We prove that a strong form of converse of the above statement also holds. Please note we could apply the Theorem 1.11.3 to prove the theorem. But we prefer to apply the Rouché theorem instead.

Theorem 2.2.2. *Let $f(z)$ be an one-one analytic function on a domain D. Then $f'(z) \neq 0$ on D.*

Proof. We suppose on the contrary that $f'(z_0) = 0$ for some z_0 and we write $f(z_0) = w_0$. We first notice that $f'(z) \not\equiv 0$. For otherwise, $f(z)$ is identically a constant, contradicting to the assumption that $f(z)$ is one-one on D.

Since the zeros of $f'(z)$ are isolated , so there is a $\rho > 0$ such that $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$. Because of the assumption that f is one-one, so

$$f(z) \neq f(z_0) \quad \text{on} \quad |z - z_0| = \rho.$$

On the other hand, $|f(z)|$ is continuous on the compact set $|z - z_0| = \rho$ so that we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| \geq \delta > 0 \quad \text{on} \quad |z - z_0| = \rho.$$

Let w' be an arbitrary point in $\{w : 0 < |w' - w_0| < \delta\}$. Then the inequality

$$|f(z) - w_0| \geq \delta > |w' - w_0|$$

holds, so that the Rouché theorem again implies that the function $f(z) - f(z_0) = f(z) - w_0$ and the function

$$[f(z) - f(z_0)] + [f(z_0) - w'] = f(z) - w'$$

have the same number of zeros inside $\{z : |z - z_0| < \rho\}$. But $f'(z_0) = 0$ so $f(z) - f(z_0)$ has at least two zeros (counting multiplicity). Hence $f(z) - w'$ also has at least two zeros (counting multiplicity) in $\{z : |z - z_0| < \rho\}$. But $f'(z) \neq 0$ in $\{z : 0 < |z - z_0| < \rho\}$, so there are at least two different zeros z_1 and z_2 in $\{z : |z - z_0| < \rho\}$ so that $f(z_1) = w'$ and $f(z_2) = w'$, thus contradicting to the assumption that $f(z)$ is one-one. □

2.3 Angle preserving mappings

We consider geometric properties of an analytic function $f(z)$ at z_0 such that $f'(z_0) \neq 0$. Let $\gamma = \{\gamma(t) : \ a \leq t \leq b\}$ a piece-wise smooth path such that $z_0 = \gamma(t_0)$ where $a \leq t_0 \leq b$ and $z'(t_0) \neq 0$, and

$$\Gamma := \{w = f(z(t)) : \ a \leq t \leq b\}.$$

That is, $\Gamma = f(\gamma)$.
It is clear that the assumption $z'(t_0) \neq 0$ above means that the path γ must have a tangent at t_0. Thus,

$$\begin{aligned} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0} &= \left.\frac{df(z)}{dz}\right|_{z=z_0} \cdot \left.\frac{dz}{dt}\right|_{t=t_0} \\ &= f'(z_0) \cdot z'(t_0) \neq 0 \end{aligned}$$

since $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$. We deduce

$$\operatorname{Arg} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0} = \operatorname{Arg} \left.\frac{df(z)}{dz}\right|_{z=z_0} + \operatorname{Arg} \left.\frac{dz}{dt}\right|_{t=t_0}.$$

Let $\theta_0 = z'(t_0)$ denote the inclination angle of the tangent to γ at z_0 and positive real axis, and let $\varphi_0 := \operatorname{Arg} \left.\frac{df[z(t)]}{dt}\right|_{t=t_0}$ denote the inclination angle of the tangent to Γ at $w_0 = f(z_0)$. Thus

$$\operatorname{Arg} f'(z_0) = \varphi_0 - \theta_0.$$

Now let $\gamma_1(t) : \ z_1(t) : \ a \leq t \leq b$ and $\gamma_2(t) : \ z_2(t) : \ a \leq t \leq b$ be two paths such that they intersect at z_0. Then

$$\varphi_1 - \theta_1 = \operatorname{Arg} f'(z_0) = \varphi_2 - \theta_2.$$

That is,

$$\varphi_2 - \varphi_1 = \theta_2 - \theta_1.$$

This shows that the difference of tangents of $\Gamma_2 = f(\gamma_2)$ and $\Gamma_1 = f(\gamma_1)$ at w_0 is equal to difference of tangents of γ_2 and γ_1 at z_0.

