Hence

$$\begin{aligned}\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right| &\le \sup_{\zeta\in[z,z_0]}|f(\zeta)-f(z_0)|\cdot\left|\frac{1}{z-z_0}\right|\int_{[z_0,z]}|d\zeta| \\ &= \sup_{\zeta\in[z,z_0]}|f(\zeta)-f(z_0)| \\ &\to 0 \quad \text{as } z\to z_0.\end{aligned}$$

Hence, $F'(z_0)=f(z_0)$. But F must be infinitely differentiable, so f is analytic on $B(a,R)$. □

1.10 Homotopy version of Cauchy's Theorem

Definition 1.10.1. Let $\gamma_0,\ \gamma_1 : [0,1]\to G$ be two closed piecewise smooth curves in a region G. Then we say that γ_0 *is homotopic to* γ_1 is there is a continuous function $\Gamma:[0,1]\times[0,1]\to G$ such that

$$\Gamma(s,0)=\gamma_0(s),\quad \Gamma(s,1)=\gamma_1(s),\quad 0\le s\le 1;$$

$$\Gamma(0,t)=\Gamma(1,t),\quad 0\le t\le 1.$$

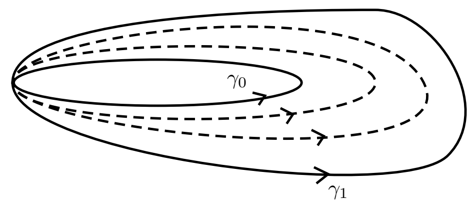

Figure 1.2: γ_0 is homotopic to γ_1

Remark. (i) If we write $\Gamma(s,t)=\gamma_t(s)$. Then the above definition does not require $\gamma_t(s)$ to be piecewise smooth.

(ii) If γ_0 is homotopic to γ_1, we write $\gamma_0\sim\gamma_1$. Note that $\sim$ defines equivalent classes on closed piecewise smooth curves in G:

(a) $\gamma_0 \sim \gamma_0$ by the identity map,

(b) If $\gamma_0 \sim \gamma_1$, then $\Lambda(s, t) = \Gamma(s, 1 - t)$ would give $\gamma_1 \sim \gamma_0$,

(c) If $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ with homotopy Γ and Λ respectively, then the homotopy $\Psi : [0, 1] \times [0, 1] \to G$ given by

$$\Psi(s, t) = \begin{cases} \Gamma(s, 2t), & 0 \leq t \leq \frac{1}{2} \\ \Lambda(s, 2t - 1), & \frac{1}{2} < t \leq 1 \end{cases}$$

shows that $\gamma_0 \sim \gamma_1$.

Definition 1.10.2. A closed piecewise smooth curve γ is said to be *homotopic to zero* if γ is homotopic to a constant curve (written $\gamma \sim 0$).

Definition 1.10.3. A region G is *a-star shaped* if the line segment $[a, z]$ lies entirely in G for each $z \in G$. We simply call G *star shaped* if G is 0-star shaped.

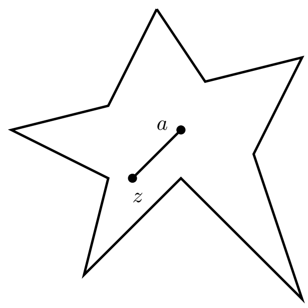

Figure 1.3: a-star shaped

Example 1.10.4. Let G be an a-star shaped region. Then every closed piecewise smooth curve γ in G is homotopic to the constant curve $\gamma_0(t) = a$.

Solution. Let

$$\begin{aligned} \Gamma(s, t) &= t\gamma_0(s) + (1 - t)\gamma_1(s) \\ &= ta + (1 - t)\gamma_1(s) \end{aligned}$$

for $0 \leq s, t \leq 1$.

It is easy to see that Γ is a homotopy between γ_1 and γ_0. $\square$

Remark. A *convex* region is a-star shaped with respect to any a that belongs to G.

Definition 1.10.5. If $\gamma_0,\ \gamma_1 : [0,1] \to G$ are two piecewise smooth curves in a region G such that $\gamma_0(0) = a = \gamma_1(1)$, $\gamma_0(1) = b = \gamma_1(1)$. We say γ_0 *is (fixed end points) homotopic to* γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $\Gamma : [0,1]^2 \to G$ such that

$$\Gamma(s,0) = \gamma_0(s), \quad \Gamma(s,1) = \gamma_1(s), \quad 0 \le s \le 1;$$

$$\Gamma(0,t) = a, \quad \Gamma(1,t) = b, \quad 0 \le t \le 1.$$

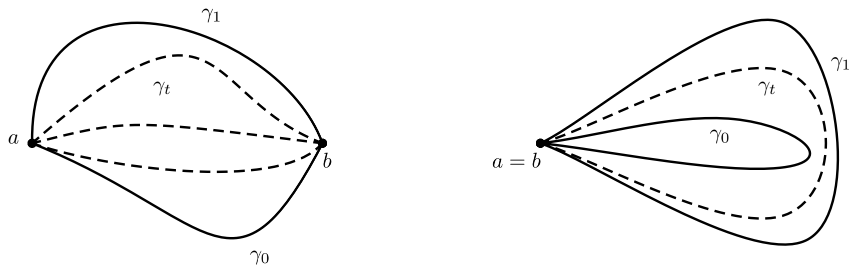

Figure 1.4: γ_0 is (fixed end points) homotopic to γ_1

Similarly, it can be verified that $\sim$ is an equivalence relation on the piecewise smooth curves satisfying the above definition. (See Conway p.93)

And note again that, the intermediate path $\gamma_s(t) = \Gamma(s,t)$ for $0 \le s \le 1$ and t fixed, need *not* be piecewise smooth.

