Complex Function Theory

MATH 5030

CHIANG, Yik Man[1]

September 2, 2017

[1]Typesetting by CHEUNG, Tsz Yung and Edmund CHIANG

Contents

Chapter 1

Analytic Functions

We shall give a brief review of the basic results in complex functions centred around Cauchy's integral formula in its general form and its immediate consequences.

1.1 Notations

$\mathbb{C} = \{z = x + iy : |x| < \infty, |y| < \infty, i^2 = -1\}$:= complex plane;
$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:= extended complex plane or Riemann sphere;
$B(z_0, r) = \{z : |z - z_0| < r\}$:= open disk;
$\overline{B}(z_0, r) = \{z : |z - z_0| \leq r\}$:= closed disk;
$\Re(z)$:= real part of z;
$\Im(z)$:= imaginary part of z.

Definition 1.1.1. 1. A set $S \in \mathbb{C}$ is *connected* if for any two points lying in S, there exist a polygonal curve lying entirely in S and connecting the points.

2. A *region* $G \in \mathbb{C}$ is an open connected set.

1.2 Cauchy-Riemann Equations

Definition 1.2.1. Let G be an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$. Then f is *differentiable at* $a \in G$ if the limit

$$\lim_{h \to \infty} \frac{f(a+h) - f(a)}{h}$$

exists; the value of the limit is denoted by $f'(a)$ which is called the *derivative of* f *at* a. If f is differentiable at each point of G, then we say f is *differentiable on* G.

Definition 1.2.2. A function $f : G \to \mathbb{C}$ is *analytic* if f is continuously differentiable on G i.e., f' is continuous at every point of G.

We shall show later (see Remark 1.11) that analyticity of f alone (i.e., without the continuity assumption) implies the continuity of f' (in a neighbourhood). That is, the function must be continuously differentiable. This is certainly not the case in real function theory; there exist many real functions such that their derivatives are not continuous. (e.g. $|x|$)

It is an easy exercise to show (from the definition) that if $f(z) = u(x,y) + iv(x,y)$ is analytic, then u and v satisfy the *Cauchy-Riemann equations* at z:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that the partial derivatives are continuous and the converse is also true.

Theorem 1.2.3. *Let u and v be real-valued functions defined on a region G and suppose that they have continuous derivatives there. Then $f : G \to \mathbb{C}$, $f = u + iv$ is analytic if and only if both u and v satisfy the Cauchy-Riemann equations.*

Proof. See Conway p.41-42. □

1.3 Line Integrals

Definition 1.3.1. A *path* in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ $(a < b)$. A path is *smooth* if γ' exists and also continuous on $[a, b]$. Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition on $[a, b]$, then a path $\gamma : [a, b] \to G$ is *piecewise smooth* if it is smooth on each subinterval $[t_{i-1}, t_i], i = 1, \ldots, n$.

Remark. We note that if $\gamma'(t) \neq 0$ implies that γ has a tangent at t. Some authors will simply assume, in addition to the existence and the continuity for the smooth curve γ, to have $\gamma' \neq 0$.

Definition 1.3.2. We define the *length* of a piecewise smooth curve to be

$$l(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

This is clearly a well-defined number. Suppose that $f : G \to \mathbb{C}$ is continuous and $\gamma[a, b] \subset G$, we define the *line integral* along γ to be the number

$$\int_\gamma f = \int_a^b f\, d\gamma = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

In fact, it can be shown that the integral always exists (see Conway p.60-62) and it is independent of any particular parametrization (see Conway p.63-64).

Definition 1.3.3. Let f and γ be defined as above. Then we define the *line integration of f along γ with respect to the arc length* as

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt. \tag{1.1}$$

The integral clearly exists since f is continuous, and γ is piecewise continuous. It is easy to verify that

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f|\,|dz|.$$

Remark. The (1.1) becomes $l(\gamma)$ if $f(t) \equiv 1$.

Theorem 1.3.4. *Let $\gamma : [a, b] \to G$ be a piecewise smooth path in a region G with initial and end points α and β. Suppose $f : G \to \mathbb{C}$ is continuous with primitive $F : G \to \mathbb{C}$ (i.e. $F' = f$), then*

$$\int_\gamma f = F(\beta) - F(\alpha). \quad (\gamma(a) = \alpha, \gamma(b) = \beta)$$

Proof. By definition of line integral above,

$$\begin{aligned} \int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F \circ \gamma)'(t)\,dt \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(\beta) - F(\alpha). \end{aligned}$$

by the Fundamental Theorem of Calculus. □

Definition 1.3.5. A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.

We deduce immediately from the above theorem that

$$\int_\gamma f = 0$$

when γ is a closed piecewise smooth path and with f as in the above theorem.

Remark. (i) All of the above definitions and results about piecewise smooth paths can be generalized to *rectifiable paths*. We shall restrict ourselves to piecewise smooth paths in the rest of the course. See Conway for more details.

(ii) Although the treatment here (and in most books) about line integral is short, complex line integral is considered to be a very important contribution from Cauchy (in a paper dated 1825).