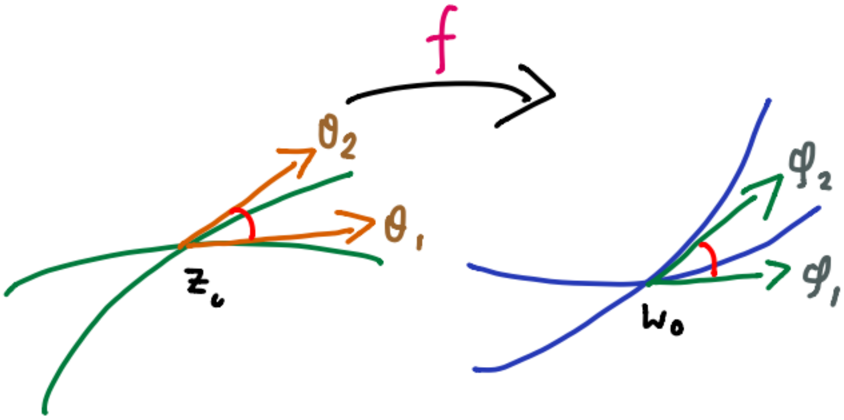

Figure 2.3: Conformal map at z_0

Definition 2.3.1. *An analytic $f : D \to \mathbb{C}$ is called* ***conformal at*** z_0 *if $f'(z_0) \neq 0$. f is called* ***conformal in*** D *if f is conformal at each point of the domain D.*

We call $|f'(z_0)|$ the **scale factor** of f at z_0.

Theorem 2.3.2. *Let $f(z)$ be analytic at z_0 and that $f'(z_0) \neq 0$. Then*

1. *$f(z)$ preserves angles (i.e.,* isogonal*) and its sense at z_0;*
2. *$f(z)$ preserves scale factor, i.e., a pure magnification at z_0 in the sense that it is* independent of directions of approach *to z_0.*

We consider a converse to the above statement.

Theorem 2.3.3. *Let $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined in a domain D with continuous u_x, u_y, v_x, v_y such that they do not vanish simultaneously. If either*

1. *f is isogonal (preserve angles) at every point in D,*
2. or *f is a pure magnification at each point in D,*

then either f or $\bar{f}$ is analytic in D.

Proof. Let $z = z(t)$ be a path passing through the point $z_0 = z(t_0)$ in D. We write $w(t) = f(z(t))$. Then

$$w'(t_0) = \frac{\partial f}{\partial x}\, x'(t_0) + \frac{\partial f}{\partial y}\, y'(t_0),$$

That is,

$$w'(t_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) z'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \overline{z'(t_0)}. \tag{2.6}$$

That is,

$$\frac{w'(t_0)}{z'(t_0)} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \frac{\overline{z'(t_0)}}{z'(t_0)}$$

where we have adopted new notation

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

If f is isogonal, then the arg $\frac{w'(t_0)}{z'(t_0)}$ is independent of $\arg z'(t_0)$ in the above expression. This renders the expression (2.6) to be independent of $\arg z'(t_0)$. Therefore, the only way for this to hold in(2.6) is that

$$0 = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

which represent the validity of the Cauchy-Riemann equations at z_0. Thus f is analytic at z_0. This establishes the first part.

We note that the right-hand side of (2.6) represents a circle of radius

$$\left|\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right|$$

centered at $\partial f/\partial z$. Suppose now that we assume that f is a pure magnification. Then the (2.6) representation this circle must either have its radius vanishes which recovers the Cauchy-Riemann equations, or the centre is at the origin, i.e.,

$$0 = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

or the equivalently $\overline{f(z)}$ is analytic at z_0 and hence over D. $\square$

Remark. If $\overline{f(z)}$ is analytic at z_0, then it means that f preserves the size of the angle but reverse its sense.

Example 2.3.4. Consider $w = f(z) = e^z$ on $\mathbb{C}$. Clearly $f'(z) = e^z \neq 0$ so that the exponential function is conformal throughout $\mathbb{C}$. Observe

$$w = e^z = e^x + e^{iy} := Re^{i\phi},$$

so that the line $x = a$ in the is mapped onto the circle $R = e^a$ in the $w-$plane, while the horizontal line $y = b$ $(-\infty < x < \infty)$ is mapped to the line $\{Re^{ib} : 0 < R < +\infty\}$. One sees that the lines $x = a$ and $y = b$ are at right-angle to each other. Their images, namely the concentric circles centred at the origin and infinite ray at angle b from the $x-$axis from the origin are also at right angle at each other. The infinite horizontal strip

$$G = \{z = x + iy : \ |y| < \pi, \ -\infty < x < \infty\}$$

is being mapped onto the slit-plane $\mathbb{C}\backslash\{z : \ z \leq 0\}$. Moreover, the image of any vertical shift of G by integral multiple of 2π under f will cover the slit-plane again. So the $f(\mathbb{C})$ will cover the slit-plane an infinite number of times.

