $\overline{B}(0,r)$ and that $\Re(g(z)=U(z)$. Then for $z=\rho e^{i\phi}$, $\rho<r$, we have

$$\begin{aligned}U(\rho e^{i\phi}) &= \frac{1}{2\pi}\int_0^{2\pi}\Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right)U(re^{i\theta})\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-\rho^2}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}U(re^{i\theta})\,d\theta.\end{aligned}$$

Let $g(z)=f(z)\prod_1^n\dfrac{r^2-\overline{a_k}z}{r(z-a_k)}$ then it is zero-free on $B(0,r)$. Hence $\log g(z)$ is analytic on $\overline{B}(0,r)$. We thus obtain

$$\begin{aligned}\log|g(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right)\log|g(re^{i\theta})|\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\Re\left(\frac{re^{i\theta+z}}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta.\end{aligned}$$

But

$$\log|g(z)| = \log|f(z)| + \sum_{k=1}^{n}\log\left|\frac{r^2-\overline{a_k}z}{r(z-a_k)}\right|$$

and the required formula now follows. □

Remark. The Poisson-Jensen formula recovers the Jensen's formula after putting $z = 0$.

Now we can give a proof of Lemma 4.7.15.

Proof of Lemma 4.7.15. The easiest way to obtain $(f'/f)^{(p)}$ is to differentiate the Poisson-Jensen formula. Suppose $f(z) \neq 0$, then $\log f(z)$ exists and Cauchy-Riemann equations gives $f'/f = \dfrac{d}{dz}\log f(z) = \dfrac{\partial}{\partial x}\Re(\log f(z)) - i\dfrac{\partial}{\partial y}\Re(\log f(z)) = \dfrac{\partial}{\partial x}\log|f(z)| - i\dfrac{\partial}{\partial y}\log|f(z)|$. We apply this formula and differentiation under the integral (see Conway p.69),

$$\log|f(z)| = -\sum_1^n \log\left|\frac{r^2 - \overline{a_k}z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta.$$

We obtain

$$\frac{f'(z)}{f(z)} = \sum_1^n \frac{1}{z-a_k} + \sum_1^n \frac{\overline{a_k}}{r^2 - \overline{a_k}z} + \frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{i\theta}}{(re^{i\theta}-z)^2}\log|f(re^{i\theta})|\,d\theta.$$

Differentiating this formula p time yields:

$$\begin{aligned}\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) &= -p!\sum_1^n \frac{1}{(a_k - z)^{p+1}} + p!\sum_1^n \frac{\overline{a_k}^{p+1}}{(r^2 - \overline{a_k}z)^{p+1}} \\ &\quad + (p+1)!\frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log|f(re^{i\theta})|\,d\theta.\end{aligned}$$

It remains to show the last two terms tend to zero as $r \to \infty$ ($n \to \infty$). We consider the integral of the last expression first, and note that the integral

$$\int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\,d\theta = 0.$$

Hence

$$-\int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log|f(re^{i\theta})|\,d\theta = \int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta.$$

Suppose $r > 2|z|$, then

$$\begin{aligned}\left|\int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}} \log|f(re^{i\theta})|\,d\theta\right| &\le \frac{2r}{(r-r/2)^{p+2}} \int_0^{2\pi} \log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta \\ &= 2^{p+3}r^{-p-1}\int_0^{2\pi} \log\frac{M(r,f)}{|f(re^{i\theta})|}\,d\theta \\ &= 2^{p+3}r^{-p-1}\int_0^{2\pi} (\log M(r,f) - \log|f(re^{i\theta})|)\,d\theta \\ &\le 2^{p+2}\int_0^{2\pi} \frac{\log M(r,f)}{r^{p+1}}\,d\theta \\ &\quad \left(\text{by } \int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \ge 0\right)\end{aligned}$$

by Jensen's formula. But $\log M(r,f)/r^{p+1} \to 0$ as $r \to \infty$ since $\lambda < p+1$ and this proves the integral tends to zero as $r \to \infty$.

We now consider an individual term in the second summand: we assume again $r > 2|z|$,

$$\left|\frac{\overline{a_k}}{r^2 - \overline{a_k}z}\right|^{p+1} \le \frac{|a_k|^{p+1}}{(r^2 - r^2/2)^{p+1}} = \frac{(2r)^{p+1}}{r^{2(p+1)}} = \left(\frac{2}{r}\right)^{p+1}.$$

Hence

$$\sum_1^n \left|\frac{\overline{a_k}}{r^2 - \overline{a_k}z}\right|^{p+1} \le 2^{p+1}\sum_1^n \frac{1}{r^{p+1}} \le 2^{p+1}\frac{n(r)}{r^{p+1}} \to 0$$

as $r \to \infty$, $n(r) \ge n(r_n)$ by Proposition 4.7.11. Thus

$$\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) \to -p! \sum_1^\infty \frac{1}{(z-a_k)^{p+1}}$$

as $r \to \infty$ and this completes the proof. □

We can rewrite the above lemma as $p \le \lambda < p+1$. We also note the following:

Theorem 4.8.2. *Let f be an entire function of finite order, then f assumes each complex number with at most one exception.*

Proof. Suppose $f(z) \neq \alpha, \beta$ for all $z \in \mathbb{C}$. Since $f - \alpha$ has the same order of growth of f and never vanish. By Theorem 4.7.16, there exists a polynomial g such that $f - \alpha = \exp(g)$. Thus $\exp(g)$ never assume $\beta - \alpha$ and so $g(z)$ never assume $\log(\beta - \alpha)$, a contradiction to the Fundamental theorem of algebra. □

Theorem 4.8.3. *Suppose the order of an entire function is finite and not equal to an integer. Then the function must have an infinite number of zeros.*

Theorem 4.8.4. *Let α be a real number. Then the function*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$$

has order $1/\alpha$.

Proof. Suppose z is real and positive. By considering

$$\frac{z}{1^\alpha} \cdot \frac{z}{2^\alpha} \cdot \cdots \cdot \frac{z}{(n-1)^\alpha} \cdot \frac{z}{n^\alpha},$$

we clearly deduce $z^n/(n!)^\alpha$ is increasing when $|z| > n^\alpha$ and it starts to decrease when $|z| < n^\alpha$. Hence the maximum of $z^n/(n!)^\alpha$ occurs when $z = n^\alpha$. Thus

$$\begin{aligned}\frac{z^n}{(n!)^\alpha} = \frac{n^{\alpha n}}{(n!)^\alpha} &= \frac{n^{n\alpha}}{(n^{n+(1/2)}e^{-n}\sqrt{2\pi}(1+o(1)))^\alpha} = \frac{e^{n\alpha}}{n^{\alpha/2}(2\pi)^{\alpha/2}(1+o(1))} \\ &= \frac{e^{\alpha}z^{1/\alpha}}{z^{1/2}(2\pi)^{\alpha/2}(1+o(1))}\end{aligned}$$

by Stirling formula (Titchmarsh, p.58).

But the order of growth of $f(z)$ must be greater than its individual term when $z > 0$. Hence $\lambda \geq 1/\alpha$.

On the other hand, $|f(z)| \leq f(|z|)$ when z is real,

$$\begin{aligned}f(z) &= \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{(n!)^\alpha} \\ &< \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{[(N+1)!N^{n-N-1}]^\alpha}.\end{aligned}$$

Suppose $N^\alpha > z$, then

$$\begin{aligned}\sum_{N+1}^{\infty} \frac{z^n}{[(N+1)!N^{n-N-1}]^\alpha} &= \frac{(N^{N+1})^\alpha}{[(N+1)!]^\alpha} \sum_{N+1}^{\infty} \frac{z^n}{(N^n)^\alpha} \\ &= \frac{(N^{N+1})^\alpha}{[(N+1)!]^\alpha} \left(\frac{z^{N+1}}{(N^{N+1})^\alpha} + \frac{z^{N+2}}{(N^{N+2})^\alpha} + \cdots \right) \\ &= \frac{z^{N+1}}{[(N+1)!]^\alpha} \left(1 + \frac{z}{N^\alpha} + \cdots \right) \\ &= \frac{z^{N+1}}{[(N+1)!]^\alpha (1 - (z/N^\alpha))}.\end{aligned}$$

Thus we have

$$f(z) < Az^N + \frac{z^{N+1}}{[(N+1)!]^\alpha (1 - (z/N^\alpha))}$$

whenever $N^\alpha > z$. Hence, by taking $N = [(2z)^{1/\alpha}]$,

$$f(z) = O(z^N) = O(z^{(2z)^{1/\alpha}}) = O(\exp(z^{(1/\alpha)+\epsilon})),$$

and we deduce the order of f does not exceed $1/\alpha$. Hence $\lambda = 1/\alpha$. $\square$

Remark. *Stirling formula:*

$$\Gamma(z) = z^{z-(1/2)} e^{-z} \sqrt{2\pi}(1 + o(1))$$

where $\Gamma(n+1) = n!$.

Exercise. What is the order of $\sum_{n=1}^{\infty} \dfrac{z^n}{n^{\alpha n}}$ for $\alpha > 0$?

Chapter 5

Periodic functions

An (analytic) function $f(z)$ is said to be **periodic** if there is a non-zero constant ω such that

$$f(z+\omega) = f(z), \quad z \in \mathbb{C}.$$

We call the number ω a **period** of the function $f(z)$.

Definition 5.0.5. We call ω a **fundamental (primitive) period** of f if $|\omega|$ is the smallest amongst all periods.