1.4 Local Cauchy Integral Formula

Theorem 1.4.1 (Local Cauchy Integral Formula)**.** *Let* $f : G \to \mathbb{C}$ *be analytic and that* $\overline{B}(a,r) \subset G$, $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. *Then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for any $z \in B(a,r)$.

To prove this theorem, we require

Proposition 1.4.2. *Let* $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ *be a continuous function. Define* $g : [c,d] \to \mathbb{C}$ *by*

$$g(t) = \int_a^b \varphi(s,t)\, ds.$$

Then g *is continuous. Moreover, if* $\dfrac{\partial \varphi}{\partial t}$ *exists and is a continuous function on* $[a,b] \times [c,d]$, *then* g *is continuously differentiable on* $[c,d]$ *and*

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\, ds. \tag{1.2}$$

Proof. Since $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ is continuous and hence it just be uniformly continuous on its domain. It follows easily that g, as defined above, must be continuous on $[c,d]$. In order to prove (1.2), it suffices to show that

$$\frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \frac{\partial \varphi}{\partial t}(s, t_0)\, ds$$

can be made arbitrarily small.

Since $\varphi_t(s,t) = \dfrac{\partial \varphi}{\partial t}(s,t)$ is continuous on $[a,b] \times [c,d]$, it must be uniformly continuous there. Thus, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi_t(s',t') - \varphi_t(s,t)| < \epsilon$$

whenever $(s'-s)^2+(t'-t)^2<\delta^2$. In particular,

$$|\varphi_t(s,t)-\varphi_t(s,t_0)|<\epsilon$$

if $a\le s\le b$ and $|t-t_0|<\delta$. Hence for $|t-t_0|<\delta$, we have

$$\left|\int_{t_0}^{t}\varphi_t(s,\tau)-\varphi_t(s,t_0)\,d\tau\right|<\epsilon|t-t_0|.$$

But the integrand of the last inequality equals, with a fixed s,

$$\begin{aligned}(\varphi(s,t)&-t\varphi_t(s,t_0))-(\varphi(s,t_0)-t_0\varphi_t(s,t_0))\\&=\varphi(s,t)-\varphi(s,t_0)-(t-t_0)\varphi_t(s,t_0).\end{aligned}$$

Hence

$$|\varphi(s,t)-\varphi(s,t_0)-(t-t_0)\varphi_t(s,t_0)|<\epsilon|t-t_0|$$

whenever $a\le s\le b$ and $|t-t_0|<\delta$. But this is precisely

$$\left|\frac{g(t)-g(t_0)}{t-t_0}-\int_a^b\varphi_t(s,t_0)\,ds\right|<\epsilon|b-a|$$

after integration with respect to s on both sides. This proves $g'(t)=\int_a^b\varphi_t(s,t)\,ds$. But φ_t is continuous and so g' must also be continuous. □

Example 1.4.3. Show that

$$\int_0^{2\pi}\frac{e^{is}}{e^{is}-z}\,ds=2\pi$$

whenever $|z|<1$.

Solution. Since $\varphi(s,t)=\dfrac{e^{is}}{e^{is}-tz}$, for $0\le t\le 1$, $0\le s\le 2\pi$, is continuously differentiable, it follows from Prop 1.4.2 that

$$g(t)=\int_0^{2\pi}\varphi(s,t)\,ds=\int_0^{2\pi}\frac{e^{is}}{e^{is}-tz}\,ds.$$

But

$$\int_0^{2\pi} \frac{ze^{is}}{(e^{is}-tz)^2}\,ds = \left.\frac{-iz}{e^{is}-tz}\right|_0^{2\pi} = \frac{-iz}{e^{2\pi i}-tz} - \frac{-iz}{e^0-tz} = 0.$$

for all $t \in [0,1]$. Hence $g(t) = \text{constant}$, and in particular,

$$g(0) = \int_0^{2\pi} \frac{e^{is}}{e^{is}-0}\,ds = 2\pi.$$

For $t = 1$, we have the required equality. □

Now, we are sufficiently prepared to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. For any $\overline{B}(a,r) \subset G$, we are required to show

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

where $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$.

Without loss of generality, it is clear that we may consider $a = 0$ and $r = 1$ only. Since the translation $f(a+rz)$ will take that $B(0,1)$ to any preassigned $B(a,r)$. Thus we aim to show

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z}\,ds, \quad z \in B(0,1).$$

Consider

$$\varphi(s,t) = \frac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z} - f(z),$$

where $t \in [0,1]$, $s \in [0,2\pi]$, $|z| < 1$. Clearly φ is continuously differentiable. Hence

$$g(t) = \int_0^{2\pi} \varphi(s,t)\,ds$$

is also continuously differentiable, and

$$\begin{aligned} g'(t) &= \int_0^{2\pi} \frac{\partial}{\partial t}\left(\frac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z)} - f(z)\right) ds \\ &= \int_0^{2\pi} \frac{(e^{is}-z)f'(z+t(e^{is}-z))e^{is}}{e^{is}-z} ds \\ &= \int_0^{2\pi} f'(z+t(e^{is}-z))e^{is}\, ds \\ &= \frac{1}{it} f(z+t(e^{is}-z))\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

for each $t \in [0,1]$. Hence $g(t) = \text{constant}$. Then

$$\int_0^{2\pi} \left(\frac{f(z)e^{is}}{e^{is}-z} - f(z)\right) ds = g(0) = g(1) = \int_0^{2\pi} \left(\frac{f(e^{is})e^{is}}{e^{is}-z} - f(z)\right) ds.$$