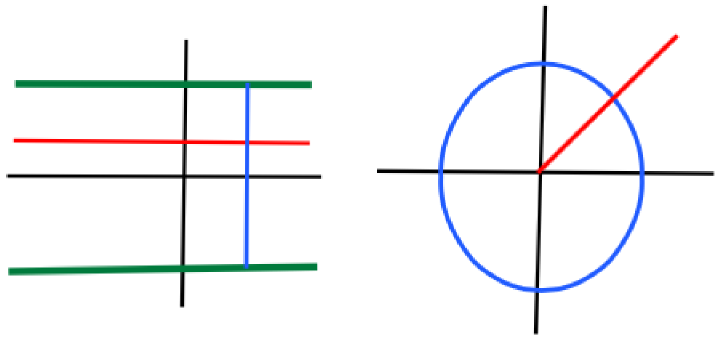

Figure 2.4: Exponential map

2.4 Möbius transformations

We study mappings initiated by A. F. Möbius (1790–1868) on the $\mathbb{C}$ that map $\mathbb{C}$ to $\mathbb{C}$ or even between $\hat{\mathbb{C}}$. Möbious considered

The mapping

$$w = f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0$$

is called a **Möbius transformation**, a **linear fractional transformation**, a **homographic transformation**. In the case when $c = 0$, then a Möbious transformation reduces to a linear function $f(z) = az + b$ which is a combination of a translation $f(z) = z + b$ and a rotation/magnification $f(z) = az$. If $ad - bc = 0$, then the mapping degenerates into a constant.

We recall that a function f having a pole of order m at z_0 is equivalent to $1/f$ to have a zero of order m at z_0 . Similarly, a function have a pole of order m at ∞ means that $1/f(\frac{1}{z})$ to have a zero of order m at $z = 0$.

The mapping w is defined on $\mathbb{C}$ except at $z = -d/c$, where $f(x)$ has a simple pole. On the other hand,

$$f(1/\zeta) = \frac{a/\zeta + b}{c/\zeta + d} = \frac{a + b\zeta}{c + d\zeta} = \frac{a}{c}$$

when $\zeta = 0$. That is, $f(\infty) = a/c$. So $f(z)$ is a one-one map between $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. One can easily check that the inverse f^{-1} of f is given by

$$f^{-1}(w) = -\frac{wd - b}{cw - a}, \quad w \neq \frac{a}{c}.$$

Thus $f^{-1} : \frac{a}{c} \mapsto \infty,\ \infty \mapsto -\frac{d}{c}$ (Since

$$f^{-1}\Big(\frac{1}{\eta}\Big) = -\frac{d/\eta - b}{c/\eta - a} = -\frac{d - b\eta}{c - a\eta} = -\frac{d}{c}$$

as $\eta = 0$. Thus $f^{-1}(\infty) = -\frac{d}{c}$. Similarly, since

$$\frac{1}{f^{-1}(w)}\Big|_{a/c} = -\frac{cw - a}{dw - b}\Big|_{w=a/c} = 0.$$

Thus $f^{-1}\Big(\frac{a}{c}\Big) = \infty$.)

Theorem 2.4.1. *The above Möbius map is conformal on the Riemann sphere.*

Proof. Let $c \neq 0$. Then

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

whenever $z \neq -\frac{d}{c}$. Hence $f(z)$ is conformal at every point except perhaps when $z = -d/c$ where f has a simple pole. So we should check if $\frac{1}{f(z)}$ is conformal at $z = -d/c$. But

$$\begin{aligned}\Big(\frac{1}{f(z)}\Big)'\Big|_{z=-d/c} &= -\frac{f'(z)}{f(z)^2}\Big|_{-d/c} = \frac{ad - bc}{(cz + d)^2} \times \Big(\frac{cz + d}{az + b}\Big)^2 \\ &= -\frac{ad - bc}{(az + b)^2}\Big|_{-d/c} = -\frac{(ad - bc)c^2}{(ad - bc)^2} = \frac{-c^2}{ad - bc} \neq 0.\end{aligned}$$

Hence f is conformal at $-d/c$, whenever $c \neq 0$.

Similarly, in order to check if f is conformal at ∞, we consider, when $c \neq 0$

$$\Big(f\Big(\frac{1}{\zeta}\Big)\Big)' = \Big(\frac{a+b\zeta}{c+d\zeta}\Big)' = \frac{bc-ad}{(c+d\zeta)^2} = \frac{bc-ad}{c^2} \neq 0$$

when $\zeta = 0$ and whenever $c \neq 0$. Hence f is conformal at ∞ if $c \neq 0$.

If $c = 0$, then we consider $f(z) = \dfrac{az+b}{d} = \alpha z + \beta$ instead. Since $f'(z) = \alpha \neq 0$ for all $z \in \mathbb{C}$, so f is conformal everywhere. It remains to consider

$$\frac{1}{f\big(1/\zeta\big)} = \frac{1}{\alpha/\zeta + \beta} = \frac{\zeta}{\alpha + \beta\zeta}.$$

Hence $f(\infty) = \infty$. We now consider the conformality at ∞:

$$\Big(\frac{1}{f(1/\zeta)}\Big)'_{\zeta=0} = \frac{\alpha}{(\alpha+\beta\zeta)^2}\Big|_{\zeta=0} = \frac{1}{\alpha} \neq 0,$$

as required. □

Exercise 2.4.1. Complete the above proof by considering the case when $c = 0$.