Theorem 1.10.6 (Cauchy's Theorem - Second version)**.** *Suppose $f : G \to \mathbb{C}$ is analytic and γ is a closed piecewise smooth curve in G such that $\gamma \sim 0$, then*

$$\int_\gamma f = 0.$$

Theorem 1.10.7 (Cauchy's Theorem - Third version)**.** *Suppose $f : G \to \mathbb{C}$ is analytic and $\gamma_0,\ \gamma_1 : [0,1] \to G$ are two closed piecewise smooth curves such that $\gamma_0 \sim \gamma_1$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Let γ_0 and γ_1 be as in the hypothesis, and $\Gamma : I^2 \to G$ ($I = [0,1]$) be the corresponding continuous function. Since I^2 is compact, Γ must be uniformly continuous on I^2. Thus $\Gamma(I^2)$ is compact and is a proper subset of G. Hence

$$d(\Gamma(I^2), \mathbb{C} \setminus G) = \inf\{|x - y| : x \in \Gamma(I^2), y \in \mathbb{C} \setminus G\} = r > 0.$$

There exists an integer $n > 0$ such that

$$|\Gamma(s', t') - \Gamma(s, t)| < r$$

whenever $|(s', t') - (s, t)|^2 < \frac{4}{n^2}$ and (s', t'), $(s, t) \in I^2$.

Set

$$J_{jk} = [\frac{j}{n}, \frac{j+1}{n}] \times [\frac{k}{n}, \frac{k+1}{n}] \quad (0 \le j, k \le n-1)$$

(this forms a partition of $I \times I$) and

$$\zeta_{jk} = \Gamma(\frac{j}{n}, \frac{k}{n}) \quad (0 \le j, k \le n).$$

As the diameter (= diagonal) of J_{jk} is $\sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} < \frac{2}{n}$, we must have $\Gamma(J_{jk}) \subset B(\zeta_{jk}, r)$ for $0 \le j, k \le n-1$. ($\cup_{jk} B(\zeta_{jk}, r)$ forms an open cover of $\Gamma(I^2)$; also it is a proper subset of G by the choice of $r > 0$.)

Let

$$Q_k = [\zeta_{0k}, \zeta_{1k}, \dots, \zeta_{nk}]$$

be the closed polygon (since $\zeta_{0k} = \zeta_{nk}$) for $0 \le k \le n$.

We will first show that

$$\int_{\gamma_0} f = \int_{Q_0} f$$

and

$$\int_{Q_n} f = \int_{\gamma_1} f,$$

then

$$\int_{Q_k} f = \int_{Q_{k+1}} f \quad (0 \le k \le n-1).$$

Thus

$$\int_{\gamma_0} f = \int_{Q_0} f = \cdots = \int_{Q_k} f = \cdots = \int_{Q_n} f = \int_{\gamma_1} f.$$

Let

$$P_{jk} = [\zeta_{jk}, \zeta_{j+1,k}, \zeta_{j+1,k+1}, \zeta_{j,k+1}, \zeta_{jk}]$$

be a closed polygon. (See Figure 1.5)

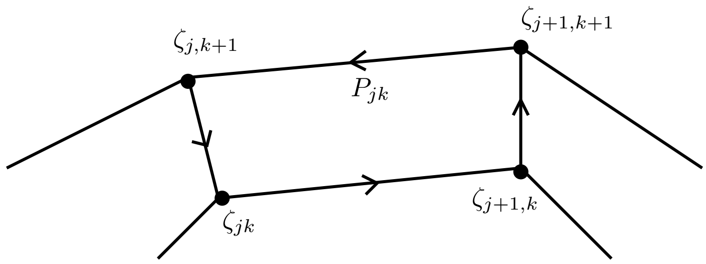

Figure 1.5: P_{jk}

But $\Gamma(J_{jk}) \subset B(\zeta_{jk}, r)$, hence $P_{jk} \subset B(\zeta_{jk}, r)$ in which f is analytic. So

$$\int_{P_{jk}} f = 0 \quad (0 \le j, k \le n-1)$$

by Theorem 1.5.3.