But

$$\int_0^{2\pi} \left(\frac{f(z)e^{is}}{e^{is}-z} - f(z)\right) ds = f(z)\int_0^{2\pi} \left(\frac{e^{is}}{e^{is}-z} - 1\right) ds = 0$$

by the Example 1.4.3 above. Hence $g(1) = 0$. And this is precisely

$$2\pi f(z) = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z} ds = \frac{1}{i}\int_\gamma \frac{f(w)}{w-z} dw.$$

The result follows. □

1.5 Consequences

We now investigate some consequences of the local Cauchy Integral formula.

Theorem 1.5.1. *Let f be analytic on $B(a,R)$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $z \in B(a, R)$ *where* $a_n = \dfrac{f^{(n)}(a)}{n!}$ *and the series has radius of convergence at least* R.

Proof. Let $r > 0$ such that $\overline{B}(a, r) \subset B(a, R)$. Suppose $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. Define $M = \max_{z\in\gamma[0,2\pi]} |f(z)|$ since $\gamma[0, 2\pi]$ is compact and f is continuous on $\gamma[0, 2\pi]$. By Theorem 1.4.1, we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad \zeta = \gamma(t) = a + re^{it}.$$

We claim that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a + a - z}\, d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)}\, d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}\sum_{k=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^k d\zeta \\ &= \sum_{k=0}^{\infty}(z-a)^k \cdot \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta := \sum_{k=0}^{\infty} a_k (z-a)^k. \end{aligned}$$

This is because

$$\left|\frac{z-a}{\zeta-a}\right| < 1 \quad \text{and} \quad \left|\frac{f(\zeta)}{\zeta-a}\left(\frac{z-a}{\zeta-a}\right)^k\right| \le \frac{M}{r}\left(\frac{|z-a|}{r}\right)^k.$$

So the series $\sum \dfrac{f(\zeta)}{\zeta-a}\left(\dfrac{z-a}{\zeta-a}\right)^k$ converges uniformly by applying M-test.

Thus we could interchange the integral and summation signs in the above computation. But the series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

can be differentiated indefinitely within its radius of convergence, and the derivatives are given by

$$f^{(n)}(z) = \sum_{k=0}^{\infty} n(n-1)\cdots(n-k+1)a_k(z-a)^{k-n}, \quad n = 1, 2, 3, \cdots$$

so that

$$f^{(n)}(a) = n!a_n.$$

Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta = a_n = \frac{f^n(a)}{n!}$$

for each $n \geq 0$. This completes the proof. □

We deduce immediately from the above theorem that

Theorem 1.5.2. *Suppose $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a,r) \subset G$. Then*

(i) f is infinitely differentiable; and

(ii)

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta, \quad \gamma(t) = a + re^{it.}$$

The next theorem is another very important result in complex analysis. It will be derived from Theorem 1.5.1 above. However, some authors prefer to derive it directly and deduce the Cauchy Integral formula as a consequence.

Theorem 1.5.3. *Let f be analytic on $B(a,R)$ and suppose γ is any closed piecewise smooth curve in $B(a,R)$. Then f has a primitive and*

$$\int_\gamma f = 0.$$

Proof. Suppose $z \in B(a, R)$ and $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ by Theorem 1.5.1. It can be easily verified that the function defined by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$$

has the same radius of convergence as that of $f(z)$. Clearly F is differentiable, and $F'(z) = f(z)$. Hence, F is a primitive of f in $B(a, R)$.

Suppose $\gamma : [a, b] \to \mathbb{C}$ is as in the assumption, then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b F'(\gamma(t)\gamma'(t)\,dt \\
&= \int_a^b \frac{d}{dt}F(\gamma(t))\,dt \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= 0
\end{aligned}$$

since γ is closed. □

1.6 Liouville's Theorem

Definition 1.6.1. We say a function f that is analytic everywhere in $\mathbb{C}$ an *entire function.*

Clearly, any entire function has the power series representation in $B(a, r)$ for any $a \in \mathbb{C}$ and any $r > 0$. So the power series must have an infinite radius of convergence.

Proposition 1.6.2. *Let G be an region. If $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is a constant on G.*

Proof. Let $z_0 \in G$ and $f(z_0) = w_0$. Set

$$A = \{z \in G : f(z) = w_0\} \subset G.$$

We aim to show that $A = G$ by proving that A is both open and closed. Then a standard topological argument gives $A = G$. Hence, f is constant on G.