Exercise 2.4.2. Show that

1. the composition of two Möbius transformations is still a Möbius transformation.
2. For each Möbius transformation f, there is an inverse f^{-1}.
3. If we denote I be the identity map, then show that the set of all Möbius transformations M forms a group under composition.

Theorem 2.4.2. *Let* $w = f(z) = \dfrac{az+b}{cz+d}$. *Then* $f(z)$ *maps any circle in the* $z-$*plane to a circle in the* $w-$*plane.*

Remark. *We regard any straight lines to be circles having infinite radii* $(+\infty)$.

Proof. We note that any $\dfrac{az+b}{cz+d}$ can be written as

$$\begin{aligned} w =& \frac{a}{c}\Big[\frac{z+b/a}{z+d/c}\Big] = \frac{a}{c}\Big[1+\frac{b/a-d/c}{z+d/c}\Big] \\ &= \frac{a}{c}\Big[1+\Big(\frac{bc/a-d}{1}\Big)\frac{1}{cz+d}\Big] \\ &= \frac{a}{c}+\Big(\frac{bc-ad}{c}\Big)\Big(\frac{1}{cz+d}\Big), \end{aligned}$$

Showing that w can be decomposed by transformations of the basic types:

1. $w = z+b$ (translation),
2. $w = e^{i\theta_0}z$ (rotation),
3. $w = kz$ ($k>0$, scaling),
4. $w = 1/z$ (inversion).

In fact, we can write the $T(z)$ as a compositions of four consecutive mappings in the forms

$$w_1 = cz+d, \qquad w_2 = \frac{1}{w_1}, \qquad w_3 = \Big(\frac{bc-ad}{c}\Big)w_2, \quad w_4 = \frac{a}{c}+w_3,$$

From the geometric view point, the translation $z+b$ or rotation $w = e^{i\theta_0}z$ all presences circles (lines). So it remains to consider scaling

$w = kz$ $(k > 0)$ and inversion $w = 1/z$.
Let us consider the circle equation (centred at $z_0 = (x_0, y_0)$ with radius R). Then

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

That is,

$$x^2 + y^2 - 2x_0 x - 2y_0 y + (x_0^2 + y_0^2 - R^2) = 0.$$

Substituting $z = x + iy,\ \overline{z} = x - iy$

$$z\overline{z} + \frac{-2}{2}(z_0 + \overline{z}_0)\frac{1}{2}(z + \overline{z}) - \frac{2}{2i}(z_0 - \overline{z}_0)\,\frac{1}{2i}(z - \overline{z}) + z_0\overline{z}_0 - R^2.$$

This can be rewritten as

$$z\overline{z} + \overline{B}z + B\overline{z} + D = 0,$$

where $B = -z_0,\ D = x_0^2 + y_0^2 - R^2$.

Conversely, suppose $B = -z_0,\ |B|^2 - D = R^2 > 0$, then the above equation represents a circle equation centred at $-B = z_0$ with radius

$$R = \sqrt{|B|^2 - D}.$$

In fact, $|z - (-B)| = \sqrt{|B|^2 - D}$. We consider the scaling : $w = kz$. The circle equation becomes

$$\frac{1}{k^2}\, w\overline{w} + \frac{\overline{B}}{k}w + \frac{B}{k}\overline{w} + D = 0.$$

Thus

$$w\overline{w} + k\overline{B}w + kB\overline{w} + k^2 D = 0$$

Clearly, $k^2 D$ is a real number, and $\sqrt{k^2\,|B|^2 - k^2 D} = k\sqrt{|B|^2 - D} > 0$. Hence the above equation is a circle equation in the $w-$plane.
It remains to consider inversion $w = 1/z$. Then the equation becomes

$$\frac{1}{w\overline{w}} + \frac{\overline{B}}{w} + \frac{B}{\overline{w}} + D = 0,$$

or

$$w\overline{w} + \frac{B}{D}w + \frac{\overline{B}}{D}\overline{w} + \frac{1}{D} = 0.$$

clearly $1/D$ is a real number, and $|B/D|^2 - 1/D = \frac{1}{D^2}(|B|^2 - D) > 0$. So the equation is a circle equation in the $w-$plane. □

2.5 Cross-ratios

Let

$$T(z) = \frac{az+b}{cz+d} \tag{2.7}$$

be a Möbius transformation, and let $w_1,\, w_2,\, w_3,\, w_4$ be the respectively images of the points $z_1,\, z_2,\, z_3,\, z_4$. Then it is routine to check that

$$w_j - w_k = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k), \quad j,\, k = 1,\, 2,\, 3,\, 4.$$