5.1 Simply periodic functions

The simplest periodic function of period ω is $e^{2\pi iz/\omega}$. Suppose Ω is a region such that if $z \in \Omega$ then $z + k\omega \in \Omega$ for all $k \in \mathbb{Z}$.

Theorem 5.1.1. *Given a meromorphic function f defined on a region Ω (as discussed above). Then there exists a unique meromorphic function F in Ω' which is the image of Ω under $e^{2\pi iz/\omega}$, such that*

$$f(z) = F\Big(e^{2\pi iz/\omega}\Big).$$

Proof. Suppose f is meromorphic in Ω in the $z-$plane with period ω. Let $\zeta = e^{2\pi iz/\omega}$. Then we define F by

$$f(z) = f(\log\zeta) = F(\zeta).$$

Then clearly F is meromorphic in the $\zeta-$plane whenever $f(z)$ is meromorphic in the $z-$plane. □

Example 5.1.2. Let $0 < |q| < 1$. Consider the function

$$f(z) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} e^{kiz}$$

which represents a $2\pi-$periodic entire function in $\mathbb{C}$. In fact, this is a complex form of a Fourier series. Let $\zeta = e^{iz}$.

$$F(\zeta) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} \zeta^k$$

which can be shown to converge in the punctured plane $0 < |\zeta| < +\infty$. Thus we have

$$f(z) = F(\zeta)$$

as asserted by the last theorem. Here we have $\omega = 2\pi$. Thus, the function F is analytic in $0 < |\zeta| < +\infty$.

More generally, if the series

$$F(\zeta) = \sum_{k=-\infty}^{\infty} c_n \zeta^k$$

converges in an annulus $r_1 < |\zeta| < r_2$, then

$$f(z) := F(\zeta) = \sum_{k=-\infty}^{\infty} c_n e^{2\pi kiz/\omega}, \quad \zeta = e^{2\pi iz/\omega},$$

is a $\omega-$periodic analytic function in the infinite horizontal strip $\{\zeta : e^{r_1} < \Im(\zeta) < e^{r_2}\}$. We can represent the coefficient

$$\begin{aligned} c_k &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{F(\zeta)}{\zeta^{k+1}}\, d\zeta, \quad r_1 < |\zeta| < r_2 \\ &= \frac{1}{\omega} \int_a^{a+\omega} f(z) e^{-2\pi kiz/\omega}\, dz, \end{aligned}$$

where a is an arbitrary in the infinite strip $\{\zeta : e^{r_1} < \Im(\zeta) < e^{r_2}\}$ and the integration is taken along any path lying in the strip.

5.2 Period module

Let M denote that set of all periods of a meromorphic function f in $\mathbb{C}$. If $\omega \neq 0$ is a period, then $n\omega$, for any integer n, obviously belongs to M. If, however, there are two distinct periods ω_1 and ω_2, then $m\omega_1 + n\omega_2$ is also a period for any integral multiples m, n. That is, $m\omega_1 + n\omega_2 \in M$. This shows that M is a **module** over the ring of integers.

We also note that the set of periods must be discrete since if there is a sequence of periods with a limit point, then this would contradicts the identity theorem for analytic functions. We are ready to answer Jacobi's first question.

Theorem 5.2.1. *A discrete module M consists of either $n\omega$ for an arbitrary integer n and $\omega \neq 0$, or $m\omega_1 + n\omega_2$ for arbitrary integers n, m and non-zero ω_1, ω_2 with $\Im(\omega_2/\omega_1) \neq 0$.*

Proof. Without loss of generality, we may assume $M \neq \emptyset$. Let $\omega = \omega_1 \in M$ and there are at most a finite number of $n\omega_1$ belong to M in a fixed $|z| \leq r$. Amongst all these ω_1, we choose the one with the smallest $|\omega_1|$.

If however, there is a period $\omega \in M$ that is not of the form $n\omega_1$ for $n \in \mathbb{Z}$. Then again we call ω_2 with $|\omega_2|$ the smallest (but not less than $|\omega_1|$). We claim that $\Im(\omega_2/\omega_1) \neq 0$. For if it were, then there is an integer n such that

$$n < \frac{\omega_2}{\omega_1} < n + 1,$$

or

$$0 < \Big|\frac{\omega_2}{\omega_1} - n\Big| < 1$$

or $|n\omega_1 - \omega_2| < |\omega_1|$. But $n\omega_1 - \omega_2$ is a period which is smaller than $|\omega_1|$. This contradicts the assumption that ω_1 is the "smallest" period.

It remains to show that any period ω must be of the form $m\omega_1 + n\omega_2$ for some integers n, m. Without loss of generality, we may assume $\Im(\omega_2/\omega_1) > 0$. Hence any complex number ω can be written as $\omega =$

$\lambda_1\omega_1+\lambda_2\omega_2$ for constants $\lambda_1,\ \lambda_2$. We claim that $\lambda_1,\ \lambda_2$ are real. Suppose

$$\begin{aligned}\omega &= \lambda_1\omega_1 + \lambda_2\omega_2,\\ \bar{\omega} &= \lambda_1\bar{\omega}_1 + \lambda_2\bar{\omega}_2.\end{aligned}$$

Then one can can find unique solutions $\lambda_1,\ \lambda_2$ since $\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0$. But then $\bar{\lambda}_1,\ \bar{\lambda}_2$ are also solutions. So they are real.

Clearly we find integers m_1 and m_2 such that

$$|\lambda_1 - m_1| \leq \frac{1}{2}, \quad |\lambda_2 - m_2| \leq \frac{1}{2}.$$

If $\omega \in M$, then so does

$$\omega' = \omega - m_1\omega_1 - m_2\omega_2.$$

But then

$$\begin{aligned}|\omega'| &< |\lambda_1 - m_1||\omega_1| + |\lambda_2 - m_2||\omega_2|\\ &\leq \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2|\\ &\leq |\omega_2|\end{aligned}$$

where the first inequality is strict since ω_2 is not a real multiple of ω_1. That is, the $|\omega'| < |\omega_2|$ while $\omega' \in M$. We conclude that ω' is an integral multiple of ω_1. This gives ω the desired form. □

5.3 Unimodular transformations

We consider the case that M is generated by two distinct ω_1 and ω_2 such that $\Im(\omega_2/\omega_1) > 0$. We recall that M consists of discrete points $n\omega_1 + m\omega_2$ where $m,\ ,n$ are integers. Suppose ω_1' and ω_2' is another pair of distinct points that also generate M. Then we must have

$$\begin{aligned}\omega_1' &= a\omega_1 + b\omega_2,\\ \omega_2' &= c\omega_1 + d\omega_2\end{aligned}$$

for some integers a, b, c, d. We can rewrite this in a matrix form:

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{5.1}$$

There is a similar matrix equation for complex conjugates:

$$\begin{pmatrix} \bar{\omega}_1' \\ \bar{\omega}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}. \tag{5.2}$$

We can combine the above two matrix equations into one:

$$\begin{pmatrix} \omega_1' & \bar{\omega}_1' \\ \omega_2' & \bar{\omega}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 & \bar{\omega}_1 \\ \omega_2 & \bar{\omega}_2 \end{pmatrix}. \tag{5.3}$$

Similarly, we can find integers a', b', c', d'

$$\begin{pmatrix} \omega_1 & \bar{\omega}_1 \\ \omega_2 & \bar{\omega}_2 \end{pmatrix}. = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' & \bar{\omega}_1' \\ \omega_2' & \bar{\omega}_2' \end{pmatrix} \tag{5.4}$$

The determinant $\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0$ since ω_2/ω_1 would have real ratio. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the determinants equal

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \pm 1$$

Definition 5.3.1. The set of all such 2×2 linear transformations with determinant ± 1 is called **unimodular**. When we restrict to determinant begin 1, it is also recognised as a subgroup of the **projective special linear group** $PSL(2, \mathbb{C})$ which we label as $\Gamma = PSL(2, \mathbb{Z})$ or just **modular group**.

It turns out that the modular group has generators

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We label the lattice generated by ω_1, ω_2 by $\Omega(\omega_1, \omega_2)$. Thus if ω_1, ω_2 by $\Omega(\omega_1', \omega_2')$ is another lattice, then two latices are connected by a unimodular transformation.

We can make the basis ω_1, ω_2 by a suitable restriction.