Let $\{z_n\}$ be a sequence in A and $z_n \to z$ as $n \to \infty$. Then by the continuity of f, we have

$$w_0 = \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(z).$$

Hence z belongs to A. This proves that A is closed.

Let $a \in A$, $B(a, \epsilon) \subset G$ and $z \in B(a, \epsilon)$. Let

$$g(t) = f(tz + (1-t)a), \quad 0 \le t \le 1.$$

Then

$$\begin{aligned}\frac{g(t) - g(s)}{t - s} &= \frac{f(tz + (1-t)a) - f(sz + (1-s)a)}{tz + (1-t)a - (sz + (1-s)a)} \\ &\quad \cdot \frac{tz + (1-t)a - (sz + (1-s)a)}{t - s} \\ &\to f'(sz + (1-s)a) \cdot (z - a) \quad \text{(Chain rule)} \\ &= 0 \cdot (z - a) = 0,\end{aligned}$$

as $t \to s$. That is $g'(s) = 0$. So $f(z) = g(1) = g(0) = f(a) = w_0$. Since $z \in B(a, \epsilon)$ is arbitrary, we conclude that $B(a, \epsilon) \subset A$. Hence A is open. This completes the proof. □

Theorem 1.6.3 (Liouville's Theorem)**.** *Any bounded entire function must reduce to a constant. That is, there is no non-constant entire function.*

Proof. Let $z \in B(z, r) \subset \mathbb{C}$. Then Theorem 1.5.2 implies

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad \gamma = z + re^{it}, \quad t \in [0, 2\pi]$$

So

$$|f'(z)| \le \frac{1}{2\pi i}\int_\gamma \frac{|f(z)|}{|\zeta - z|^2}|ire^{it}|\,dt$$
$$\le \frac{\text{upper bound of } |f|}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

Hence $f'(z) = 0$ for every $z \in \mathbb{C}$.

Alternatively,

$$|a_n| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta\right|$$
$$\le \frac{\text{upper bound of } |f|}{r^n} \to 0 \quad \text{as} \quad r \to \infty$$

for each $n \ge 1$. Hence

$$f(z) = \sum a_n(z-a)^n = a_0 = \text{constant}.$$

□

Definition 1.6.4. Let $f : G \to \mathbb{C}$ and $a \in G$ such that $f(a) = 0$. Then a is a *zero of f with multiplicity $m \ge 1$* if there is an analytic function g such that $f(z) = (z-a)^m g(z)$ and $g(a) \ne 0$.

We deduce the following important theorem from the Louville Theorem.

Theorem 1.6.5 (Fundamental Theorem of Algebra)**.** *Every polynomial $P(z) = a_n z^n + \cdots + a_0$ can be factored as*

$$P(z) = c(z-b_1)^{k_1}\cdots(z-b_m)^{k_m},$$

where c is a constant, $b_1, \ldots, b_m$ are the zeros of P and $k_1 + \cdots + k_m = n$.

Proof. It suffices to show that P has at least one zero if it is non-constant, so that we have $P(z) = (z-a)g(z)$, and then obtain the general form via induction on the degree of P.

So let us suppose that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$F(z) := \frac{1}{P(z)}$$

is an entire function on $\mathbb{C}$. But $F(z) \to 0$ uniformly as $z \to \infty$ along all possible paths, so we can find an $M' > 0$ and $R > 0$ such that $|F(z)| < M$ for $z \in \mathbb{C} \setminus B(0, R)$.

Notice that F is also continuous on $\overline{B}(0, R)$ since P has no zeros there. Hence we may find a $M'' > 0$ such that $|F| < M''$ on $\overline{B}(0, R)$ since the closed disk is a compact set and F is continuous on it.

Let $M = \max\{M', M''\}$, we see that $|F| < M$ for all $z \in \mathbb{C}$. So F, and hence P, must reduce to a constant by Louville's theorem. It contradicts to the assumption that P is an arbitrary polynomial. □

1.7 Maximum Modulus Theorem

Theorem 1.7.1 (Isolated Zero Theorem)**.** *Let G be a region, $f : G \to \mathbb{C}$ be analytic. if the set $Z := \{z \in G : \ f(z) = 0\}$ has a limit point in G, then $f \equiv 0$ in G.*

Proof. Let a be a limit point of $Z := \{z \in G : \ f(z) = 0\}$. Then we can find a sequence $\{z_n\}$ in G, $z_n \to a$ and $f(z_n) = 0$. Since

$$0 = \lim_{n\to\infty} f(z_n) = f(a),$$

so $f(a) = 0$. Theorem 1.5.1 implies that for some $R > 0$ such that $B(a, R) \subset G$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a_k)^k$$

Suppose that there is an integer N where $0 = a_0 = a_1 = \cdots = a_{N-1}$ but $a_N \neq 0$. Then we can write

$$f(z) = (z - a)^N g(z),$$

in $B(a,\, R)$ where g is analytic there and $g(a) \neq 0$. But since g is analytic and hence continuous in $B(a,\, R)$, we can find $0 < r < R$ such that $g(z) \neq 0$ in $B(a,\, r)$. But since a is a limit point, so there is a $b \in B(a,\, r)$ different from a such that $0 = f(b) = (b-a)^N g(b) \neq 0$. A contradiction. So no such integer N can be found. Thus, the set

$$A := \{z \in G : \; f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

is non-empty.