Then

$$(w_1 - w_3)(w_2 - w_4) = \frac{(ad-bc)^2}{\Pi_{j=1}^4 (cz_j + d)}(z_1 - z_3)(z_2 - z_4) \tag{2.8}$$

Similarly, we have

$$(w_1 - w_4)(w_2 - w_3) = \frac{(ad-bc)^2}{\Pi_{j=1}^4 (cz_j + d)}(z_1 - z_4)(z_2 - z_3). \tag{2.9}$$

Dividing the (2.8) by (2.9) yields

$$\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{2.10}$$

Definition 2.5.1. Let $z_1,\, z_2,\, z_3,\, z_4$ be four distinct numbers in $\mathbb{C}$. Then

$$(z_1,\, z_2,\, z_3,\, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} \tag{2.11}$$

is called the **cross-ratio** of the four points. If, however, when any one of z_1, z_2, z_3, z_4 is ∞, then the cross-ratio becomes

$$\begin{aligned}
(\infty,\, z_2,\, z_3,\, z_4) &:= \frac{z_2 - z_4}{z_2 - z_3},\\
(z_1,\, \infty,\, z_3,\, z_4) &:= \frac{z_1 - z_3}{z_1 - z_4},\\
(z_1,\, z_2,\, \infty,\, z_4) &:= \frac{z_2 - z_4}{z_1 - z_4},\\
(z_1,\, z_2,\, z_3,\, \infty) &:= \frac{z_1 - z_3}{z_2 - z_3},
\end{aligned}$$

respectively.

The equation (2.10) implies that we have already proved the following theorem.

Theorem 2.5.2. *Let T be any Möbius transformation. Then*

$$(Tz_1,\, Tz_2,\, Tz_3,\, Tz_4) = (z_1,\, z_2,\, z_3,\, z_4). \tag{2.12}$$

Remark. The above formula means that the cross-ratio of four points is preserved under any Möbius transformation $T(z)$.

Example 2.5.3. We note that the cross-ratio when written as

$$(z,\, z_2,\, z_3,\, z_4) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)} = \frac{z - z_3}{z - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$$

is a Möbius transformation of z that maps the points z_2, z_3, z_4 to 1, 0, ∞ respectively.

Theorem 2.5.4. *Let z_1, z_2, z_3 and w_1, w_2, w_3 be two sets of three arbitrary complex numbers. Then there is a unique Möbius transformation $T(z)$ that satisfies $T(z_i) = w_j$, $j = 1, 2, 3$.*

Proof. The cross-ratio formula

$$\frac{w-w_3}{w-w_4} : \frac{w_2-w_3}{w_2-w_4} = \frac{z-z_3}{z-z_4} : \frac{z_2-z_3}{z_2-z_4}$$

does the trick. □

Example 2.5.5. *Find a Möbius transformation w that maps -1, i, 1 to -1, 0, 1 respectively.*
It follows that

$$\frac{w-0}{w-1} : \frac{-1-0}{-1-1} = \frac{z-i}{z-1} : \frac{-1-i}{-1-1}.$$

So

$$\frac{2w}{w-1} = \frac{z-i}{z-1}\Big(\frac{1}{1+i}\Big).$$

Hence

$$w = \frac{1+iz}{i+z}.$$

Arrangements

The above arrangement of the four points z_1, z_2, z_3, z_4 in the construction of our cross-ratio is not special. One can try the remaining twenty three different permutations of z_1, z_2, z_3, z_4 in the construction. However, we note that

$$\lambda := (z_1,\, z_2,\, z_3,\, z_4) = (z_2,\, z_1,\, z_4,\, z_3) = (z_3,\, z_4,\, z_1,\, z_2) = (z_4,\, z_3,\, z_2,\, z_1)$$

so that the list reduces to six only. They are given by

$$(z_2,\, z_3,\, z_1,\, z_4) = \frac{\lambda-1}{\lambda}, \quad (z_3,\, z_1,\, z_2,\, z_4) = \frac{1}{1-\lambda}$$

$$(z_2,\, z_1,\, z_3,\, z_4) = \frac{1}{\lambda}, \quad (z_3,\, z_2,\, z_1,\, z_4) = \frac{\lambda}{\lambda-1}, \quad (z_1,\, z_3,\, z_2,\, z_4) = 1-\lambda.$$

The above list contains all six distinct values for the cross-ratio for distinct z_1, z_2, z_3, z_4. If, however, two of the points z_1, z_2, z_3, z_4 coincide, then the list of values will reduce further. More precisely, if $\lambda = 0$ or 1, then the list reduces to three, namely 0, 1, ∞. If $\lambda = -1$, $-1/2$ or 2, then the list reduces to three again with values -1, $1/2$, 2. There is another possibility that

$$\lambda = \frac{1 \pm i\sqrt{3}}{2}.$$

See exercise.

Moreover, if we put $z_2 = 1$, $z_3 = 0$, $z_4 = \infty$, then the cross-ratio becomes

$$(\lambda,\, 1,\, 0,\, \infty) = \lambda,$$

which means that λ is a **fixed point** of the map.

