Chapter 2

Conformal mappings

2.1 Stereographic Projection

One known problem with numbers in the complex plane $\mathbb{C} = \{(x, y) : -\infty < x, y < +\infty\}$ do not have an ordering like the real numbers on the real-axis $\mathbb{R}$. Riemann's (1826-1866) idea is to add an ideal point, denoted by ∞, to $\mathbb{C}$ to obtain an extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This construction can get around the problem of ordering. The resulting $\hat{\mathbb{C}}$ is compact which can be vasualised by the following construction.

We show that there is an one-to-one correspondence between

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Let $N = (0, 0, 1)$ and $z \in \mathbb{C}$. If we join the straight line between N and z, the straight line intersects the sphere S at $Z = (x_1, x_2, x_3)$ say. The construction clearly exhibits an one-to-one correspondence between $S \setminus \{N\}$ and $\mathbb{C}$. Note that $Z \to N$ as $|z| \to \infty$. We may associate N with ∞ and obtain the bijection between S and $\widehat{\mathbb{C}}$. This is known as the *Stereographic projection*.

Suppose $P(x_1, x_2, x_3) = Z \in S$ associates with $z = (x, y) \in \widehat{\mathbb{C}}$. Then we may associate z the notation P with coordinate $(x, y, 0)$.

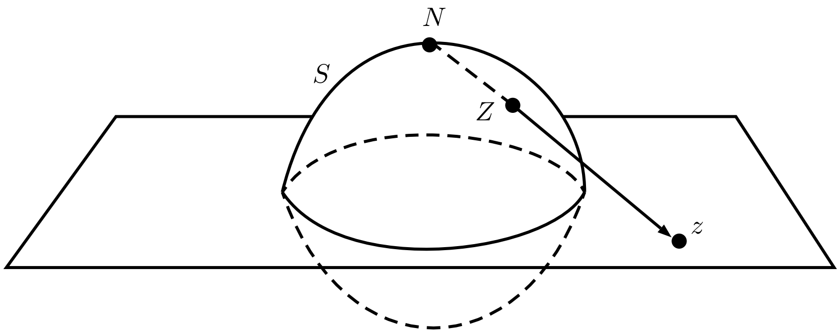

Figure 2.1: Riemann sphere

Then we have, by considering similar triangles formed by the line segment NP and projecting onto the $x-$, $y-$ and $z-$axes respectively,

$$\frac{|NP|}{|NZ|} = \frac{x}{x_1} = \frac{y}{x_2} = \frac{1}{1-x_3}, \tag{2.1}$$

so that

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}.$$

Then

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3},$$

hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Then

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2}, \qquad x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)}.$$

This clearly shows a one-one correspondence between $S\backslash(0, 0, 1)$ and $\mathbb{C}$ with the $N = (0, 0, 1)$ corresponds to ∞. We also note that the upper hemisphere where $x_3 > 0$ corresponds to $|z| > 1$ and the lower hemisphere of S corresponds to $|z| < 1$. An advantage with this Riemann sphere model is that it puts all complex numbers including '∞' in equal footing since any number can be rotated to N and vice-verse.

From a geometrical viewpoint, it is evident that every (infinite) straight line in the $z-$plane is transformed into a circle on S that passes through the North pole N, and conversely. Hence, every circle (straight line included) on the $z-$plane corresponds to a circle/straight line on S.

Theorem 2.1.1. *A circle on the Riemann sphere is mapped under the Stereographic projection into a circle (including a straight line) of the* $\mathbb{C}$*, and conversely.*

Proof. Show that

1. a circle equation that lies on the Riemann sphere is an equation of the form
$$ax_1 + bx_2 + cx_3 = d$$
subject to $0 \leq c < 1$ and $a^2 + b^2 + c^2 = 1$ (this is the intersection of the plane and the unit sphere).

2. the above equation can be rewritten in the form
$$a(z + \bar{z}) - ib(z - \bar{z}) + c(|z|^2 - 1) = d(|z|^2 + 1)$$

3. the above equation can be further rewritten into the form
$$(d - c)(x^2 + y^2) - 2ax - 2by + d + c = 0,$$

which is clearly a circle equation in the $\mathbb{C}$ and it becomes a straight line. equation if and only if $c = d$. □

That is, a circle on the Riemann sphere S corresponds to either a circle or a straight line on $\mathbb{C}$. In the case the circle on S passes through the North pole $N = (0, 0, 1)$, then the corresponding straight line (also considered as an unbounded circle passes through) to ∞.