We now show $\int_{\gamma_0} f = \int_{Q_0} f$, where

$$Q_0 = [\zeta_{00}, \zeta_{10}, \ldots, \zeta_{n0}].$$

Let $\sigma_j(s) = \gamma_0(s)$ for $\dfrac{j}{n} \le s \le \dfrac{j+1}{n}$, $(0 \le j \le n-1)$. (See Figure 1.6)

Clearly $\sigma_j + [\zeta_{j+1,0}, \zeta_{j0}]$ is a closed piecewise smooth curve in $B(\zeta_{j0}, r)$ and so

$$\int_{\sigma_j + [\zeta_{j+1,0}, \zeta_{j0}]} f = 0.$$

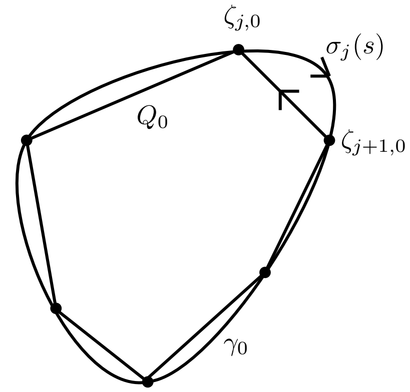

Figure 1.6: $\sigma_j(s)$

That is

$$\int_{\sigma_j} f = -\int_{[\zeta_{j+1,0},\zeta_{j0}]} f = \int_{[\zeta_{j0},\zeta_{j+1,0}]} f$$

So

$$\int_{\gamma_0} f = \sum_{j=0}^{n-1} \int_{\sigma_j} f = \sum_{j=0}^{n-1} \int_{[\zeta_{j0},\zeta_{j+1,0}]} f = \int_{Q_0} f.$$

Similarly, we can prove $\int_{\gamma_1} f = \int_{Q_n} f$. Finally, we show $\int_{Q_k} f = \int_{Q_{k+1}} f \quad (0 \leq k \leq n-1)$. Clearly we have $0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f$.

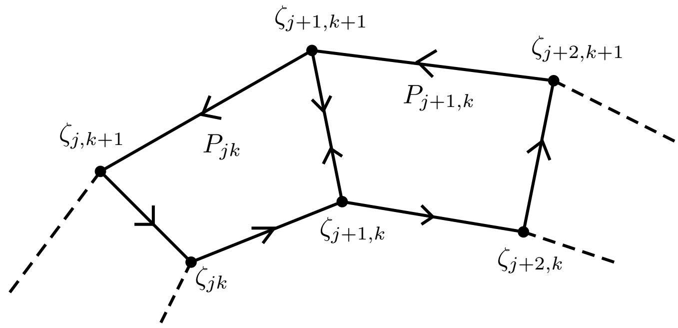

Figure 1.7: P_{jk} and $P_{j+1,k}$

It follows from the Figure 1.7 that

$$\int_{[\zeta_{j+1,k},\zeta_{j+1,k+1}]} f$$

of $\int_{P_{jk}} f$ cancels the

$$\int_{[\zeta_{j+1,k+1},\zeta_{j+1,k}]} f$$

of $\int_{P_{j+1,k}} f$. Thus

$$0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f = \int_{Q_k} f - \int_{Q_{k+1}} f.$$

□

Theorem 1.10.8. *Let γ be a closed piecewise smooth curve in G with $\gamma \sim 0$. Then $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus G$.*

Proof. The proof follows from Theorem 1.10.6. Since $\dfrac{1}{z-a}$ is analytic on G if $a \in \mathbb{C} \setminus G$,

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta = 0.$$

□

We note that the converse of Theorem 1.9.8 is not true. That is, there exist a γ such that $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus G$ but it is not true that $\gamma \sim 0$. (See exercise). Thus Theorem 1.9.2 and 1.9.3 are more general than Theorem 1.10.6 and 1.10.7.

Theorem 1.10.9. *If γ_0 and γ_1 are two piecewise smooth curves joining a to b and $\gamma_0 \sim \gamma_1$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Since $\gamma_0 \sim \gamma_1$, so there exists a continuous map $\Gamma : I^2 \to \mathbb{C}$ such that

$$\Gamma(s, 0) = \gamma_0(s), \quad \Gamma(s, 1) = \gamma_1(s), \quad 0 \le s \le 1;$$

$$\Gamma(0, t) = a, \quad \Gamma(1, t) = b, \quad 0 \le t \le 1.$$

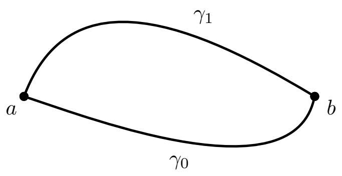

Figure 1.8: $\gamma_0 \sim \gamma_1$

Because $\gamma_0 - \gamma_1$ is a closed piecewise smooth curve, we define

$$\gamma(s) = \begin{cases} \gamma_0(3s), & 0 \le s \le \frac{1}{3} \\ b, & \frac{1}{3} < s \le \frac{2}{3} \\ \gamma_1(3-3s), & \frac{2}{3} < s \le 1. \end{cases}$$

Next we show $\gamma \sim 0$ by claiming that $\Lambda : I^2 \to G$ is a suitable function:

$$\Lambda(s,t) = \begin{cases} \Gamma(3s(1-t), t), & 0 \le s \le \frac{1}{3} \\ \Gamma(1-t, 3s-1+2t-3st), & \frac{1}{3} < s \le \frac{2}{3} \\ \gamma_1((3-3s)(1-t)), & \frac{2}{3} < s \le 1. \end{cases}$$

Note that

$$\Lambda(s,t) = \gamma_t(s), \quad \Lambda(s,0) = \gamma_0 - \gamma_1, \quad \Lambda(s,1) = a = b.$$

It is easy to see that Λ is continuous at $s = \frac{1}{3}$; and at $s = \frac{2}{3}$ because $\Gamma(1-t,1) = \gamma_1(1-t)$. So, Λ is continuous on I^2.