Theorem 5.3.2. *Let* $\tau = \omega_2/\omega_1$. *If*

1. $\Im(\omega_2/\omega_1) > 0$,
2. $-\frac{1}{2} < \Re(\tau) \le \frac{1}{2}$,
3. $|\tau| \ge 1$,
4. $\Re(\tau) \ge 0$ *when* $|\tau| = 1$,

then the τ *is uniquely determined.*

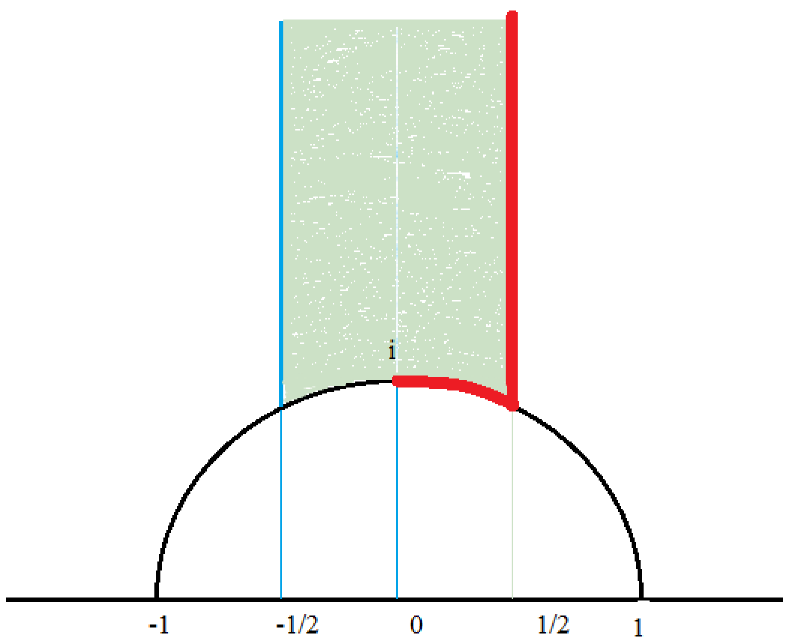

Figure 5.1: Fundamental region of modular function

It is clear that the region defined by the criteria (1-4) in the theorem is not an open. But it still call it a **fundamental region**. If it happens that $\Im(\omega_2/\omega_1) < 0$, then we could replace (ω_1, ω_2) by $(-\omega_1, \omega_2)$

without changing the assumption $-\frac{1}{2} < \Re(\tau) \leq \frac{1}{2}$. The assumption (2) is also arbitrary in the sense if $-\frac{1}{2} \leq \Re(\tau) < \frac{1}{2}$, then we could use $(\omega_1, \omega_1 + \omega_2)$. Finally, if the last assumption (4) is replaced by $\Re(\tau) < 0$ when $|\tau| = 1$, then we consider $(-\omega_2, \omega_1)$ instead.

Proof. Let τ' be

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers and such that $ad - bc = \pm 1$. We recall that the above Möbius transformation that maps the upper half $\tau-$plane onto itself if the determinant is $+1$ and onto the lower half $\tau-$plane if he determinant is -1. A simple calculation gives

$$\Im(\tau') = \frac{\pm\Im(\tau)}{|c\tau + d|^2} \tag{5.5}$$

where the $\pm$ accords to that of $ad - bc$.

Suppose both the τ', τ situate inside the fundamental region. We want to show that $\tau' = \tau$. Without loss of generality, we may assume that $ad - bc = 1$, and $\Im(\tau') \geq \Im(\tau)$. This means that

$$|c\tau + d| \leq 1.$$

Since c, d are integers, so there are not too many cases to check.

If $c = 0$, then $d = \pm 1$. The condition $ad - bc = 1$ implies $ad = 1$. So either $a = d = 1$ or $a = d = -1$, so that the equation (5.5) becomes

$$\tau' = \tau \pm b.$$

But both τ', τ satisfy the assumption (2) which implies that

$$|b| = |\Re(\tau') - \Re(\tau)| < 1.$$

Thus $b = 0$ and $\tau' = \tau$.

Suppose now that $c \neq 0$. We have

$$|\tau + d/c| \leq 1/|c|.$$

We claim that $|c| = 1$. For suppose $|c| \geq 2$, then $|\tau + d/c| \leq \frac{1}{2}$ meaning that τ is closer to the d/c (real axis) than $1/2$. This contradicts the assumption (3) that $|\tau| \geq 1$. Thus $c = \pm 1$ and

$$|\tau \pm d| \leq 1.$$

But since τ situates in the fundamental region, so either $d = 0$ or $d = \pm 1$. In the latter, the $|\tau + 1| \leq 1$ has no solution there (the only point being $e^{2\pi i/3}$ is outside the fundamental region). The other inequality $|\tau - 1| \leq 1$ has the only one solution $e^{i\pi/3}$ and it becomes an equality and $|c\tau + d| = 1$. We deduce from (5.5) that $\Im(\tau') = \Im(\tau)$ and hence $\tau' = \tau$. Suppose $d = 0$ and $|c| = 1$. So $|\tau| \leq 1$. This together with the assumption (3) $|\tau| \geq 1$ imply that $|\tau| = 1$. Hence

$$\tau' = \frac{a\tau + b}{c\tau} = \frac{a}{c} + \frac{b}{c\tau} = \frac{a}{c} + \frac{-1}{\tau}$$

since $bc = -1$. Hence

$$\tau' = \pm a - \frac{1}{\tau} = \pm a - \bar{\tau}.$$

But then $\Re(\tau') = \pm a - \Re(\bar{\tau}) = \pm a - \Re(\tau)$ so that

$$\Re(\tau' + \tau) = \pm a$$

which is an integer. This is possible only if $a = 0$. Thus $\tau' = -1/\tau$ and the only solution for this equation in the fundamental region is when $\tau' = \tau = i$ (since $|\tau| = 1$). □

5.4 Doubly periodic functions

Definition 5.4.1. Let ω_1 and ω_2 be two distinct non-zero complex numbers such that $\Im\omega_1/\omega_2 > 0$. An **elliptic function** f is a meromorphic function on $\mathbb{C}$ such that

$$f(z + \omega_1) = f(z), \quad f(z + \omega_2) = f(z)$$

for any two distinct periods ω_1 and ω_2.

That is, $f(z+\omega) = f(z)$ whenever $\omega = n\omega_1 + m\omega_2$ for any integers n, m. Thus we may shift the vertex $z = 0$ of the parallelogram to any point a and the above statement still hold. We denote such parallelogram by P_a with vertices a, $a+\omega_1$, $a+\omega_2$, $a+\omega_1+\omega_2$.

We note that the $\tau = \omega_2/\omega_1$ when restricted to the fundamental region described in the Theorem 5.3.2 is unique.

Theorem 5.4.2. *An elliptic function without poles must be a constant.*

Proof. Being without poles, so an elliptic function f is bounded on the period spanned by $\{0, \omega_1, \omega_2, \ \omega_1+\omega_2\}$ which is a compact set. Hence f is a bounded entire function. Thus f is constant by Liouville's theorem. □

Theorem 5.4.3. *The sum of the residues of an elliptic function is zero.*

Proof. Without loss of generality, we may choose a so that none of the poles falls on the boundary of P_a. Hence

$$\sum \operatorname{Res} f(\text{poles}) = \frac{1}{2\pi i}\int_{P_a} f(z)\, dz = 0$$

since the integral along the opposite sides of the parallelogram have equal magnitudes but with opposite signs. □

Definition 5.4.4. The sum of orders of the poles of an elliptic function in its period parallelogram is called **the order of the function**.

We deduce that the order of an elliptic function in a period parallelogram is at least two. That is, an elliptic function cannot have a single simple pole in a period parallelogram.

Theorem 5.4.5. *A non-constant elliptic function has an equal number of poles and zeros in its period parallelogram.*

Proof. Note that the quotient $f'(z)/f(z)$ is an elliptic of function of the same periods as f. But then the last theorem asserts that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{P_a} \frac{f'(z)}{f(z)}\, dz \\ &= (\text{no. of zeros}) - (\text{no. of poles}) \end{aligned}$$

in the period parallelogram P_a. □

By considering the function

$$\frac{f(z)}{f(z)-a},$$

we deduce immediately that

Theorem 5.4.6. *An elliptic function of order $m \geq 2$ assumes every value m times in the period parallelogram (counted according to multiplicities).*

Theorem 5.4.7. *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be the zeros and poles of an elliptic function f in a period parallelogram respectively. Then*

$$\sum_{k=1}^{n}(a_k - b_k) = n\omega_1 + m\omega_2$$

for some integers n, m.

We sometime write the above conclusion in the abbreviated form $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k \pmod{M}$.

Proof. By choosing a suitable a we assume that there is no zeros and poles of f that lie on the boundary of the period parallelogram P_a. It follows from the residue theorem that

$$\frac{1}{2\pi i}\int_{\partial P_a} \frac{zf'(z)}{f(z)}\, dz = \sum_{k=1}^{n}(a_k - b_k).$$

However, the integral has another interpretation: consider the integrations of the integrand from a to $a+\omega_1$ and then from $a+\omega_2$ to $a+\omega_1+\omega_2$. So

$$\begin{aligned}
&\frac{1}{2\pi i}\Big(\int_a^{a+\omega_1} - \int_{a+\omega_2}^{a+\omega_1+\omega_2}\Big)\frac{zf'(z)}{f(z)}\,dz\\
&\qquad = \frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{zf'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{(\zeta+\omega_2)f'}{f}\,d\zeta\\
&\qquad = -\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(z)}{f(z)}\,dz
\end{aligned}$$

where the last integral is the winding number of f along the path from a to $a+\omega_1$. Hence the integral is an integral multiple of ω_2. Similar calculation over the second and the fourth sides gives an integral multiple of ω_1. This completes the proof. □

5.5 Weierstrass elliptic functions

We start to construct doubly periodic functions. Since there is no non-constant doubly periodic function with a single pole. Otherwise, such an elliptic function would contradict the sum of residues in a period parallelogram is zero. Thus a simplest elliptic function f has a double pole or at least two simple poles with opposite residues in a period parallelogram. Without loss of generality, we may assume in the former that this double pole locates at the origin $z=0$ (so that the function has zero residue at $z=0$). Moreover, we see that the function

$$f(z)-f(-z)$$

has no pole in a period parallelogram. So it must be a constant. But since $f(\omega_1/2)-f(-\omega_1/2)=0$ so that $f(-z)=f(z)$ implying that f must be an even function. We denote such an elliptic function by $\wp(z)$. Hence we have the following expansion

$$\wp(z)=\frac{1}{z^2}+a_1z^2+a_2z^4+\cdots$$

around the origin. To actually construct such an elliptic function, we could resort to **Mittag-Leffler** theorem. But in this special case we could construct such functions directly. That is, we have

Theorem 5.5.1. *Let ω_1, ω_2 be such that $\Im(\omega_2/\omega_1) \neq 0$. Then the function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\neq 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big) \tag{5.6}$$

where $\omega = n\omega_1 + n\omega_2$ for all integers n, m with $(n, m) \neq (0, 0)$, is an elliptic function with fundamental periods ω_1, ω_2.