Exercise 2.1.1. Show that if z and w are two points in $\mathbb{C}$ so that their images lie on two diametrically opposite points on the Riemann sphere, then
$$w\bar{z} + 1 = 0.$$

Theorem 2.1.2. *The stereographic projection is isogonal (i.e., the mapping preserves angles).*

Proof. The statement of the theorem means that the tangents of two curves in the $\mathbb{C}$ intersect at point z_0 is equal to the angle made by two tangents at the corresponding intersection point of two image curves on the Riemann sphere. We shall make two assumptions:

1. that the Stereographic projection preserves tangents. We skip the detail verification of this fact. But this is not difficult to see since the Stereographic projection is a smooth map,

2. that without loss of generality that the two curves in $\mathbb{C}$ are (infinite) straight lines.

Suppose the two straight line equations are given by

$$\begin{aligned} a_1x + a_2y + a_3 = 0 \quad (x_3 = 0); \\ b_1x + b_2y + b_3 = 0. \quad (x_3 = 0) \end{aligned} \tag{2.2}$$

It follows from (2.1) that the two plane equations become respectively,

$$\begin{aligned} a_1X_1 + a_2X_2 + a_3(X_3 - 1) = 0; \\ b_1X_1 + b_2X_2 + b_3(X_3 - 1) = 0. \end{aligned}$$

In the limiting case when $X_3 = 1$, we have the two tangent plane equations

$$\begin{aligned} a_1X_1 + a_2X_2 = 0; \\ b_1X_1 + b_2X_2 = 0. \end{aligned} \tag{2.3}$$

at $N(0, 0.1)$ parallel to the $\mathbb{C}$. Clearly the angle between the two curves in (2.2) is the same angle between the two lines in (2.3).

Note that any two intersecting circles in general positions on S can be rotated so that the intersection point passes through the North pole N. This consideration takes care of the preservation of the angle of intersection of two curves in general position in $\mathbb{C}$ under the Stereographic projection. □

Theorem 2.1.3. *Let z_1, z_2 be two points in $\mathbb{C}$ and Z_1, Z_2 be their images on the Riemann sphere S under the Stereographic projection. We denote $a(Z_1, Z_2)$ to be arc length between Z_1 and Z_2. Then*

$$\lim_{z_2 \to z_1} \frac{a(Z_1, Z_2)}{|z_1 - z_2|} = \frac{2}{1 + |z_1|^2}. \tag{2.4}$$

That is, the ratio depends on position only. So the Stereographic projection is called a pure magnification.

We easily deduce from the above theorem that

Theorem 2.1.4. *Let $C = \{z = z(s) : \ 0 \le s \le L\}$ be a piecewise smooth curve in $\mathbb{C}$. Let Γ be the image curve of C on the Riemann sphere under the Stereographic projection. Then the length $\ell(\Gamma)$ of Γ is given by*

$$\ell(\Gamma) = \int_0^L \frac{2|dz(s)|}{1 + |dz(s)|^2}.$$

Let $d(Z_1, Z_2)$ denote the *chordal distance* between Z_1 and Z_2 on S. We also write

$$\chi(z_1, z_2) := d(Z_1, Z_2).$$

where z_1, z_2 are the corresponding points in $\mathbb{C}$.

Theorem 2.1.5. *Let $z_1, z_2 \in \mathbb{C}$. Then*

$$\chi(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}. \tag{2.5}$$

Since

$$\chi(z_1, z_2) := d(Z_1, Z_2) \approx a(Z_1, Z_2)$$

as $z_1 \to z_2$. So the Theorem 2.1.3 follows from the equation (2.5) in the limit $z_2 \to z_1$.

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ and none equal to ∞. We construct a plane passing through the following three points:

$$(0, 0, 1), \quad (x_1, y_1, 0), \quad (x_2, y_2, 0).$$

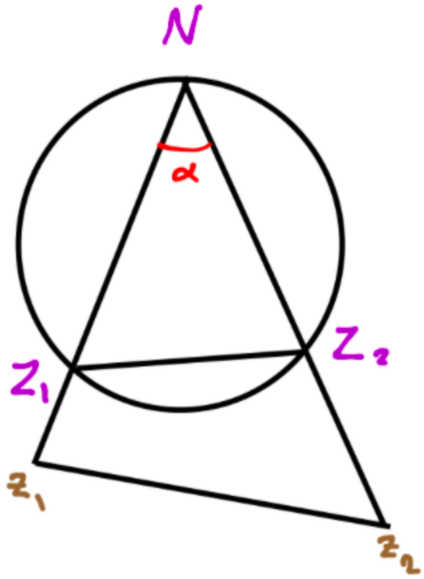

Figure 2.2: Riemann sphere slide

Then we have the above figure.