Hence

$$0 = \int_\gamma f = \int_{\gamma_0} f - \int_{\gamma_1} f.$$

$\square$

Definition 1.10.10. An open set G is called *simply connected* if it is connected and every closed curve in G is homotopic to zero (i.e., $\gamma \sim 0$).

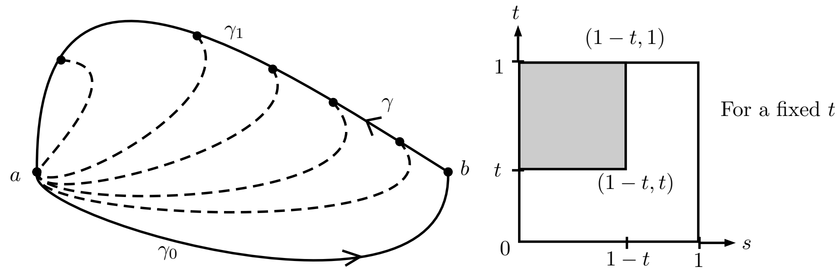

Figure 1.9: $\Lambda(s,t)$ and $[0,1-t]\times[t,1]$

So we have the following version of Cauchy's Theorem.

Theorem 1.10.11 (Cauchy's Theorem - Fourth version)**.** *If G is simply connected, then $\int_\gamma f = 0$ for every closed piecewise smooth curve and every analytic f.*

The notion of simply connected region lies much deeper than it appears. We shall study this in a more detailed way in a later chapter (pending). Here we chiefly want to prove some immediate consequences of analytic function defined on simply connected region.

Theorem 1.10.12. *Suppose the region G is simply connected, and $f : G \to \mathbb{C}$ is analytic. Then f has a primitive on G.*

Proof. Let $a \in G$ and $\gamma : [0,1] \to G$ be a piecewise smooth curve (if closed, then by Theorem 1.5.3 immediately) in G where $\gamma(0) = a$.

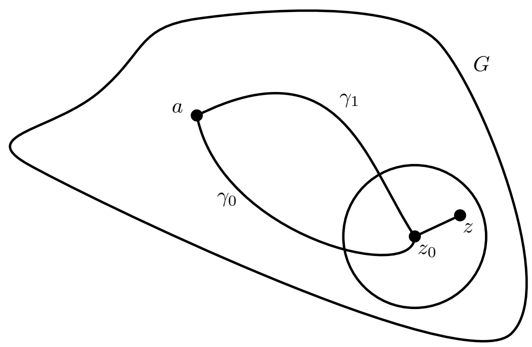

Figure 1.10: $\gamma_0 - \gamma_1$

Define an expression $F(z) = \int_\gamma f(\zeta)\, d\zeta$. We first verify that F is well-defined.

Since $\gamma_0 - \gamma_1 \sim 0$, Cauchy's Theorem implies that

$$\int_{\gamma_0-\gamma_1} f\, d\zeta = \int_{\gamma_0} f\, d\zeta - \int_{\gamma_1} f d\zeta = 0.$$

Hence F is independent on the choice of γ. Thus F is a well-defined function.

To show F is analytic and $F' = f$, we consider $r > 0$ so small such that $B(z_0, r) \subset G$. Replace γ by $\gamma + [z_0, z]$ in F:

$$F(z) = \int_{\gamma+[z_0,z]} f.$$

Then we have

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, d\zeta.$$

By the similar argument in the proof of Morera's Theorem, we can deduce that $F' = f$ and F is analytic. □

The next result lies deeper.

Theorem 1.10.13. *Let G be simply connected and $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = e^{g(z)}$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, then we may choose g such that $g(z_0) = w_0$. So simply connected region implies every non-vanishing analytic function can have a logarithm.*

Proof. Since f has no zeros, and $\dfrac{f'}{f}$ is analytic on G. By Theorem 1.10.12, we let g to be a primitive of $\dfrac{f'}{f}$. Consider

$$\frac{d}{dz}\left(\frac{f}{e^g}\right) = \frac{f' - g'f}{e^g} = 0.$$

Thus $f = (\text{constant})e^g = e^{g+c}$, where c is a constant.

So, if $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may find a suitable integer k such that $w_0 = g(z_0) + c + 2\pi k$. Now define $\tilde{g} = g + c + 2\pi k$, which is the required function. □

Remark. The converse of the above statement also hold, namely that, G is a simply connected region if every non-vanishing analytic function f can be represented as $f = e^g$ for same analytic function g on G. We refer to [1] or [2] for the detail.

1.11 Open Mapping Theorem

Definition 1.11.1. If γ is a closed piecewise smooth curve in G such that $n(\gamma; w) = 0$ for each $w \in \mathbb{C} \setminus G$. We call such curve *homologous to zero* ($\gamma \approx 0$).

The following contour shows that although $\gamma \sim 0$ implies $\gamma \approx 0$, the converse is not true. One can verify that following figure has $\gamma \approx 0$ but $\gamma \not\sim 0$ since $n(\gamma;\, a) = 0 = n(\gamma;\, b)$. The contour was first written down independently by C. Jordan (1887) and L. Pochhammer (1890).