Proof. We first show that the infinite sum

$$\sum_{\omega\neq 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big)$$

does converges away from the poles. So let $|\omega| > 2|z|$. Then

$$\Big|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big| = \Big|\frac{z(2\omega - z)}{\omega^2(z-\omega)^2}\Big| \leq \frac{10|z|}{|\omega|^3}.$$

It remains to consider the sum

$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3} = \sum_{(n,\,m)\neq(0,\,0)} \frac{1}{|n\omega_1 + m\omega_2|^3} \tag{5.7}$$

converges.

We let S_1 to denote the part of the infinite sum that runs through the points

$$\pm\omega_1, \quad \pm(\omega_1+\omega_2), \quad \pm\omega_2, \quad \pm(\omega_1-\omega_2)$$

over the lattice that are closest to the origin $(0, 0)$. There are exactly eight points. Let D and d be the longest and shortest distances of the eight points to the origin $(0, 0)$. Then we have

$$\frac{8}{D^3} \leq S_1 \leq \frac{8}{d^3}.$$

The sum S_2 over the second layer has $2 \times 8 = 16$ lattice points. But then

$$\frac{16}{(2D)^3} \le S_2 \le \frac{16}{(2d)^3}.$$

Similarly, the sum S_3 is over $3 \times 8 = 24$ lattice points. Hence

$$\frac{24}{(3D)^3} \le S_3 \le \frac{24}{(3d)^3}.$$

For S_n, we have $8n$ lattice points so that

$$\frac{8}{D^3n^2} = \frac{8n}{(nD)^3} \le S_n \le \frac{8n}{(nd)^3} = \frac{8}{d^3n^2}.$$

The above analysis is sufficient to guarantee that the $\wp$ converges uniformly in any compact subset of $\mathbb{C}$ with the lattice points ω and 0 removed.

Then

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega\neq 0} \frac{2}{(z-\omega)^3} = -2\sum_{\omega} \frac{1}{(z-\omega)^3}.$$

This shows that the $\wp'$ is doubly periodic. We deduce that

$$\wp(z+\omega_1) - \wp(z), \quad \wp(z+\omega_2) - \wp(z)$$

are both constants. We further note that the $\wp(z)$ as defined above is an even function. Substitute $z = -\omega_1$ and $z = -\omega_2$ into the above formulae shows that the two constants can only be zero. We deduce that $\wp(z)$ is doubly periodic. □

5.6 Weierstrass's Sigma and Zeta functions

Since the sum (5.7)

$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3}$$

converges, so we can form a Hadamard product

$$\sigma(z) = \sigma(z\,|\omega_1,\,\omega_2) := z \prod_{\omega\neq 0} \Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big). \tag{5.8}$$

Thus the infinite product converges uniformly in any compact subset of $\mathbb{C}$, so it represents an entire function (of order 2). It is not an elliptic function, for it would reduce to a constant otherwise. The function is called **Weiestrass's Sigma function**.

We further note that

$$\begin{aligned}\sigma(z) = z \prod_{m,\,n} \Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big)\\ \times \prod_{m',\,n'} \Big(1+\frac{z}{\omega}\Big)\exp\Big(-\frac{z}{\omega}+\frac{z^2}{2\omega^2}\Big)\end{aligned}$$

where the m', n' in the second infinite product indicate the $-m$, $-n$, for a corresponding pair m, n in the first infinite product. But the signs interchanged when z is replaced by $-z$. Hence

$$\sigma(-z) = -\sigma(z)$$

(because of the factor z) showing that the Sigma function $\sigma(z)$ is an odd function.

We now take logarithmic derivative on both sides of the Sigma function. This gives

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\Big)$$

We define the **Weiestrass's Zeta function** to be

$$\zeta(z) = \frac{d}{dz}\log\sigma(z).$$

Notice that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{z+\omega}-\frac{1}{\omega}+\frac{z}{\omega^2}\Big),$$

hence

$$\begin{aligned}\zeta(-z) &= -\frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{-z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2}\Big)\\ &= \frac{1}{z} + \sum_{\omega\neq 0}\Big(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\Big)\\ &= -\zeta(z).\end{aligned}$$

So $\zeta(z)$ is an odd function. Although the Zeta function is meromorphic, it is not an elliptic function. For it has a residue 1 at the only pole in each period parallelogram.

We now connect the Weierstrass's Sigma function and the elliptic function $\wp(z)$. It should be self-evident that

$$\wp(z) = -\frac{d}{dz}\zeta(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big).$$

Pseudo-periodicity of Zeta function

Since $-\zeta'(z) = \wp(z) = \wp(z+\omega_1) = -\zeta'(z+\omega_1)$. So

$$\zeta(z+\omega_1) = \zeta(z) + 2\eta_1, \tag{5.9}$$

for a suitable η_1. Let $z = -\omega_1/2$ in the above relation. We deduce

$$2\eta_1 = \zeta(\omega_1/2) - \zeta(-\omega_1/2) = 2\zeta(\omega_1/2)$$

because $\zeta(z)$ is odd. Hence $\eta_1 = \zeta(\omega_1/2)$. Similarly, if

$$\zeta(z+\omega_2) = \zeta(z) + 2\eta_2, \tag{5.10}$$

then $\eta_2 = \zeta(\omega_2/2)$. We also observe that $(\eta_1, \eta_2) \neq (0, 0)$ for if it were, then $\zeta(z)$ being doubly periodic would be an elliptic function, contradicting to our earlier conclusion.

The above relations (5.9) and (5.10) are called **pseudo-periodicity** of ζ.

Theorem 5.6.1. *Let η_1, η_2 be defined by $\eta_j = \zeta(\omega_j)$ $(j = 1, 2)$. Then*

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i.$$

Proof. We consider a contour following the parallelogram defined by

$$P := \big[-\frac{\omega_1}{2} - \frac{\omega_2}{2},\ \frac{\omega_1}{2} - \frac{\omega_2}{2},\ \frac{\omega_1}{2} + \frac{\omega_2}{2},\ -\frac{\omega_1}{2} + \frac{\omega_2}{2},\ -\frac{\omega_1}{2} - \frac{\omega_2}{2}\big]$$

Because the $\zeta(z)$ has a residue 1 at the only simple pole $z = 0$ inside the contour P, so Residue's theorem implies

$$\begin{aligned}
2\pi i &= \int_P \zeta(z)\,dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\,dz + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\,dz \\
&\quad + \int_{[\frac{\omega_1}{2}+\frac{\omega_2}{2},\ -\frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\,dz + \int_{[-\frac{\omega_1}{2}+\frac{\omega_2}{2},\ -\frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\,dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} \zeta(z)\,dz + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\,dz \\
&\quad - \int_{[-\frac{\omega_1}{2}+\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\,dz - \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ -\frac{\omega_1}{2}+\frac{\omega_2}{2}]} \zeta(z)\,dz \\
&= \int_{[-\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}-\frac{\omega_2}{2}]} [\zeta(z) - \zeta(z+\omega_2)]\,dz \\
&\quad + \int_{[\frac{\omega_1}{2}-\frac{\omega_2}{2},\ \frac{\omega_1}{2}+\frac{\omega_2}{2}]} [\zeta(z) - \zeta(z-\omega_1)]\,dz \\
&= (\omega_1)(-2\eta_2) + (\omega_2)(2\eta_1)
\end{aligned}$$

as required. □

The above relationship

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i.$$

is known as **Legendre's relation**.

Pseudo-periodicity of Sigma function

It follows from integrating

$$\zeta(z+\omega_1)=\zeta(z)+2\eta_1,$$

that

$$\sigma(z+\omega_1)=Ae^{2\eta_1 z}\sigma(z).$$

for some non-zero A. Putting $z=-\frac{\omega_1}{2}$ in the above equation yields

$$A=e^{\eta_1\omega_1}\frac{\zeta(\omega_1/2)}{\zeta(-\omega_1/2)}=-e^{\eta_1\omega_1}$$

since $\sigma(z)$ is an odd function. Hence

$$\sigma(z+\omega_1)=-e^{\eta_1\omega_1}e^{2\eta_1 z}\sigma(z)=-e^{\eta_1(\omega_1+2z)}\sigma(z).$$

Similarly, we have

$$\sigma(z+\omega_2)=-e^{\eta_2(\omega_2+2z)}\sigma(z).$$

Exercise 5.6.1. Let ω_3 be the period of $\wp(z)$ defined by $\omega_1+\omega_2+\omega_3=0$. Show that

1. $\eta_1+\eta_2+\eta_3=0$,
2. $\sigma(z+\omega_3)=-e^{\eta_3(\omega_3+2z)}\sigma(z)$,
3. $\pi i=\eta_2\omega_3-\eta_3\omega_2=\eta_3\omega_1-\eta_1\omega_3=\eta_1\omega_2-\eta_2\omega_1$.