We next show that A is both closed and open. Let z belongs to the closure of A, and $\{z_k\} \subset A$ converges to z. Since each $f^{(n)}$ is continuous, it follows that $0 = \lim_{k\to\infty} f^{(n)}(z_k) = f^{(n)}(z)$. Hence $z \in A$ and A is closed.

Let $a \in A$ and $B(a,\, R) \subset G$. Then $f(z) = \sum a_k(z-a)^k$ in $B(a,\, R)$, and $f^{(n)}(a) = 0$ for each n. So $f(z) = 0$ in $B(a,\, R)$. Then clearly $B(a,\, R) \subset A$. Hence A is open. Since A is non-empty, so $A = G$. □

Corollary 1.7.1.1 (Identity Theorem)**.** *If $f = g$ on a sequence of points having a limit point in G, then $f \equiv g$ on G.*

Theorem 1.7.2 (Maximum Modulus Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ is analytic. If there exists a point $a \in G$ such that $|f(z)| \leq |f(a)|$ for all $z \in G$, then f is constant.*

Proof. Let z_0 be an arbitrary point in G such that $|f(z_0)| = |f(a)|$, $B(z_0, r) \subset G$, $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

By Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt. \end{aligned}$$

We may suppose that $|f|$ is non-constant on $\partial B(z_0, r)$ for some $r > 0$. Hence, there exists a $t_0 \in [0, 2\pi]$ and $\delta > 0$ such that

$$|f(z_0 + re^{it})| < M = |f(a)| \quad \text{on } [t_0 - \delta, t_0 + \delta].$$

Hence

$$\begin{aligned} M = |f(z_0)| &\leq \left| \frac{1}{2\pi} \int_{t \in [0,2\pi] \setminus [t_0 - \delta, t_0 + \delta]} f(z_0 + re^{it})\, dt \right| \\ &\quad + \left| \frac{1}{2\pi} \int_{t \in [t_0 - \delta, t_0 + \delta]} f(z_0 + re^{it})\, dt \right| \\ &< \frac{M}{2\pi}(2\pi - 2\delta) + \frac{M}{2\pi} 2\delta = M. \end{aligned}$$

A contradiction since $M \not< M$. Hence $|f| \equiv M$ in $B(z_0, r)$, then f is constant in $B(z_0, r)$ (Use $f' = u_x + iv_x$ and Proposition 1.6.2). Now, since $B(z_0, r)$ is non-empty open subset of G, then by the Identity Theorem, f is constant on G. □

Theorem 1.7.3 (Minimum Modulus Theorem)**.** *Let $f : G \to \mathbb{C}$ be analytic and G is a region. If there exists $a \in G$ such that $|f(z)| \geq |f(a)|$ for all $z \in G$, then either f is a constant or $f(a) = 0$ i.e. a is zero of f.*

Proof. Exercise. □

1.8 Branch of the Logarithm

Definition 1.8.1. Let G be a region and $f : G \to \mathbb{C}$ is continuous. We call $f(z)$, *a branch of the logarithm* if $e^{f(z)} = z$ for every $z \in G$.

If $e^w = z$, then we write $w = \log z = f(z)$. But $e^{w + 2\pi ik} = e^w = z$ for every integer k. Hence for each z, the equation $e^w = z$ has an infinite number of solution for $w = \log|z| + i(\arg z + 2\pi k)$. Let $G = \mathbb{C} \setminus \{x : x \leq 0\}$ and $-\pi < \arg z < \pi$. The function

$$f(z) = \log|z| + i \arg z, \quad z \in G$$

is called the *principal branch of the logarithm.* The *other branches* of the logarithm are given by

$$f_k(z) = \log|z| + i \arg z$$

for $(2k-3)\pi < \arg z < (2k-1)\pi.$, $k \in \mathbb{Z} \setminus \{1\}$. (Principal branch $f = f_1$, i.e. $k = 1$)

The principal branch of the logarithm is analytic on $\mathbb{C} \setminus \{x : x \leq 0\}$.

Proposition 1.8.2. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed piecewise smooth curve and assume that $a \notin \gamma$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - a} \in \mathbb{Z}.$$

This proposition seems trivial since

$$\int_\gamma \frac{d\zeta}{\zeta - a} = \int_\gamma d(\log(\zeta - a)) = \int_\gamma d(\log|\zeta - a|) + i \int_\gamma d(\arg(\zeta - a)).$$

When γ has described a complete revolution, $\gamma(t)$ returns to its initial position, so the first integral $\int_\gamma d(\log|\zeta - a|) = 0$; and $i \int_\gamma d(\arg(\zeta - a))$ gives $2\pi i k$, where k is the number of the complete revolutions that γ around a. However, the function $\arg(\zeta - a)$ is *not uniquely determined*, so the above argument is not precise.