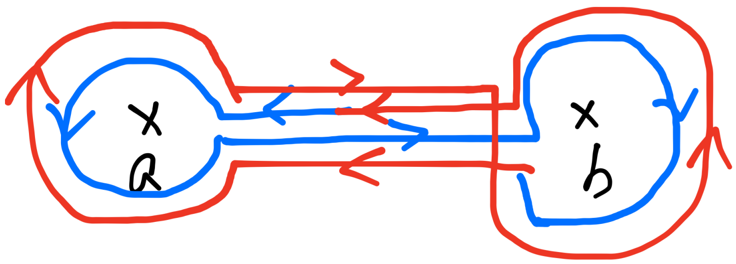

Figure 1.11: Pochhammer contour

Remark. The *Beta function* is defined by the integral

$$B(x,\, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \tag{1.3}$$

where $\Re(x)$ and $\Re(y) > 0$ so that the integral converges. However, if we remove the restriction$\Re(x)$ and $\Re(y) > 0$, then we can still compute the beta function via Pochhammer contour to

$$B(x,\, y) = \int_{\text{(Pochhammer)}}^{(1+,\,0+,\,1-,\,0-)} t^{x-1}(1-t)^{y-1}\, dt = \frac{-e^{\pi i(x+y)}4\pi^2}{\Gamma(1-x)\Gamma(1-y)\Gamma(x+y)}\, dt. \tag{1.4}$$

See [6] for the detail.

By using Cauchy's Theorem, we shall see below some topological results of different natures.

Theorem 1.11.2. *Let G be a region and $f : G \to \mathbb{C}$ analytic on G with zeros $a_1, \ldots, a_m$ (counted with multiplicity). If γ is a closed piecewise smooth curve in G such that $a_k \notin \gamma$ for each k, and if $\gamma \approx 0$ in G, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\, d\zeta = \sum_{k=1}^{m} n(\gamma; a_k).$$

Proof. According to previous discussion,

$$f(z) = (z-a_1)\cdots(z-a_m)g(z), \quad g(z) \neq 0, \quad z \in G.$$

Then for $z \neq a_1, \ldots, a_m$, we have

$$\frac{f'}{f}(z) = \frac{1}{z-a_1} + \cdots + \frac{1}{z-a_m} + \frac{g'}{g}.$$

So

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(\zeta)\, d\zeta &= \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta-a_1} + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta-a_m} + \int_\gamma \frac{g'}{g}\, d\zeta \\ &= n(\gamma; a_1) + \cdots + n(\gamma; a_m) + \int_\gamma \frac{g'}{g}\, d\zeta.\end{aligned}$$

Since $\gamma \approx 0$ and $\dfrac{g'}{g}$ is analytic on G, by the Cauchy Theorem - First version, we have $\int_\gamma \dfrac{g'}{g}\, d\zeta = 0$. This completes the proof. □

Corollary 1.11.2.1. *Let f, G and γ be as in the preceding theorem except that $a_1, \ldots, a_m$ are the roots of $f(z) = \alpha$ (counted according to multiplicity). Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - \alpha} \, d\zeta = \sum_{k=1}^{m} n(\gamma; a_k).$$

We next prove the important Open Mapping Theorem. But we first need the following theorem.

Theorem 1.11.3. *Let $f : G \to \mathbb{C}$ be analytic where $f(a) = \alpha$. Suppose $f - \alpha$ has a zero of multiplicity m. Then we can find an $\epsilon > 0$ and a $\delta > 0$ such that for all ξ in $0 < |\zeta - a| < \delta$, the equation $f(z) = \xi$ has exactly m simple roots in $0 < |z - a| < \epsilon$. (A simple root of $f(z) = \xi$ is a zero of $f - \xi$ with multiplicity 1.)*

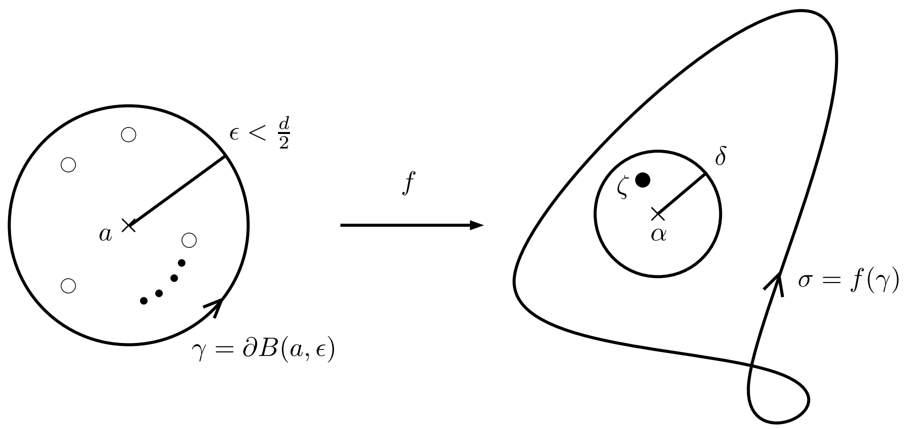

Figure 1.12: $f : G \to \mathbb{C}$, $f(a) = \alpha$

Proof. Let

$$d = \inf_{w \in \mathbb{C} \setminus G} \{|a - w|\}.$$

Since the zero a of $f - \alpha$ is isolated, we may choose $\epsilon < \dfrac{d}{2}$ such that $f(z) - \alpha \neq 0$ in $0 < |z - a| < \epsilon$. Then we have the representation

$$F(z) = f(z) - \alpha = (z - a)^m g(z)$$

over the disk $B(a, \epsilon)$, where g is analytic and $g \neq 0$ there.