Theorem 2.5.6. *Let z_1, z_2, z_3, z_4 be four distinct points in $\hat{\mathbb{C}}$. Then their cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle (including a straight line).*

Proof. Let $Tz = (z_1, z_2, z_3, z)$.

We first prove that if z_1, z_2, z_3, z_4 lie on a circle/straight-line in $\hat{\mathbb{C}}$, then Tz is real. But by the fundamental property that T is the unique Möbius map that maps z_1, z_2, z_3 onto 0, 1, ∞. Hence T is real on $T^{-1}\mathbb{R}$. It remains to show that the whole circle/straight-line passing through z_1, z_2, z_3 has Tz real.

If Tz is real, then we have $Tz = \overline{Tz}$. Hence

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}.$$

Cross multiplying yields

$$(a\bar{c} - c\bar{a})|w^2| + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - d\bar{b} = 0$$

which is a straight-line if $a\bar{c}-c\bar{a}=0$ (and hence $a\bar{d}-c\bar{b}\neq 0$). Moreover, in the case when $a\bar{c}-c\bar{a}\neq 0$, the above equation can be written in the form

$$\left|w+\frac{\bar{a}d-\bar{c}b}{\bar{a}c-\bar{c}a}\right|=\left|\frac{ad-bc}{\bar{a}c-\bar{c}a}\right|,$$

which is an equation of a circle. □

Exercise 2.5.1. Verify that

$$(\lambda,\,1,\,0,\,\infty)=\lambda.$$

Then use this identity to give a different proof of the above theorem: $(z_1,\,z_2,\,z_3,\,z_4)$ is real if and only if the four points $z_1,\,z_2,\,z_3,\,z_4$ lie on a circle.

Exercise 2.5.2. Show that if one of $z_2,\,z_3,\,z_4$ is ∞, the corresponding cross-ratio still maps the triple onto $1,\,0,\,\infty$. Namely the

$$(z,\,\infty,\,z_3,\,z_4):=\frac{z-z_3}{z-z_4},$$
$$(z,\,z_2,\,\infty,\,z_4):=\frac{z_2-z_4}{z-z_4},$$
$$(z,\,z_2,\,z_3,\,\infty):=\frac{z-z_3}{z_2-z_3},$$

2.6 Inversion symmetry

We already know that the point z and its conjugate $\bar{z}$ are symmetrical with respect to the real-axis. If we take the real-axis into a circle C by a Möbius transformation T, then we say that the points $w=Tz$ and $w^*=T\bar{z}$ are *symmetric with respect to C*. Since the symmetry is a geometric property, so the w and w^* are independent of T. For suppose there is another Möbius transformation that maps the real-axis onto the C, then the composite map $S^{-1}T$ maps the $\mathbb{R}$ onto itself. Thus the images,

$$S^{-1}w = S^{-1}Tz, \quad S^{-1}w^* = S^{-1}T\bar{z}$$

are obviously conjugates. Hence we can define

Definition 2.6.1. Two points z and z^* are said to be *symmetrical with respect to the circle* C passing through z_1, z_2, z_3 if and only if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

In order to see what is the relationship between z and z^*, we consider the following special case.

Example 2.6.2. When $z_3 = \infty$. Then the symmetry yields

$$\frac{z^* - z_2}{z - z_4} = \frac{\bar{z}_2 - \bar{z}_4}{\bar{z}_1 - \bar{z}_4}.$$

That is,

$$|z^* - z_2| = |z - z_2|$$

first showing that the z and z^* are equal distances to z_2 (which is arbitrary on C). And

$$\Im\Big(\frac{z^* - z_2}{z_1 - z_2}\Big) = -\Im\Big(\frac{z - z_2}{z_1 - z_2}\Big)$$

finally showing that the z and z^* are on *different sides* of C.

Theorem 2.6.3. *Let z and z^* be symmetrical with respect to a circle C of radius R and centred at a. Then*

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a.$$

Proof. We note that

$$(z_j - a)\overline{(z_j - a)} = R^2, \quad j = 1, 2, 3.$$

Thus we have

$$\begin{aligned}
\overline{(z, z_1, z_2, z_3} &= \overline{(z - a, z_2 - a, z_3 - a, z_3 - a)} \\
&= \Big(\bar{z} - \bar{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \Big) \\
&= \Big(\frac{R^2}{z - a}, z_1 - a, z_2 - a, z_3 - a\Big) \\
&= \Big(\frac{R^2}{z - a} + a, z_1, z_2, z_3, \Big) \\
&:= (z^*, z_2, z_3, z_3)
\end{aligned}$$

as required. □

We deduce immediately that

Theorem 2.6.4. *A Möbius transformation carries a circle C_1 into a circle C_2 also transforms any pair of symmetric points of C_1 into a pair of symmetric points of C_2.*

Remark. 1. $(z^* - a)(\bar{z} - \bar{a}) = R^2$,

2. The symmetry point $a^* = \infty$ for the centre a above.

3. The expression

$$\frac{z^* - a}{z - a} = \frac{R^2}{(\bar{z} - \bar{a})(z - a)} > 0$$

implying that z and z^* lie on the same half-line from a.