5.7 The differential equation satisfied by $\wp(z)$

We recall the following expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=0}^{\infty} a_k z^k = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$$

around the origin since $\wp$ is an even function, so there are no odd coefficients in the Laurent expansion. Notice that for z sufficiently small,

$$\begin{aligned}\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{1}{\omega^2(1-z/\omega)^2} - \frac{1}{\omega^2} \\ &= \frac{1}{\omega^2}\sum_{k=1}^{\infty} k\Big(\frac{z}{\omega}\Big)^{k-1} - \frac{1}{\omega^2} \\ &= 2\,\frac{z}{\omega^3} + 3\,\frac{z^2}{\omega^4} + 4\,\frac{z^3}{\omega^5} + 5\,\frac{z^4}{\omega^6} + \cdots\end{aligned} \tag{5.11}$$

This implies that

$$a_2 = 3\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad a_4 = 5\sum_{\omega\neq 0}\frac{1}{\omega^6},$$

and so on. So

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + O(z^6)$$

where the $O(z^6)$ represents a function analytic at $z = 0$ with a zero of order 6. Hence

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + O(z^5).$$

Notice that,

$$\wp^3(z) = \frac{1}{z^6} + 3\,\frac{a_2}{z^2} + 3a_4 + O(z^2)$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_z}{z^2} - 16a_4 + O(z^2)$$

so that

$$\begin{aligned}\wp'(z)^2 - 4\wp^3(z) &= -20\frac{a_2}{z^2} - 28a_4 + O(z^2)\\ &= -20a_2\wp(z) - 28a_4 + O(z^2).\end{aligned}$$

This shows that the function

$$\Phi(z) := \wp'(z)^2 - 4\wp^3(z) + 20a_2\wp(z) + 28a_4$$

has a double zero around the origin $z = 0$ and hence analytic there. Moreover, the construction of the function Φ asserts that it is also an elliptic function with periods ω_1 and ω_2. That is, the $\Phi(z)$ is analytic at every ω which are the only potential singularities. So the $\Phi(z)$ is an entire function in $\mathbb{C}$. So it must reduce to a constant which mush equals to 0 (because the function has a double zero at $z = 0$.).

Let us summarise the above discussion into a theorem.

Theorem 5.7.1. *The elliptic function $\wp(z)$ with periods ω_1 and ω_2 satisfies the differential equation*

$$y'(z)^2 = 4y^3(z) - g_2y(z) - g_3 \tag{5.12}$$

where

$$g_2 := 20a_2 = 60\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad g_3 = 28a_4 = 140\sum_{\omega\neq 0}\frac{1}{\omega^6}.$$

We actually can have

Theorem 5.7.2. *$\wp(z)$ has Laurent expansion of the form*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k},$$

where

$$G_k = \sum_{\omega\neq 0}\frac{1}{\omega^k}, \quad k \geq 3$$

is called the Eisenstein series of order n.

Proof. Exercise. □

That is, $g_2 = 60G_4$ and $g_3 = 140G_6$.

Exercise 5.7.1. Show that

1. $$\wp''(z) = 6\wp^2 - \frac{1}{2}g_2.$$

2. $$\wp^{(3)} = 12\wp\wp'$$

3. $$\wp^{(4)} = 120\wp^3 - 18g_2\wp - 12g_3.$$

Exercise 5.7.2. Recall the Taylor expansion

$$\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} c_1 z^2 + c_2 z^4 + \cdots + c_n z^{2n} + \cdots .$$

Show that

$$(n-2)(2n+3)c_n = 3(c_1 c_{n-2} + c_2 c_{n-3} + \cdots c_{n-2} c_1), \quad n \geq 3.$$

Hence prove that each c_n is a polynomial in g_2 and g_3 with positive rational coefficients.

Exercise 5.7.3. Show that

1. $\sigma(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda\sigma(z\,|\omega_1,\, \omega_2)$,
2. $\zeta(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda^{-1}\zeta(z\,|\omega_1,\, \omega_2)$,
3. $\wp(\lambda z\,|\lambda\omega_1,\, \lambda\omega_2) = \lambda^{-2}\wp(z\,|\omega_1,\, \omega_2)$.

Three roots of $\wp'(z)$

We shall revisit the differential equation

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$$

obtained above.

We also recall that

$$\wp'(z) = -2\sum_{\omega} \frac{1}{(z-\omega)^3},$$

and it is therefore clear that the $\wp'$ is an odd elliptic function. Hence

$$\wp'(\omega_1/2) = \wp'(-\omega_1/2) = -\wp'(\omega_1/2)$$

and this immediately implies that $\wp'(\omega_1/2) = 0$. Similarly,

$$\wp'(\omega_2/2) = 0.$$

Notice that

$$\wp'(\omega_1/2 + \omega_2/2) = \wp'(-\omega_1/2 - \omega_2/2) = -\wp'(\omega_1/2 + \omega_2/2).$$

Hence $\wp'(\omega_1/2 + \omega_2/2) = 0$. Recall that $\omega_1 + \omega_2 + \omega_3 = 0$. Then

$$-\wp'(\omega_3/2) = \wp'(-\omega_3/2) = \wp'(\omega_1/2 + \omega_2/2) = 0.$$

Since the $\omega_3/2$ are incongruent module to $\omega_1/2$ and $\omega_2/2$ within a period parallelogram, so we have shown that all the three simple roots of $\wp'(z)$ there (since $\wp'$ has order 3 there).

Let

$$\wp(\omega_1/2) = e_1, \quad \wp(\omega_2/2) = e_2, \quad \wp(\omega_3/2) = e_3.$$

Since $\wp'(\omega_1/2) = 0$, so the elliptic function $\wp(z) - e_1$, which is of order 2, has a double root at $\omega_1/2$. So it cannot vanish at any other point in the period parallelogram. This implies that $e_1 \neq e_2$, and $e_1 \neq e_3$. Similarly, $e_2 \neq e_3$ so that all three numbers e_1, e_2, e_3 are distinct. It follows that

$$\wp'(z)^2/[(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)]$$

has no zero in any period parallelogram and hence in $\mathbb{C}$. Thus the quotient is a constant C, say. Hence

$$\wp'(z)^2 = C(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Comparing with the lowest term above with that in (5.12) implies that $C = 4$ which gives the desired

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Moreover, the e_1, e_2, e_3 are three roots of the algebraic equation $y^2 = 4x^3 - g_2x - g_3$.

Exercise 5.7.4. Verify

1. $e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2$,
2. $e_1e_2e_3 = \frac{1}{4}g_3$,
3. $e_1^2 + e_2^2 + e_3^2 = \frac{1}{2}g_2$.

5.8 Elliptic integrals

The differential equation

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$$

gives the solution $w = \wp(z)$. We can invert the z by

$$z = \int^w \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}.$$

More precisely,

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}},$$

where the path of integration is the path of $\wp$ on a path from z_0 to z avoiding the zeros and poles of $\wp'(z)$.

There is already a similar elliptic integral we encountered earlier under the conformal mapping of the upper half-plane $\mathbb{H}$ onto a rectangle:

$$f(z) = \alpha \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + \beta.$$

The Jacobian sine elliptic function is $w = \operatorname{sn}(z)$ is the function behind.

Chapter 6

Modular functions

This chapter is a brief introduction to modular functions.

We recall that the **modular group** consists of the set of all Möbius transformations of the from

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where a, b, c, d are integers such that (WLOG) $ad-bc = 1$. This group is denoted by Γ. Such a Möbius transformation can be represented in a matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

Definition 6.0.1. An analytic function λ which satisfies

$$\lambda\Big(\frac{az+b}{cz+d}\Big) = \lambda(z),$$

where the Möbius transformation belongs to the modular group is called an **automorphic function**.

Recall that for a given Weierstrass elliptic function $\wp(z)$, we have

$$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad e_3 = \wp(\omega_3/2),$$

where $\omega_1 + \omega_2 + \omega_3 = 0$.

6.1 The function $\lambda(\tau)$

We observe that scaling of the periods ω_k ($k = 1, 2, 3$) by $t\omega_k$ results in

$$\wp\Big(t\frac{\omega_k}{2}\Big) = \frac{1}{t^2}\wp\Big(\frac{\omega_k}{2}\Big) = \frac{1}{t^2}e_k, \quad k = 1, 2, 3.$$

Thus the function

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}, \tag{6.1}$$

is a function of $\tau := \omega_2/\omega_1$. Since the $e_j \neq e_k$ whenever $j \neq k$, so the $\lambda(\tau)$ is an analytic function in the upper half-plane $\Im(\tau) > 0$. Moreover,

$$\lambda(\tau) \neq 0, 1$$

since $e_2 \neq e_3$ and $e_1 \neq e_3$ respectively.

Applying knowledge from theta function, one can actually write

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} = 16q \prod_{k=1}^{\infty} \Big(\frac{1 + q^{2k}}{1 + q^{2k-1}}\Big)^8,$$

where $q = e^{i\pi\tau}$.

Congruent subgroup of mod 2

Suppose our initial ω_1, ω_2 is replaced by

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1, \\ \omega_1' &= c\omega_2 + d\omega_1. \end{aligned} \tag{6.2}$$

But since the $\wp(z)$ is invariant with respect to any modular transformation, so it follows from the differential equation

$$\begin{aligned} \wp'(z)^2 &= 4\wp^3(z) - g_2\wp(z) - g_3 \\ &= (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \end{aligned}$$

the corresponding e_k ($k = 1, 2, 3$) are permuted (and so changing the value of λ) under a unimodular transformation.