We deduce from the Riemann sphere S that

$$d(N,\, z_1) = \sqrt{1+|z_1|^2}, \qquad d(N,\, z_2) = \sqrt{1+|z_2|^2}.$$

One can see from similar triangles consideration on the Riemman sphere S that

$$\frac{x_1}{x} = \frac{1-x_3}{1} = \frac{x_2}{y}.$$

Hence

$$1+|z|^2 = 1+x^2+y^2 = 1+\frac{x_1^2}{(1-x_3)^2}+\frac{x_2^2}{(1-x_3)^2}$$
$$= \frac{2(1-x_3)}{(1-x_3)^2} = \frac{2}{1-x_3}.$$

and

$$\frac{d(N,\, Z)}{d(N,\, z)} = \frac{1-x_3}{1} = \frac{2}{1+|z|^2}.$$

holds. This gives raise to

$$d(N,\, Z_1) = \frac{2}{\sqrt{1+|z_1|^2}}, \qquad d(N,\, Z_2) = \frac{2}{\sqrt{1+|z_2|^2}}.$$

We conclude that

$$d(N,\, z_1)d(N,\, Z_1) = 2 = d(N,\, z_2)d(N,\, Z_2).$$

Hence the triangle $\Delta N z_1 z_2$ and $\Delta N Z_1 Z_2$ are similar. Hence

$$\frac{d(Z_1,\, Z_2)}{d(z_1,\, z_2)} = \frac{d(N,\, Z_2)}{d(N,\, z_1)}.$$

It follows from the above consideration that

$$d(Z_1,\, Z_2) = d(z_1,\, z_2) \cdot \frac{d(N,\, Z_2)}{d(N,\, z_1)} = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

as required. □

We are ready to prove Theorem 2.1.3.

We observe the relation

$$\frac{d(Z_1,\, Z_2)}{a(Z_1,\, Z_2)} = \frac{\sin\alpha}{\alpha},$$

holds, where α is the angle between the line segments NZ_1 and NZ_2 from the above figure. Hence

$$\frac{a(Z_1,\, Z_2)}{|z_1 - z_2|} = \frac{d(Z_1,\, Z_2)}{|z_1 - z_2|} \approx \frac{\chi(z_1,\, z_2)}{|z_1 - z_2|} \to \frac{2}{1+|z_1|^2}$$

as $z_2 \to z_1$.

We also note that

$$\chi(z_1,\, \infty) = \lim_{z_2\to\infty} \chi(z_1,\, z_2) = \frac{2}{\sqrt{1+|z_1|^2}},$$

which follows from the Riemann sphere (geometric) or the Theorem 2.1.5 (algebraic) considerations. Thus we define the chordal distance to be

$$\chi(z, z') = \begin{cases} \dfrac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}, & z, z' \in \mathbb{C} \\ \dfrac{2}{\sqrt{1+|z|^2}}, & z' = \infty. \end{cases}$$

Alternative derivation

of the chordal distance. Suppose $(x_1, x_2, x_3) \in S$ associates with $z = (x,\, y) \in \widehat{\mathbb{C}}$ and $(x_1', x_2', x_3') \in S$ associates with $z' \in \widehat{\mathbb{C}}$.

Then the distance or the length of the chord joining (x_1, x_2, x_3) and (x_1', x_2', x_3') on S is given by

$$\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}.$$

On the other hand,

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 = 2 - 2(x_1x_1' + x_2x_2' + x_3x_3').$$

Exercise 2.1.2. Show that

$$\begin{aligned} x_1x_1' &+ x_2x_2' + x_3x_3' \\ &= \frac{(z+\bar{z})(z'+\bar{z}') - (z-\bar{z})(z'-\bar{z}') + (|z|^2-1)(|z'|-1)}{(1+|z|^2)(1+|z'|^2)} \\ &= \frac{(1+|z|^2)(1+|z'|^2) - 2|z-z'|^2}{(1+|z|^2)(1+|z'|^2)} \end{aligned}$$

Exercise 2.1.3. Verify the formaula for chordal distance using the above formala.

Exercise 2.1.4. Verify that $\chi(z_1,\, z_2) = \chi(\bar{z}_1,\, \bar{z}_2) = \chi(1/z_1,\, 1/z_2)$.

Exercise 2.1.5. Describe a $\varepsilon-$neighbourhood of a pont z_0 in the chordal metric.

Metric space

The chordal distance $\chi(z_1, z_2)$ defines a metric on $\hat{\mathbb{C}}$. This is because

1. $\chi(z_1, z_2) \geq 0$ and with equality if and only if $z_1 = z_2$;
2. $\chi(z_1, z_2) = \chi(z_2, z_1)$;
3. $\chi(z_1, z_3) \leq \chi(z_1, z_2) + \chi(z_2, z_3)$,

where the third item follows from

Exercise 2.1.6. Let a, b, $c \in \mathbb{C}$. Then

$$(a-b)(1-\bar{c}c) = (a-c)(1+\bar{c}b) + (c-b)(1+\bar{c}a).$$

Exercise 2.1.7. Show that the above metric space is complete.