Proof. One of the easiest proofs available is to consider the function

$$g(t) = \int_0^t \frac{\zeta'(t)}{\zeta(t) - a}\, dt.$$

Note that

$$g(1) = \int_0^1 \frac{\zeta(t)}{\zeta(t) - a}\, dt = \int_\gamma \frac{d\zeta}{\zeta - a}.$$

We aim to show that $\dfrac{e^{g(t)}}{\zeta(t)-a}$ is constant on $[0,1]$. Consider

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^{g(t)}}{\zeta(t)-a}\right) &= \frac{g'(t)e^g}{\zeta(t)-a} - \frac{\zeta'(t)e^g}{(\zeta(t)-a)^2} \\ &= e^g\left(\frac{\zeta'(t)}{(\zeta(t)-a)^2} - \frac{\zeta'(t)}{(\zeta(t)-a)^2}\right) \\ &= 0\end{aligned}$$

for $t \in [0,1]$. Thus

$$\frac{e^{g(0)}}{\zeta(0)-a} = \frac{e^{g(1)}}{\zeta(1)-a} \implies e^{g(0)} = e^{g(1)}.$$

But $g(0) = 0$, so $e^{g(1)} = 1$.
Hence

$$g(1) = \int_0^1 \frac{\zeta'(t)}{\zeta(t)-a}\,dt = \int_\gamma \frac{d\zeta}{\zeta - a} = 2\pi i k$$

for some integer k. Then the result follows.

□

Definition 1.8.3. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed and piecewise smooth curve, and $a \notin \gamma$. We define

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - a}$$

to be the *index of* γ *with respect to* a or the *winding number of* γ *around* a.

Suppose $\gamma(t) : [0,1] \to \mathbb{C}$ is a curve, we define $-\gamma(t) = \gamma(1-t)$. If $\sigma : [0,1] \to \mathbb{C}$ is another curve such that $\gamma(1) = \sigma(0)$, then $\gamma + \sigma$ means

$$(\gamma+\sigma)(t) = \begin{cases}\gamma(2t), & 0 \le t \le \frac{1}{2} \\ \sigma(2t-1), & \frac{1}{2} < t \le 1.\end{cases}$$

It is left as an exercise to verify that

(i) $n(-\gamma; a) = -n(\gamma; a)$

(ii) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$.

Proposition 1.8.4. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed and piecewise smooth curve, and $a \notin \gamma$. Then $n(\gamma; a)$ is constant for any a belongs to a bounded component of $\mathbb{C} \setminus \gamma$, and zero for a belongs to the unbounded component.*

Remark. There is only one unbounded component since γ is a compact set.

Proof. Let a and b belong to the same component D of $\mathbb{C} \setminus \gamma$. Since $n(\gamma; a)$ and $n(\gamma; b)$ both equal to some integers, it suffices to prove $n(\gamma; a)$ is continuous on D. (Then, $n(\gamma; D)$ is connected, and since $n(\gamma; D) \subset \mathbb{Z}$, $n(\gamma; D)$ is a constant integer only.)

Let $d = \min_{\zeta \in \gamma}\{|\zeta - a|, |\zeta - b|\}$. Then, by definition,

$$
\begin{aligned}
|n(\gamma; a) - n(\gamma; b)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{\zeta - a} - \frac{1}{\zeta - b} \right) d\zeta \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{a - b}{(\zeta - a)(\zeta - b)} d\zeta \right| \\
&\le \frac{1}{2\pi} \int_\gamma \frac{|a - b|}{|(\zeta - a)(\zeta - b)|} |d\zeta| \\
&\le \frac{|a - b|}{2\pi d^2} \int_\gamma |d\zeta| \\
&= \frac{|a - b|}{2\pi d^2} l(\gamma) \to 0,
\end{aligned}
$$

as $|a - b| \to 0$. Hence $n(\gamma; a)$ is continuous on any components of $\mathbb{C} \setminus \gamma$.

For a belongs to the unbounded component of $\mathbb{C} \setminus \gamma$, let $d = \min_{\zeta \in \gamma}\{|\zeta - a|\}$ By the above argument, we have

$$
\begin{aligned}
|n(\gamma; a)| &= \frac{1}{2\pi} \left| \int_\gamma \frac{d\zeta}{\zeta - a} \right| \\
&\le \frac{1}{2\pi d} l(\gamma).
\end{aligned}
$$

But $\min_{\zeta\in\gamma}\{|\zeta - a|\} \to \infty$ as $a \to \infty$. Hence, $|n(\gamma; a)| \to 0$ as $a \to \infty$. Since $n(\gamma; a)$ is constant and so $n(\gamma; a) = 0$ in this unbounded component because $n(\gamma; a)$ was proved to be continuous. □

1.9 Cauchy's Theorem

We next prove the general Cauchy's Integral formula and Cauchy's theorem. In particular, we give conditions on $n(\gamma; a)$ so that the Cauchy's theorem holds.