The identity map for the e_k $(k = 1, 2, 3)$ from the following unimodular transformation. If we choose the a, b, c, d such that $a \equiv 1 \equiv d \mod 2$ and $b \equiv 0 \equiv c \mod 2$, then this imply

$$\frac{\omega_1'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_2'}{2} \equiv \frac{\omega_2}{2}. \qquad \mod M$$

So the e_k $(k = 1, 2, 3)$ remain fixed. We may rephrase the above by writing

$$\lambda\Big(\frac{a\tau + b}{c\tau + d}\Big) = \lambda(\tau), \quad \text{when} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mod 2. \tag{6.3}$$

The collection of unimodular transformations can easily be seen to form a group, called the **congruence subgroup** mod 2 of the modular group. In general a function f that satisfies the equation $f(M\tau) = f(\tau)$ is called *automorphic*. An automorphic function with respect to a subgroup of the full modular group is called a **(elliptic) modular function**.

Incongruent subgroup of mod 2

It is sufficient to consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mod 2 \tag{6.4}$$

since the other ones can be composed from these two. The equation (6.3) would therefore be violated. Indeed, in the first case above

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1 + \omega_2}{2}, \quad \frac{\omega_1'}{2} \equiv \frac{\omega_1}{2}, \qquad \mod M$$

we have, so that $e_2 \leftrightarrow e_3$ (they are interchanged), e_1 remains fixed. We have the $\lambda(\tau)$ becomes

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} \longmapsto \frac{\lambda(\tau)}{\lambda(\tau) - 1} = \frac{e_2 - e_3}{e_1 - e_3}.$$

But the corresponding unimodular transformation is $\tau \to \tau+1$. Hence

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}.$$

This is called **Jacobi's imaginary transformation**. The second transformation corresponds to

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_1'}{2} \equiv \frac{\omega_2}{2}, \qquad \bmod M \tag{6.5}$$

so that $e_1 \leftrightarrow e_2$ and e_3 remains unchanged. We see that

$$\lambda(\tau) = \frac{e_3-e_2}{e_1-e_2} \longmapsto 1-\lambda(\tau) = \frac{e_3-e_1}{e_2-e_1}$$

the corresponding unimodular transformation is $\tau \to -1/\tau$. Hence

$$\lambda\Big(-\frac{1}{\tau}\Big) = 1-\lambda(\tau).$$ [1]

Remark. We note that the choice of the matrices representations (6.4) are far from unique. For example, if we rewrite (6.5) with

$$\frac{\omega_2'}{2} \equiv \frac{\omega_1}{2}, \quad \frac{\omega_1'}{2} \equiv -\frac{\omega_2}{2}, \qquad \bmod M \tag{6.6}$$

then we would have matrix representation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \bmod 2$$

instead of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \bmod 2.$$

[1] This formula is called Jacobi's imaginary transformation formula (1828).

6.2 Growth properties of $\lambda(\tau)$

We normalise the choice $\omega_1 = 1$ and $\omega_2 = \tau$ for ease of later discussion. We observe that

Theorem 6.2.1. *The elliptic modular function* (6.1) $\lambda(\tau)$ *is real when* τ *is purely imaginary.*

Proof. This essentially follows from the definition of the e_k, namely

$$e_3 - e_2 = \sum_{m,\, n=-\infty}^{\infty} \Big(\frac{1}{(m-\frac{1}{2}+(n-\frac{1}{2})\tau)^2} - \frac{1}{(m+(n-\frac{1}{2})\tau)^2}\Big),$$

and

$$e_1 - e_2 = \sum_{m,\, n=-\infty}^{\infty} \Big(\frac{1}{(m-\frac{1}{2}+n\tau)^2} - \frac{1}{(m+(n-\frac{1}{2})\tau)^2}\Big),$$

where the double series are absolutely convergent. If $\tau = it$ $(t > 0)$, then clearly, the above sums remain unchanged with τ is replaced by $-\tau = \bar{\tau}$. This establishes the theorem. □

Theorem 6.2.2. *The elliptic modular function* (6.1) $\lambda(\tau)$ *satisfies*

1. $\lambda(\tau) \to 0$ *as* $\Im(\tau) \to +\infty$ *uniformly with respect to the* $\Re(\tau)$,
2. *more precisely,*

$$\lambda(\tau)/e^{i\pi\tau} \to 16, \quad \Im(\tau) \to +\infty, \tag{6.7}$$

3. $\lambda(\tau) \to 1$ *as* $\tau \to 0$ *along the imaginary axis.*

Proof. Let us quote the elementary Mittag-Leffler expansion formula:

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{m=-\infty}^{\infty} \frac{1}{(z-m)^2}.$$

Applying this expansion in the definition of e_k summing first over m yields

$$e_3 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \Big(\frac{1}{\cos^2 \pi(n-\frac{1}{2})\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \Big)$$

and

$$e_1 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \Big(\frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \Big).$$

Notice that the terms $|\sin n\pi\tau|$ and $|\cos n\pi\tau|$ are comparable to $e^{|n|\pi\Im(\tau)}$ so that the above sums are uniformly convergent as $n \to \pm\infty$ when $\Im(\tau) \geq \delta > 0$ (for some $\delta > 0$). This also means that we could take limit on individual terms of the above sum as $\Im(\tau) \to +\infty$. This yields

$$e_3 - e_2 \to 0, \quad e_1 - e_2 \to \pi^2, \quad \Im(\tau) \to +\infty,$$

and hence $\lambda(\tau) \to 0$ as $\Im(\tau) \to +\infty$ as asserted. If we let $\tau \to 0$ along the imaginary axis, then we easily deduce from the equation $\lambda(-1/\tau) = 1 - \lambda(\tau)$ that $\lambda(\tau) \to 1$.

We note that the leading terms (i.e., $n = 0,\ 1$) of the above sum for $e_3 - e_2$ are given by

$$2\pi^2 \Big(\frac{4e^{\pi i\tau}}{(1+e^{\pi i\tau})^2} + \frac{4e^{\pi i\tau}}{(1-e^{\pi i\tau})^2} \Big).$$

This concludes the part (2). □

6.3 Covering property of $\lambda(\tau)$

Let

$$\Omega := \big\{z :\ 0 < \Re(z) < 1,\ \Im(z) > 0\big\} \backslash \big\{z :\ |z-1| \geq 1/2\big\} \tag{6.8}$$

We are ready to deal with

Theorem 6.3.1. *The modular function*

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

is a one-one conformal mapping $\lambda : \Omega \to \mathbb{H}$. Moreover, the mapping extends continuously to the boundary of Ω so that

1. *the image of $\partial\Omega$ is real-valued; and*
2. *the boundary points $\tau = 0,\ 1,\ \infty$ correspond to $\lambda = 1,\ \infty,\ 0$;*
3. *the $\lambda(\tau)$ is monotone on $\partial\Omega$ so that $\lambda(\partial\Omega) = (-\infty,\ \infty)$ in such a way that*
   - $\lambda : -\infty \uparrow 0$ *over* $[1,\ 1 + i\infty)$;
   - $\lambda : 0 \uparrow 1 = \lambda(0)$ *over* $(i\infty,\ 0]$;
   - $\lambda : 1 \uparrow +\infty$ *over* $\frac{1}{2} + \frac{1}{2}e^{i\theta}$ *where* $\theta : -\pi \uparrow \pi$.

Proof. We first investigate the behaviour of $\lambda(\tau)$ on the boundary of Ω. We recall from Theorem 6.2.2 that $\lambda(z)$ is real on imaginary axis. So the transformation $\tau + 1$ maps the imaginary axis onto the $\Re(\tau) = 1$. So

$$\lambda(it + 1) = \frac{\lambda(it)}{1 - \lambda(it)}$$

is therefore *real for all* $t > 0$. Moreover, the map $1/\tau$ maps the $\Re(\tau) = 1$, i.e., $\tau = 1 + it$ $(t > 0)$ onto the circle $|\tau - \frac{1}{2}| = \frac{1}{2}$[2] Let $\tau' = \frac{1}{2} + \frac{1}{2}e^{i\theta}$. Let

$$\tau = \frac{1}{1 - \tau'}, \quad \tau' = 1 - \frac{1}{\tau},$$

and

$$\tau = 1 + i\frac{\sin\theta}{1 + \cos\theta}, \quad \Re(\tau) = 1.$$

[2] $\left|1 - \frac{1}{\tau} - \frac{1}{2}\right|^2 = \left|1 - \frac{1}{1+it} - \frac{1}{2}\right|^2 = (\frac{1}{2})^2.$

Then the image of λ on $|\tau - \frac{1}{2}| = \frac{1}{2}$ can "pull-back" by the transformation:

$$\begin{aligned}\lambda(\tau') = \lambda\Big(1 - \frac{1}{\tau}\Big) &= \frac{\lambda(-1/\tau)}{1-\lambda(-1/\tau)} = \frac{1-\lambda(\tau)}{1-(1-\lambda(\tau))}\\ &= \frac{1}{\lambda(\tau)} - 1,\end{aligned}$$

hence showing that λ (where τ lies on the $\Re(\tau) = 1$) is again real on $|\tau - \frac{1}{2}| = \frac{1}{2}$ by the first case. Hence we have established that $\lambda(\tau)$ is real-valued on the whole boundary of Ω.