We briefly mention the issue of orientation. Suppose we have a circle C. Then there is an analytic method to distinguish the inside/outside of the circle by the cross-ratio. Since the cross-ration

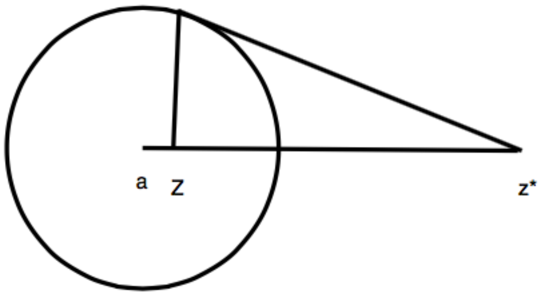

Figure 2.5: Inversion: z and z^*

is invariant with respect to any Möbius transformation, so it is sufficient to consider the inside/outside issue of the real-axis $\mathbb{R}$ since we can always map the circle C onto the $\mathbb{R}$. Let us write

$$(z_1,\, z_2,\, z_3,\, z) = \frac{az+b}{cz+d}$$

where a, b, c, d are real coefficients (since z_1, z_2, $z_3 \in \mathbb{R}$). Then

$$\Im(z,\, z_1,\, z_2,\, z_3) = \frac{ad-bc}{|cz+d|^2}\, \Im z.$$

Suppose we choose $z_1 = 1$, $z_2 = 0$ and $z_3 = \infty$. Then a previous formulai

$$(z,\, 1,\, 0,\, \infty) = z$$

implies that $\Im(z,\, 1,\, 0,\, \infty) = \Im z$, so that $\Im(i,\, 1,\, 0,\, \infty) > 0$ and $\Im(-i,\, 1,\, 0,\, \infty) < 0$. The ordered triple, namely $1,\, 0,\, \infty$ clearly indicates that the point i is on the **right** of $\mathbb{R}$ (in that order) and the other point $-i$ is on the **left** of $\mathbb{R}$ (in that order). But any circle C can be brought to the real-axis $\mathbb{R}$ while keeping the cross-ratio unchanged. So we have

Definition 2.6.5. Let C be a given circle in $\hat{\mathbb{C}}$. An **orientation** of C is determined by the direction of a triple z_1, z_2, z_3 (i.e., $z_1 \mapsto z_2 \mapsto z_3$

) lying on C. Let $z \notin C$. The point z is said to lie on the **right** of C if $\Im(z, z_1, z_2, z_3) > 0$ of the oriented circle. The point z is said to lie on the **left** of C if $\Im(z, z_1, z_2, z_3) < 0$ of the oriented circle.

Definition 2.6.6. We define an **absolute orientation** for each finite circle with respect to ∞ in the sense that the ∞ is on its *right* (we call this *outside*), otherwise, on its *left* (we call this *inside*).

2.7 Explicit conformal mappings

Example 2.7.1. *Find a Möbius mapping that maps the upper half-plane $\mathbb{H}$ onto itself.*
Suppose $f(z) = \dfrac{az+b}{cz+d}$ maps the upper half-plane onto itself.
Then $f(z)$ must map any three points $\{x_1, x_2, x_3\}$ on the x-axis in the order $x_1 < x_2 < x_3$ respectively to three points $u_1 < u_2 < u_3$ on real-axis. It follows that is "no turning" on the real-axis, thus implying that

$$\arg f'(x_1) = 0 \quad or \quad f'(x_1) > 0.$$

But

$$f'(x_1) = \frac{ad - bc}{(cx_1 + d)^2} > 0,$$

implying that $ad - bc > 0$. Moreover, one can solve for the coefficients a, b, c and d by solving

$$u_i = \frac{ax_i + b}{cx_i + d}, \quad i = 1, 2, 3.$$

One notices that a, b, c and d are therefore all real constants. Since f must map $\hat{\mathbb{C}}$ one-one onto $\hat{\mathbb{C}}$, the upper half-plane onto itself. Thus we deduce that

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc > 0.$$

Conversely, suppose

$$w = f(z) = \frac{az+b}{cz+d},$$

where a, b, c and d are real and $ad - bc > 0$. Then for all real x,

$$f'(x) = \frac{ad-bc}{(cx+d)^2} > 0, \quad \textit{and} \quad \arg f'(x) = 0.$$

That is, there is "no turning" on the real-axis. Therefore w must map the real-axis onto the real-axis, and hence Therefore w must map the upper half-plane onto upper half-plane.