Proposition 1.9.1. *Let γ be a piecewise smooth curve and φ is a function continuous on γ. For each $m \geq 1$, define, for $z \notin \gamma$*

$$F_m(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^m}\, d\zeta.$$

Then, F_m is analytic on $\mathbb{C} \setminus \gamma$ and $F_m' = mF_{m+1}$.

Proof. We first show that F_m is continuous on $\mathbb{C}\setminus\gamma$. Since γ is compact and φ is continuous on γ, we may let $M = \max_{z\in\gamma} |\varphi(z)|$.

Let a and b belong to the same component (if any) of $\mathbb{C} \setminus \gamma$. Then, as in the proof for $n(\gamma; a)$,

$$\begin{aligned} |F_m(a) - F_m(b)| &= \left| \int_\gamma \left(\frac{\varphi(\zeta)}{(\zeta - a)^m} - \frac{\varphi(\zeta)}{(\zeta - b)^m} \right) d\zeta \right| \\ &\leq M \int_\gamma \left| \frac{1}{(\zeta - a)^m} - \frac{1}{(\zeta - b)^m} \right| \cdot |d\zeta|. \end{aligned}$$

So, it remains to estimate the function inside the integrand: Since

$$A^m - B^m = (A - B)(A^{m-1} + A^{m-2}B + \cdots + AB^{m-2} + B^{m-1}).$$

Putting $A = \dfrac{1}{\zeta - a}$ and $B = \dfrac{1}{\zeta - b}$, and let $d = \min_{\zeta\in\gamma}\{|\zeta - a|, |\zeta - b|\}$, gives

$$|F_m(a) - F_m(b)| \leq mM \frac{|a - b|}{d^{m+1}} l(\gamma) \to 0 \quad \text{as } a \to b.$$

Hence, F_m is continuous on $\mathbb{C} \setminus \gamma$.

Let $a,\ b \in \mathbb{C} \setminus \gamma$ and $A,\ B$ as defined above. Then

$$\begin{aligned}
\frac{F_m(a) - F_m(b)}{a-b} &= \frac{1}{a-b}\int_\gamma \varphi(\zeta)(A-B)(A^{m-1} + A^{m-2}B + \cdots + AB^{m-2} + B^{m-1})\,d\zeta \\
&= \frac{1}{a-b}\int_\gamma \varphi(\zeta)(a-b)AB(A^{m-1} + A^{m-2}B + \cdots \\
&\qquad + AB^{m-2} + B^{m-1})\,d\zeta \\
&= \int_\gamma \varphi(\zeta)(A^m B + A^{m-1}B^2 + \cdots + AB^m)\,d\zeta \\
&\longrightarrow \int_\gamma \varphi(\zeta)(B^{m+1} + B^{m+1} + \cdots + B^{m+1})\,d\zeta \\
&= m\int_\gamma \varphi(\zeta)B^{m+1}\,d\zeta \\
&= m\int_\gamma \frac{\varphi(\zeta)}{(\zeta-b)^{m+1}}\,d\zeta \\
&= F_m'(b)
\end{aligned}$$

as $a \to b$.

Hence, F_m is analytic with its derivative, which is given at the end in the above expression. □

Theorem 1.9.2 (Cauchy's Integral Formula - First version)**.** *Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If γ is a closed piecewise smooth curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \gamma$,*

$$n(\gamma; a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}\,d\zeta.$$

Proof. Define $\varphi : G \times G \to \mathbb{C}$ by

$$\varphi(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & \text{if } z \neq w \\ f'(z), & \text{if } z = w. \end{cases}$$

(Exercise: Show φ is continuous and $z \mapsto \varphi(z,w)$ is analytic.)

Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Then H is open since $n(\gamma; w)$ is continuous on $\mathbb{C} \setminus \gamma$ and integer-valued i.e. $\{0\}$ is open in $\mathbb{Z}$. From the definition of G and H, we deduce that $\mathbb{C} = G \cup H$ and $G \cap H \neq \emptyset$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\gamma \varphi(z, \zeta)\, d\zeta, & \text{if } z \in G \\ \int_\gamma \dfrac{f(\zeta)}{\zeta - z}\, d\zeta, & \text{if } z \in H. \end{cases}$$

Next, we verify that g is well-defined on $G \cap H$.

$$\begin{aligned} \int_\gamma \varphi(z, \zeta)\, d\zeta &= \int_\gamma \frac{f(z) - f(\zeta)}{z - \zeta}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta}{\zeta - z}\, d\zeta - f(z) \cdot 2\pi i\, n(\gamma; z) \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \because z \in G \cap H \end{aligned}$$

Hence, g is a well-defined function on $\mathbb{C}$.

It follows from Proposition 1.9.1 that g is an entire function, and from Proposition 1.8.4, H must contain the unbounded component of $\mathbb{C} \setminus \gamma$ (because if $n(\gamma; w) = 0$, then $w \in H$). For z belongs to the unbounded component, we have

$$\lim_{z \to \infty} g(z) = \lim_{z \to \infty} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_\gamma f(\zeta) \lim_{z \to \infty} \frac{1}{\zeta - z}\, d\zeta = 0$$

since f is bounded on γ and $\lim_{z \to \infty} \dfrac{1}{\zeta - z} = 0$ uniformly.