Since our aim is to prove $\lambda : \Omega \to \mathbb{H}$ is a one-one conformal map, so we choose an arbitrary point w_0 in $\mathbb{H}$. Then Theorem 6.2.2 (1) guarantees that there exists a number $t_0 > 0$ so that

$$w_0 \neq \lambda(\tau) = \lambda(s + it)$$

for $t \geq t_0$.

Let us consider the images of the horizontal line segment

$$L_0 := \{s + it_0 : \ 0 \leq s \leq 1\}$$

under the modular transformations λ:

1. $-1/\tau$: L_0 is mapped onto a circle C_0 tangent to the point $\tau = 0$ in the upper half-plane. Clearly, the "smaller" the circle is when the larger the $t_0 > 0$ is chosen;

2. $1 - 1/\tau$: L_0 is mapped onto a circle C_1 tangent to the point $\tau = 1$ in the upper half-plane. Clearly, the "smaller" the circle is when the larger the $t_0 > 0$ is chosen again.

We recall that the region of Ω is a "triangle" with all three angles zero (one at ∞). Let us "cut off the three angles" by removing the portions

- $\Im(\tau) > t_0$;

- the whole disc filled from C_0 tangent at $\tau = 0$ constructed above;
- the whole disc filled from C_1 tangent at $\tau = 1$ constructed above.

We write Ω_0 to denote the remaining region of Ω. Since $\lambda(\tau) \to 1$ as $\tau \to 0$ (Theorem 6.2.2 (1)), so $\lambda(-1/\tau) \approx 1$ uniformly on C_0 as $t_0 \to +\infty$.

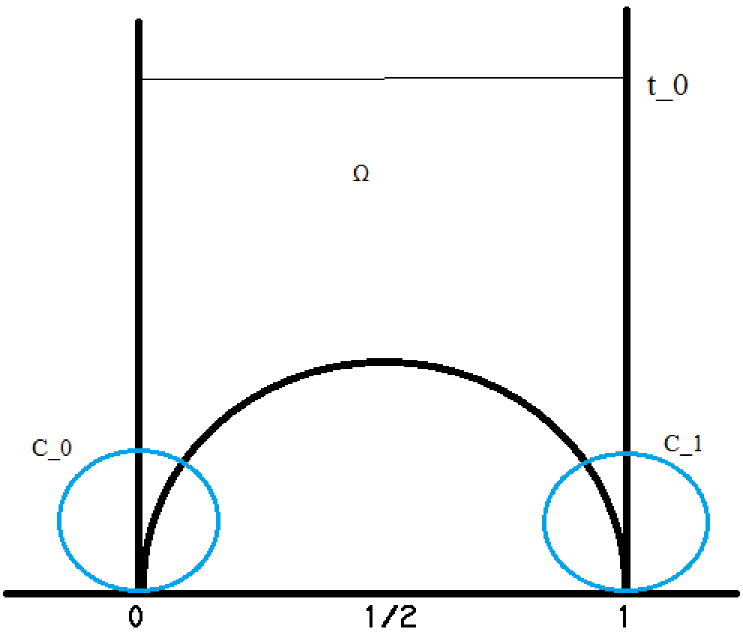

Figure 6.1: Non-Euclidean triangle with three angles 0

On the other hand, Theorem 6.2.2 (2) asserts that when τ' is close to C_1 when $\tau' \approx 1$,

$$\begin{aligned}\lambda(\tau') = \lambda(1 - 1/\tau) &= 1 - 1/\lambda(\tau)\\ &\approx 1 - \frac{1}{16}e^{-i\pi(s+it_0)}\\ &= 1 + \frac{1}{16}e^{\pi t_0 + i\pi(1-s)},\end{aligned}$$

for $0 \leq s \leq 1$, so that this is approximately a semi-circle in the upper half-plane. This together with earlier analysis shows that in the limit as $\Omega_0 \to \Omega$ as $t_0 \to +\infty$ that

$$\begin{aligned} n\big(\lambda(\partial\Omega);\, w_0\big) &= \frac{1}{2\pi i}\int_{\lambda(\partial\Omega)} \frac{d\lambda}{\lambda - w_0} \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\lambda'(T)}{\lambda(T) - w_0}\, dT \\ &= 1. \end{aligned}$$

Hence each w_0 in $\mathbb{H}$ has been "taken" once and once only by $\lambda(\tau)$ inside Ω, and none of those points with $\Im(w_0) < 0$ are taken by λ in Ω. It is clear that $\lambda(0) = 1$, $\lambda(1) = \infty$ and $\lambda(\infty) = 0$.

The above analysis shows that $\lambda : \Omega \to \mathbb{H}$ is a one-one conformal map also implies that $\lambda(\tau)$ is *monotone* on $\partial\Omega$. For suppose not, then there would be a boundary point a on $\partial\Omega$ at which $\lambda'(a) = 0$. But then, in a neighbourhood of a in Ω, we have

$$\lambda(z) = \lambda(a) + \frac{\lambda^{(k)}(a)}{k!}(z - a)^k[1 + O(z - a)],$$

where $k \geq 2$, so it is evident that the image of such neighbourhood could not lie entirely within $\mathbb{H}$. A contradiction. □

Corollary 6.3.1.1. *Let Ω' denote the region that is the mirror image of Ω reflected along the imaginary axis in $\mathbb{H}$. Then the modular function maps the Ω' onto the lower half-plane, and $\lambda(\bar{\Omega} \cup \Omega') = \mathbb{C}\backslash\{0,\, 1\}$.*

Remark. We call the modular function λ a *universal cover* of $\mathbb{C}\backslash\{0,\, 1\}$.

Exercise 6.3.1. Show that if $\tau' = \frac{1}{2} + \frac{1}{2}e^{i\theta}$, $\tau' = 1 - \frac{1}{\tau}$, then

$$\tau = 1 + i\frac{\sin\theta}{1 - \cos\theta}.$$

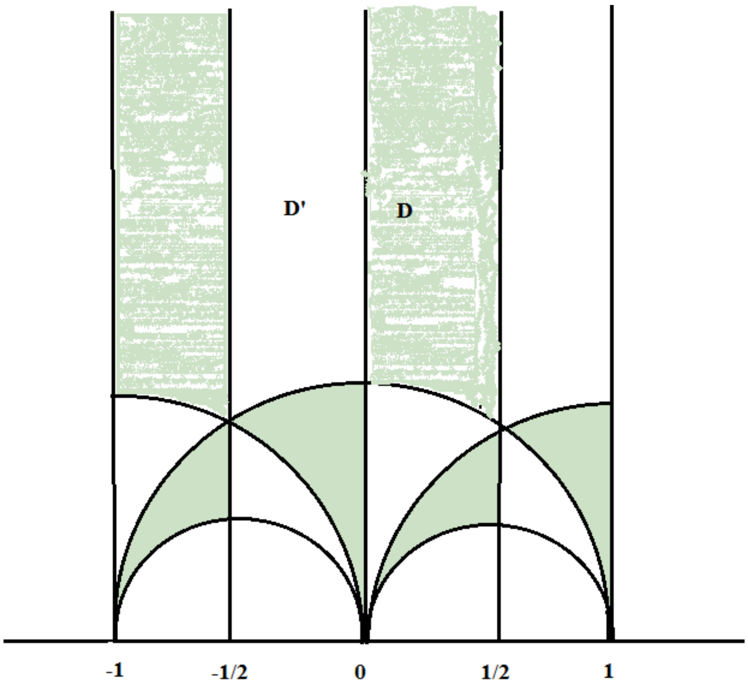

Figure 6.2: D is the "Right-half" of FR, D' its mirror-image. The figure shows regions that are reflections of D and D'

It is routine to check that the six shaded regions in the above figure are images of the Fundamental Region D under the following transformations:

$$\tau, \quad -\frac{1}{\tau}, \quad \tau - 1, \quad \frac{1}{1-\tau}, \quad \frac{\tau-1}{\tau}, \quad \frac{\tau}{1-\tau}. \tag{6.9}$$

which we denote by $S_1,\ S_2, \cdots, S_6$. They form a complete set of incongruent unimodular transformations (i.e., members of modular group) mod 2, in the sense that each unimodular transformation is congruent mod 2 to one of the S_k. Let we denote S_k^{-1} $(k = 1, \cdots, 6)$ to denote the corresponding inverses. Then it can be checked that they map the region D' (the "left-half" of the FD) onto the unshaded regions of the above figure. One see immediately that the union of 12 images of $\bar{D}$ and $\bar{D}'$ covers $\bar{\Omega} \cup \bar{\Omega}'$ (here the closure refers to the $\mathbb{H}$ only).

Let Ω' be the mirror image of Ω reflected along the imaginary axis.

Theorem 6.3.2. *Every point* τ *in the upper half-plane* $\mathbb{H}$ *is equivalent under the congruence subgroup* mod 2 *to exactly one point in* $\bar{\Omega} \cup \Omega'$.

Proof. Let τ be an arbitrary point in $\mathbb{H}$. Then according to Theorem 5.3.2 that there is a unimodular transformation S such that $S\tau$ in D, say. But there is a S_k^{-1} such that $S \equiv S_k^{-1} \mod 2$, i.e., $T = S_k S \equiv I \mod 2$. But then $T\tau = S_k(S\tau)$ belongs to one of 12 regions and hence in $\bar{\Omega} \cup \bar{\Omega}'$. A similar reasoning also applies if $S\tau \in D'$. Hence $T\tau \in \bar{\Omega} \cup \bar{\Omega}'$ in either cases. Hence $T\tau \in \bar{\Omega} \cup \Omega'$.