Let γ be the boundary of $B(a, \epsilon)$, and write $\sigma = f(\gamma)$. Since $\mathbb{C} \setminus \sigma$ is open, we can find a component of $\mathbb{C} \setminus \sigma$ containing α, and a number $\delta > 0$ such that $B(\alpha, \delta)$ is a proper subset of this component.

Consider

$$\begin{aligned} n(\sigma; \alpha) &= \frac{1}{2\pi i} \int_\sigma \frac{dw}{w - \alpha} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{F'(\zeta)}{F(\zeta)} \, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{m}{\zeta - a} \, d\zeta + \frac{1}{2\pi i} \int_\gamma \frac{g'(\zeta)}{g(\zeta)} \, d\zeta \\ &= m + 0 = m \end{aligned}$$

since γ is closed and $g \neq 0$ on $B(a, \epsilon)$. (So $\dfrac{g'}{g}$ has a primitive.)

According to Proposition 1.8.4, $n(\sigma; \zeta)$ is a constant on this component for each $\xi \in B(\alpha, \delta) \setminus \{\alpha\}$. Theorem 1.11.2 gives

$$\begin{aligned} n(\sigma; \xi) &= \frac{1}{2\pi i} \int_\sigma \frac{dw}{w - \xi} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - \xi} \, d\zeta \\ &= \sum_{k=1}^{n} n(\gamma; a_k) \end{aligned}$$

where a_k for $k = 1, \ldots, n$ are the zero of $f - \xi$ in $B(a, \epsilon)$. But γ is a circle, so $n(\gamma; a_k) = 1$ for $1 \leq k \leq n$. But then we must have $m = n$. Theorem 1.11.2 again implies that each of these zeros a_k is a simple root of $f - \xi$. This completes the proof. □

We deduce immediately the following important result.

Theorem 1.11.4 (Open Mapping Theorem)**.** *Let f be a non-constant analytic function defined on a region G. Then f is an* open mapping, *i.e. f maps open sets onto open sets.*

Proof. Suppose $U \subset G$ is open. To show $f(U)$ is open, it suffices to find a $\delta > 0$ for each $\xi \in f(U)$ such that $B(\xi, \delta) \subset f(U)$. But this follows easily from Theorem 1.11.3 that there exist ϵ, $\delta > 0$ such that $B(a, \epsilon) \subset U$, $B(\alpha, \delta) \subset f(B(a, \epsilon))$. In fact, only part of the conclusion in Theorem 1.11.3 is used. $\square$

We now can give a second proof for the maximum modulus theorem.

Theorem 1.11.5 (Maximum Modulus Theorem)**.** *Let G be a region and $f : G \to \mathbb{C}$ is analytic. If there exists a point a in G such that $|f(z)| \le |f(a)|$ for all $z \in G$, then f is constant.*

Second proof (Topological argument). Suppose $\alpha \in f(G)$ and $f(a) = \alpha$, $a \in G$. Then we can find a $\delta > 0$ such that $B(\alpha, \delta) \subset f(G)$ by open mapping theorem. Hence there exist points in $B(\alpha, \delta)$ with modulus strictly longer than $|\alpha|$. Hence $\max |f(z)|$ cannot occur at an interior of G. $\square$

We now consider the definition of an analytic function. Since the main result we use is Morera's Theorem, we could do this immediately after the proof of Morera's Theorem.

Recall that $f : G \to \mathbb{C}$ is *analytic* on G if f is continuously differentiable.

Theorem 1.11.6. *Let G be an open set and $f : G \to \mathbb{C}$ is differentiable. Then f must be analytic on G. That is, f is differentiable if and only if f is continuous differentiable.*

Proof. According to the statement of the theorem, it suffices to show f' is continuous. But by using Morera's Theorem, we can show that f is analytic directly. See, for examples, [1], [2], [3] for a proof of Goursat's Theorem. $\square$

Remark. It follows from Theorem 1.11.6 that we could define analytic function simply that f is merely differentiable (without continuity) at each point of an open set G.

1.12 Isolated Singularities

We have proved that every zero of an analytic function must be isolated; and as indicated that this property is not shared by real functions. The next natural question is about the singularities of analytic functions, i.e., the nature of points a such that $f(a)$ undefined, such as $f(a) = \infty$. The following is a list of examples:

1. $$\sqrt{z-1}$$
   has a (square-root) branch point at $z = 1$.

2. $$\ln(z-1)$$
   has a logarithmic branch point at $z = 1$.

3. $$e^{1/(z-1)}$$
   has an essential singularity at $z = 1$ (see below).

4. $$\tan[\ln(z-1)]$$
   has a non-isolated essential singularity at $z = 1$ (see below).