So, there exists an $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$, and since g is bounded on the compact set $\overline{B}(0, R)$, then g is a bounded entire function. Hence g is constant by Liouville's Theorem. Thus, $g(z) = 0$ for all $z \in \mathbb{C}$.

That is, for $a \in G \setminus \gamma$,

$$\begin{aligned} 0 = g(a) &= \int_\gamma \frac{f(\zeta) - f(a)}{\zeta - a}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - a} - f(a) \cdot 2\pi i\, n(\gamma; a). \end{aligned}$$

This completes the proof. □

Theorem 1.9.3 (Cauchy's Integral Formula - Second version)**.** *Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed piecewise smooth curves in G such that*

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$$

for all $w \in \mathbb{C} \setminus G$, then for all $a \in G \setminus \gamma$ and $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$,

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - a}\, d\zeta.$$

Proof. The proof is similar to that of Theorem 1.9.2 except to define suitable φ, H and g. □

Theorem 1.9.4 (Cauchy's Theorem - First version)**.** *Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed piecewise smooth curves in G such that*

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$$

for all $w \in \mathbb{C} \setminus G$, then

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

Proof. Put $f(z)(z - a)$ instead of $f(z)$, and then apply Theorem 1.9.3. □

Theorem 1.9.5 (Morera's Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ be a continuous function such that*

$$\int_T f = 0$$

for every closed triangular curve T in G, then f is analytic on G.

Remark. A closed triangular curve is a closed three sides polygon.

Proof. It suffices to show that f has a primitive on each open disks in G. In fact, we may assume $G = B(a,R)$ since G is open.

Let $z \in B(a,R)$ and define

$$F(z) = \int_{[a,z]} f.$$

Suppose $z_0 \in B(a,R)$, then

$$F(z) = \left(\int_{[a,z_0} + \int_{[z_0.z]}\right) f.$$

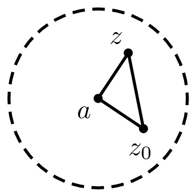

Figure 1.1: $B(a,R)$

So

$$\begin{aligned}\frac{F(z)-F(z_0)}{z-z_0} &= \frac{1}{z-z_0}\int_{[z_0,z]} f \\ &= \frac{1}{z-z_0}\int_{[z_0.z]} (f(\zeta)-f(z_0))\,d\zeta + f(z_0).\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &\le \sup_{\zeta\in[z,z_0]} |f(\zeta)-f(z_0)| \cdot \left|\frac{1}{z-z_0}\right| \int_{[z_0,z]} |d\zeta| \\ &= \sup_{\zeta\in[z,z_0]} |f(\zeta)-f(z_0)| \\ &\to 0 \quad \text{as } z \to z_0.\end{aligned}$$

Hence, $F'(z_0) = f(z_0)$. But F must be infinitely differentiable, so f is analytic on $B(a,R)$. □

1.10 Homotopy version of Cauchy's Theorem

Definition 1.10.1. Let $\gamma_0,\ \gamma_1 : [0,1] \to G$ be two closed piecewise smooth curves in a region G. Then we say that γ_0 *is homotopic to* γ_1 is there is a continuous function $\Gamma : [0,1]\times[0,1] \to G$ such that

$$\Gamma(s,0) = \gamma_0(s), \quad \Gamma(s,1) = \gamma_1(s), \quad 0 \le s \le 1;$$

$$\Gamma(0,t) = \Gamma(1,t), \quad 0 \le t \le 1.$$

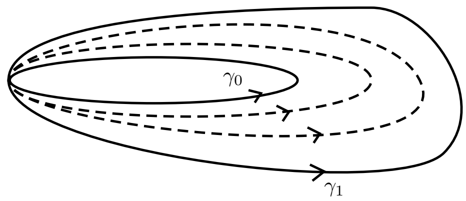

Figure 1.2: γ_0 is homotopic to γ_1

Remark. (i) If we write $\Gamma(s,t) = \gamma_t(s)$. Then the above definition does not require $\gamma_t(s)$ to be piecewise smooth.

(ii) If γ_0 is homotopic to γ_1, we write $\gamma_0 \sim \gamma_1$. Note that $\sim$ defines equivalent classes on closed piecewise smooth curves in G:

(a) $\gamma_0 \sim \gamma_0$ by the identity map,

(b) If $\gamma_0 \sim \gamma_1$, then $\Lambda(s,t) = \Gamma(s,1-t)$ would give $\gamma_1 \sim \gamma_0$,

(c) If $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ with homotopy Γ and Λ respectively, then the homotopy $\Psi : [0,1]\times[0,1] \to G$ given by

$$\Psi(s,t) = \begin{cases} \Gamma(s,2t), & 0 \le t \le \frac{1}{2} \\ \Lambda(s,2t-1), & \frac{1}{2} < t \le 1 \end{cases}$$

shows that $\gamma_0 \sim \gamma_1$.