The uniqueness follows from the fact that the $S_1, \cdots, S_6$ as well as $S_1^{-1}, \cdots, S_6^{-1}$ are incongruent mod 2. □

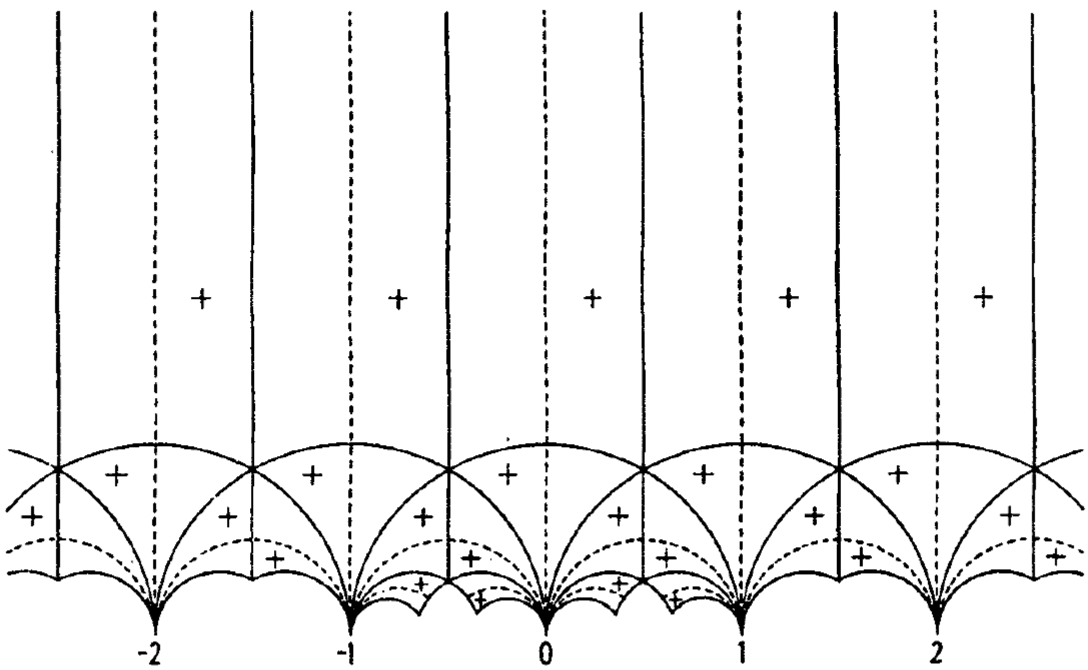

Figure 6.3: Taken from page 426 of E. T. Copson

Chapter 7

Picard's theorem

7.1 Monodromy

The terminologies below are used to handle multi-valued functions.

An analytic function f defined on a region Ω that constitute (f, Ω) is called a **function element**. A **global analytic function** is a collection of function elements (f, Ω). Two function elements (f_1, Ω_1), (f_2, Ω_2) are **direct analytic continuations** of each other if $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ over $\Omega_1 \cap \Omega_2$. There need not be any direct analytic continuation of f_1 from Ω_1 to Ω_2. But the continuation must be unique if there is such a continuation (by the identity theorem since $\Omega_1 \cap \Omega_2 \neq \emptyset$).

Suppose the chain (f_1, Ω_1), (f_2, Ω_2), $\cdots (f_n, \Omega_n)$ are analytic continuations of each other so that $\Omega_{k-1} \cap \Omega_k \neq \emptyset$ for each k. Then we say (f_n, Ω_n) is **an analytic continuation** of (f_1, Ω_1). This is an equivalence relation. The equivalence classes are called **global analytic functions**. We label the global analytic determined by the function element by $\mathbf{f}$. However, a global analytic function $\mathbf{f}$ can have many function element (f, Ω) on Ω. In fact, we call each function element (f, Ω) **a branch** of $\mathbf{f}$.

We now replace a region Ω by a single pont ζ, and we say that two function elements (f_1, ζ_1) and (f_2, ζ_2) are **equivalent** if and only if $\zeta_1 = \zeta_2$ and $f_1 = f_2$ in a neighbourhood of ζ_1 $(= \zeta_2)$. This is again an equivalence relation. In this case, the equivalence classes are called **germs** or **germs of analytic functions**. Each germ determines a unique ζ, called the **projection** of the germ, which we denote by $\mathbf{f}_\zeta$. Thus a function element (f, Ω) gives raise to a germ $\mathbf{f}_\zeta$ for each $z \in \Omega$.

Theorem 7.1.1 (Monodromy theorem)**.** *If the two arcs γ_1, γ_2 are homotopic in a region Ω, and if a given germ $\boldsymbol{f}$ at the initial point can be continued along all arcs in Ω, then the continuations of this germ along γ_1 and γ_2 lead to the same germ at the end point.*

We refer the reader to Ahlfors for its proof.

7.2 Picard's theorem

A value $a \in \mathbb{C}$ is called a **lacunary value** of an analytic function f if $f(z) \neq a$ for all z in the region Ω where f is defined. The exponential function e^z has $z = 0$ as the only lacunary value.

Theorem 7.2.1 (Picard (1879))**.** *An entire function that has at least two finite lacunary values reduces to a constant.*

Proof. WLOG, we may assume that f has two lacunary values a, b and that $a = 0$ and $b = 1$. For we could consider the function

$$F(z) = \frac{f(z) - a}{b - a},$$

otherwise. The main idea is to construct a global analytic function $\mathbf{h}$ such that its function elements (h, Ω) satisfies

$$\Im(h(z)) > 0, \quad \lambda(h(z)) = f(z), \quad z \in \Omega,$$

where $\lambda(z)$ is the elliptic modular function constructed in the last chapter. Then we want to show that $\mathbf{h}$ can be continued to along all paths. Since the $\mathbb{C}$ is simply connected, so the monodromy theorem asserts that $\mathbf{h}$ defines an entire function.

Theorem 6.3.1 asserts that there is a $\tau_0 \in \Omega$ such that

$$\lambda(\tau_0) = f(0).$$

Since the $\lambda(\tau)$ is conformal, so $\lambda'(\tau_0) \neq 0$. Therefore we can find a local inverse λ_0^{-1} of λ over a neighbourhood Δ_0 of $f(0)$ where

$$\lambda(\lambda_0^{-1}(w)) = w, \quad w \in \Delta_0$$

and

$$\lambda_0^{-1}(f(0)) = \tau_0.$$

By continuity there is a neighbourhood Ω_0 of $z = 0$ where $f(z) \in \Delta_0$. Hence we can define

$$h(z) = \lambda_0^{-1}(f(z)), \quad z \in \Omega_0.$$

Hence we have a function element (h, Ω_0). We next show that the germ **h** that the function element determines can be continued in all possible ways to become an entire function and that the continuation of (h, Ω_0) has

$$\Im(h(z)) > 0$$

throughout the continuation. If this continuation is not possible, then we can find a path $\gamma[0, t_1]$ such that h can be continued and $\Im(h)$ remains positive up the $t < t_0$, where

- either the h cannot be continued to t_1 (which is not possible),
- or

$$\Im(h(z)) \to 0, \quad t \to t_1.$$

Let us take a closer look at $t = t_1$. Then there is a $\tau_1 \in \mathbb{H}$ and $\lambda(\tau_1) = f(\gamma(t_1))$. There is a local inverse λ_1^{-1} over a neighbourhood Δ_1 of $f(\gamma(t_1))$ such that

$$\lambda_1^{-1}(f(\gamma(t_1)) = \tau_1.$$

Let Ω_1 be a neighbourhood of $\gamma(t_1)$ (in the $z-$plane) so that $f(z) \in \Delta_1$ when $z \in \Omega_1$.

Let $t_2 < t_1$ be so chosen that $\gamma(t_2) \in \Omega_1$ for $t \in [t_2, t_1]$. But $\lambda(\tau_2) = f(\gamma(t_2))$ can simultaneously be computed by

- $$\tau = h(\gamma(t_2))$$
- and
$$\tau = \lambda_1^{-1} f(\gamma(t_2))$$

indicating that the τ is "pull-back" from two different branches of $\mathbf{h}$. So Theorem 6.3.2 asserts that there is an elliptic modular transformation S that belongs to congruence subgroup mod 2 such that

$$S[\lambda_1^{-1} f(\gamma(t_2))] = h(\gamma(t_2)).$$

Thus we could define a continuation function element (h_1, Ω_1) by

$$h_1(z) = S[\lambda_1^{-1} f(z))], \quad z \in \Omega_1$$

so that h_1 and hence $\mathbf{h}$ can be continued to $\gamma(t_1)$ with

$$\lambda(h_1(z)) = f(z), \quad \Im(h_1)(z) > 0.$$

As we have anticipated that we have constructed a global analytic function $\mathbf{h}$ so that
$$\lambda(h(z)) = f(z)$$
for all function element (h, Ω). Now consider the funcion

$$e^{ih(z)},$$

which is an entire function with $|e^{ih}| \leq 1$ bounded since $\Im(h) > 0$. Liouville's theorem implies that the h in

$$\lambda(h(z)) = f(z)$$

is a constant. Hence f must also reduce to a constant. □

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