2.2 Analyticity revisited

Local properties of one-one analytic functions

We recall that if $f : E \to P$ and there correspond only one point in E for every point in P under this f, then we say the map f is **injective**. This defines a function g on P, denoted by $z = g(w)$, called the **inverse function or inverse mapping** of f. In particular, we see that $g[f(z)] = z$.

Let $w = f(z) = u(x, y) + iv(x, y)$. Then one can view f as a mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

What is a criterion that guarantee the existence of an inverse mapping for the above mapping?

Standard material from calculus courses asserts that if

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \neq 0, \qquad \text{at } z_0 = (x_0,\, y_0),$$

then the Implicit function theorem asserts that an inverse function of f exists there. That is, if the *Jacobian* is non-zero at z_0. But then the Cauchy-Riemann equations give

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |f'(z_0)|^2.$$

This leads to the following statement.

Theorem 2.2.1. *Let $f(z)$ be an analytic function on a domain D such that $f'(z_0) \neq 0$. Then there is an analytic function $g(w)$ defined in a neighbourhood $N(w_0)$ of $w_0 = f(z_0)$ such that $g(f(z)) = z$ throughout this neighbourhood.*

Proof. Since

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = |f'(z_0)|^2 \neq 0,$$

so the Implicit Function theorem asserts that is a neighbourhood $N(w_0)$ of $w_0 = f(z_0)$ in which f has a local inverse at w_0

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} x(u,\, v) \\ y(u,\, v) \end{pmatrix}.$$

Moreover, the analytic Implicit Function theorem asserts that the stronger conclusion that since f is analytic at z_0 so the $g(w)$ is analytic at w_0. $\square$

We prove that a strong form of converse of the above statement also holds. Please note we could apply the Theorem 1.11.3 to prove the theorem. But we prefer to apply the Rouché theorem instead.

Theorem 2.2.2. *Let $f(z)$ be an one-one analytic function on a domain D. Then $f'(z) \neq 0$ on D.*

Proof. We suppose on the contrary that $f'(z_0) = 0$ for some z_0. We first notice that $f'(z) \not\equiv 0$. For otherwise, $f(z)$ is identically a constant, contradicting to the assumption that $f(z)$ is one-one on D.
Since the zeros of $f'(z)$ are isolated , so there is a $\rho > 0$ such that $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$. Because of the assumption that f is one-one, so

$$f(z) \neq f(z_0) \quad \text{on} \quad |z - z_0| = \rho.$$

On the other hand, $|f(z)|$ is continuous on the compact set $|z-z_0| = \rho$ so that we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| \geq \delta > 0 \quad \text{on} \quad |z - z_0| = \rho.$$

Let w' be an arbitrary point in $\{w : 0 < |w' - w_0| < \delta\}$. Then the inequality

$$|f(z) - w_0| \geq \delta > |w' - w_0|$$

holds, so that the Rouché theorem again implies that the function $f(z) - f(z_0) = f(z) - w_0$ and the function

$$[f(z) - f(z_0)] + [f(z_0) - w'] = f(z) - w'$$

have the same number of zeros inside $\{z : |z - z_0| < \rho\}$. But $f'(z_0) = 0$ so $f(z) - f(z_0)$ has at least two zeros (counting multiplicity). Hence $f(z) - w'$ also has at least two zeros (counting multiplicity) in $\{z : |z - z_0| < \rho\}$. But $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$, so there are at least two different zeros z_1 and z_2 in $\{z : |z - z_0| < \rho\}$ so that $f(z_1) = w'$ and $f(z_2) = w'$, thus contradicting to the assumption that $f(z)$ is one-one. □

Bibliography

[1] L. V. Ahlfors, *Complex Analysis*, 3rd Ed., McGraw-Hill, 1979.

[2] J.B. Conway, *Functions of One Complex Variable*, 2nd Ed., Springer-Verlag, 1978.

[3] F. T.-H. Fong, *Complex Analysis*, Lecture notes for MATH 4023, HKUST, 2017.

[4] E. Hille, *Analytic Function Theory*, Vol. I, 2nd Ed., Chelsea Publ. Comp., N.Y., 1982

[5] E. Hille, *Analytic Function Theory*, Vol. II, Chelsea Publ. Comp., N.Y., 1971.

[6] E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*, 4th Ed., Cambridge Univ. Press, 1927