We can deal with a small selection of singularities in this course. In the case where $f(a) = \infty$, the standard way to investigate the problem is to consider $F(z) = \dfrac{1}{f}$ at a i.e. $F(a) = \dfrac{1}{\infty} = 0$. Since any zeros are isolated, we may assume F has no zeros in $0 < |z - a| < \delta$ for some $\delta > 0$. So F has only one zero at a i.e. any singularities of f with $f(a) = \infty$ must be isolated (just like the zeros). It turns out that there are only a few types of singularities for analytic functions, and the easiest way to study them is by considering the power series expansions of the functions around the singularities.

Theorem 1.12.1 (Laurent Series, 1843)**.** *Let $f(z)$ be analytic function in an annulus $\Gamma(a; R_1, R_2) = \{z : R_1 < |z - a| < R_2\}$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

and the series converges uniformly in $\Gamma(a; R_1, R_2) = \{z : R_1 < |z-a| < R_2\}$. The coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta$$

where γ is any circle in $\Gamma(a; R_1, R_2)$ centred at a, and for all integers n.

Proof. Let r_1 and r_2 be two real numbers such that $R_1 < r_1 < r_2 < R_2$, and σ be a straight line segment joining the boundary of $\Gamma(a; r_1, r_2)$ and passing through a. Let $\gamma_1(t) = a + r_1 e^{it}$, and $\gamma_2(t) = a + r_2 e^{it}$ for $t \in [0, 2\pi]$, then any closed curve inside $\gamma := \gamma_2 + \sigma - \gamma_1 - \sigma$ is ~ 0. By Cauchy's formula we obtain, for $z \in \Gamma(a; r_1, r_2)$,

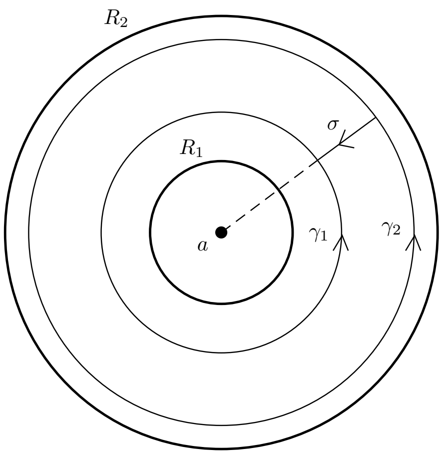

Figure 1.13: $\gamma := \gamma_2 + \sigma - \gamma_1 - \sigma$

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_2} + \int_\sigma - \int_{\gamma_1} - \int_\sigma\right)\frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(z - a)\left(1 - \frac{\zeta-a}{z-a}\right)}\,d\zeta \\
&= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \\
&\qquad + \sum_{n=0}^{\infty}(z-a)^{-n+1}\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta-a)^n\,d\zeta \\
&= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{-\infty}^{-1}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta-a)^{-n-1}\,d\zeta \\
&= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{-\infty}^{-1} a_n(z-a)^n \quad \text{(uniform convergence)}
\end{aligned}
$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \quad \text{for } n \geq 0$$

and

$$a_n = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \quad \text{for } n \leq -1.$$

Let $\gamma = a + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r_1 < r < r_2 < R_2$. By constructing suitable contours involving γ, we may bring the above two line integrals over γ_2 and γ_1 respectively to the common curve γ. Thus we obtain the formula for a_n as stated in the theorem. $\square$

Remark. We remark that Laurent expansion of an analytic function in a punctured disk gives a beautiful generalization of Taylor expansion of analytic function.

Looking at the Laurent expansion of the functions in the above theorem, there are several possibilities:

(i) $a_k = 0$ for all $k \leq -n$ for some integer $n > 0$; the point a is called *a pole of order* n;

(ii) there are infinitely many $a_k \neq 0$, $k \leq -1$; the point a is called *an essential singularity* of f at a;

(iii) $a_k = 0$ for all $k \leq -1$, then a is called a *removable singularity* of f at a.

- If f has a pole of order n, then

$$f(z) = \sum_{k=1}^{n} \frac{a_k}{(z-a)^k} + \sum_{k=0}^{\infty} a_k(z-a)^k$$

  where the sum $\sum_{k=1}^{n} a_k/(z-a)^k$ is called the *principal part of f at a*, and $|f| \to \infty$ in the manner of $O(|z-a|^{-n})$ as $z \to a$.

- If f has a removable singularity at a, then $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$ in $0 < |z-a| < \delta$ (some $\delta > 0$). But we clearly have $f \to a_0$ as $z \to a$, thus we may define a new function at a by $g(z) = f(z)$ for $0 < |z-a| < \delta$ and $g(z) = a_0$ at $z = a$. Then g is an analytic function in $|z-a| < \delta$. Thus f is almost analytic at a if it has a removable singularity at a and so from this point of view, this case is less interesting.

We shall discuss the implication of pole later. The behaviour of f near an essential singularity is very different. It is *not* true that $|f| \to \infty$ as $z \to a$.

Example 1.12.2. 1. The $\sin z/z$ has a removable singularity at $z = 0$.

2. The Euler-Gamma function $\Gamma(z)$ has simple poles at each of negative integers (see a later chapter).

3. The Weierstrass function $\wp(z)$ has double poles at the vertices of its fundamental period parallelograms (see a later chapter).

4. The $e^{1/z}$, $\sin\big(1/z\big)$ and $\cos(1/z)$ all have an essential singularity at $z = 0$.

5. Show the following Laurent expansion

$$e^{\frac{1}{2}(z-1/z)} = \sum_{-\infty}^{\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\theta - \sin\theta)\, d\theta.$$

Theorem 1.12.3 (Casorita-Sokhotskii-Weierstrass-1864)**.** *Suppose f has an essential singularity at a. Then for every $\delta > 0$, $\overline{f(\Gamma(a; 0, \delta))} = \mathbb{C}$.*

The statement of this theorem is equivalent to given any ρ, $\epsilon > 0$ and any $c \in \mathbb{C}$, there is a point z inside $0 < |z - a| < \rho$ in which $|f(z) - c| < \epsilon$. That is to say, given any c, f tends to c as the limit as z tends to a through a suitable sequence of complex numbers.

Proof. We first show that f is unbounded on any punctured disks $\Gamma(a; 0, \delta)$.

Suppose $|f(z)| \leq M$ for all $z \in \Gamma(a; 0, \delta)$. Let $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$, then

$$\begin{aligned}
|a_n| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta\right| \quad \text{for } n \leq -1\\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(a + Re^{it})}{(Re^{it})^{n+1}} iRe^{it}\, dt\right|\\
&\leq MR^{-n}\\
&\to 0 \quad \text{as } R \to 0.
\end{aligned}$$

Hence $a_n = 0$ for all $n \le -1$ and f has a removable singularity at most. A contradiction.

Let us now assume that $\delta > 0$ is chosen so small that $f - c$ has no zero in $\Gamma(a; 0, \delta)$. Then the function $\phi(z) = \dfrac{1}{f - c}$ is analytic in $\Gamma(a; 0, \delta)$. We claim that ϕ has an essential singularity at a. For if ϕ has a pole at a, then $f = \dfrac{1}{\phi} + c$ would be analytic at a; while if ϕ has a removable singularity, then f either has pole or analytic at a. This is a contradiction.

We now apply the result obtained above to ϕ i.e. ϕ is unbounded on $\Gamma(a; o, \delta)$, so $|f - c| = 0$ on $\Gamma(a; 0, \delta)$. That is, given $\varepsilon > 0$, there exists $z \in \Gamma(a; 0, \delta)$ such that

$$|\phi(z)| > \ 1/\varepsilon,$$

i.e.,

$$|f(z) - c| = |1/\phi(z)| < \varepsilon.$$

So we could find a sequence $\varepsilon_n = 1/n$ and $\{\delta_n\}$ such that $\delta_n \to 0$ and $z_n \in \Gamma(a; 0, \delta_n)$ so that $z_n \to a$ for and $f(z_n) \to c$. This completes the proof. □

1.13 Rouché's theorem

This is an application of the argument principle discussed earlier.

Theorem 1.13.1 (E. Rouché). *Let $f(z)$ and $g(z)$ be analytic in the domain D containing the closed, piece-wise smooth curve γ. Suppose*

$$|f(z)| > |g(z)|, \qquad \textit{for all} \ \ z \in \gamma.$$

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicity, in the domain enclosed by γ.

Proof. It is evident from the assumption that $|f(z)| > |g(z)|$, for all $z \in \gamma$ that both $f(z)$ and $f(z) + g(z)$ do not have zeros on γ.
The argument principle assets that

$$\begin{aligned}\Delta_\gamma \arg\big(f(z) + g(z)\big) &= \Delta_\gamma \arg\Big[f(z)\Big(1 + \frac{g(z)}{f(z)}\Big)\Big] \\ &= \Delta_\gamma \arg f(z) + \Delta_\gamma \arg\Big(1 + \frac{g(z)}{f(z)}\Big).\end{aligned}$$

But since

$$1 > \left|\frac{g(z)}{f(z)}\right| = \left|\Big(\frac{g(z)}{f(z)} + 1\Big) - 1\right|,$$

on γ. It follows that $1 + \frac{g(z)}{f(z)}$ can never circle around $w = 0$. Hence

$$\Delta_\gamma \arg(f + g) = \Delta_\gamma \arg f(z) + 0.$$

Thus

$$N_{f+g} = \frac{1}{2\pi i} \int_\gamma \frac{(f+g)'(z)}{f(z) + g(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = N_f$$

inside γ, as required. $\square$

Example 1.13.2. If $f(z)$ has zero of order two at a, and a pole of order 3 at b, where both a and b are inside γ, then

$$\Delta_\gamma \arg f(z) = 2\pi(2 - 3) = -2\pi.$$

Example 1.13.3. Determine the number of roots of

$$z^7 - 4z^3 + z - 1 = 0$$

in $|z| < 1$.
On $|z| = 1$, we write

$$f(z) = -4z^3, \qquad g(z) = z^7 + z - 1.$$

Then $|f(z)| = 4$ and $|g(z)| \leq |z|^7 + |z| + 1 = 3$ Hence $|f(z)| > |g(z)|$ on $|z| = 1$. Thus Rouché's theorem asserts that $f + g$ has the same number of zeros as that of $f = -4z^3$ in $|z| < 1$. Thus there are 3 zeros inside $|z| < 1$.

Exercise 1.13.1. Prove the open mapping theorem for analytic function by applying Rouché's theorem.

See next chapter for an hint.