Exercise 2.7.1. *Prove directly, that is without applying f', that it is necessary sufficient that $ad - bc > 0$ for*

1. *f maps $\mathbb{H}$ into $\mathbb{H}$;*
2. *that the above map is "onto".*

Example 2.7.2. *Construct a Möbius mapping f that maps upper half-plane into upper half-plane such that $0 \mapsto 0$ and $i \mapsto 1 + i$.*
According to the last example, we must have

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc > 0,$$

where a, b, c and d are real. Since $f(0) = 0$ implying that $b = 0$. On the other hand,

$$1 + i = f(i) = \frac{ai}{ci+d} = \frac{i}{ei+f},$$

say. That is, $e - f = 0$ and $e + f = 1$, or $e = f = \frac{1}{2}$. Hence

$$w = \frac{2z}{z+1}.$$

Example 2.7.3. *Show that a Möbius mapping f that maps the upper half-plane $\mathbb{H}$ onto $\triangle = \{z: \ |z| < 1\}$ if and only if*

$$w = f(z) = e^{i\theta_0}\frac{z-\alpha}{z-\overline{\alpha}}, \quad \Im\alpha > 0, \quad \theta_0 \in \mathbb{R}.$$

Suppose $f : \mathbb{H} \to \triangle$. It follows that f must map the $x-$axis onto $|w| = 1$. Let us consider the images of $z = 0$, 1 and ∞. Since $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$. Thus $1 = |f(0)| = \left|\frac{b}{d}\right|$, implying $|b| = |d|$. We also require $f(\infty)$ to lie on $|w| = 1$ which is necessary finite. But we know from a previous discussion that

$$|f(\infty)| = \left|f\Big(\frac{1}{\zeta}\Big)\right|_{\zeta=0} = \left|\frac{a+b\zeta}{c+d\zeta}\right|_{\zeta=0} = \left|\frac{a}{c}\right| = 1,$$

implying that $|a| = |c|$. So

$$w = \frac{az+b}{cz+d} = \frac{a}{c} \times \frac{z+b/a}{z+d/c} = \frac{a}{c}\frac{z-z_0}{z-z_1}$$

where $|z_0| = |b/a| = |d/c| = |z_1|$. Since $|a/c| = 1$, so there exists a real θ_0 such that $\frac{a}{c} = e^{i\theta_0}$. Thus

$$w = e^{i\theta_0}\,\frac{z-z_0}{z-z_1}, \quad |z_0| = |z_1|.$$

Consider

$$1 = |f(1)| = \left|\frac{z-z_0}{z-z_1}\right|$$

implying $|z - z_0| = |z - z_1|$ or

$$(1-z_1)(1-\overline{z}_1) = (1-z_0)(1-\overline{z}_0).$$

Notice that $|z_1| = |z_0|$. Hence

$$1 - z_1 - \overline{z}_1 + |z_1|^2 = 1 - z_0 - \overline{z}_0 + |z_0|^2.$$

Thus

$$2\Re(z_1) = z_1 + \overline{z}_1 = z_0 + \overline{z}_0 = 2\Re(z_0)$$

or $\Re(z_1) = \Re(z_0)$. *Hence* $z_1 = z_0$ *or* $z_1 = \overline{z}_0$. *We must have* $z_1 = \overline{z}_0$, *for if* $z_1 = z_0$, *then* $f(z)$ *is identically a constant. Thus*

$$f(z) = e^{i\theta_0}\Big(\frac{z - z_0}{z - \overline{z_0}}\Big).$$

Since $f(z_0) = 0$ *so* $\Im(z_0) > 0$.
Conversely, suppose

$$f(z) = e^{i\theta}\Big(\frac{z - \alpha}{z - \overline{\alpha}}\Big), \quad z \in \mathbb{H}.$$

Then $|w| < |f| = \Big|\dfrac{z - \alpha}{z - \overline{\alpha}}\Big| < 1$. *If* z *lies on the lower half-plane, then* $|w| < |f| = \Big|\dfrac{z - \alpha}{z - \overline{\alpha}}\Big| > 1$. *If* z *lies on the real axis, then* $|w| = \Big|\dfrac{z - \alpha}{z - \overline{\alpha}}\Big| = 1$. *Since* f *maps* $\hat{\mathbb{C}}$ *to* $\hat{\mathbb{C}}$ *in a one-one manner, so* f *must maps the* $\mathbb{H}$ *onto* $|w| < 1$.

Remark. *If* $\Im(z_0) = \Im(\alpha) < 0$, *then* f *maps the upper half-plane onto the lower half-plane.*

Exercise 2.7.2. *Find a Möbius transformation* $w : \mathbb{H} \to \triangle, \quad i \mapsto 0$. *So*

$$w = f(z) = e^{i\theta_0}\Big(\frac{z - i}{z + i}\Big).$$

Exercise 2.7.3. *Let* $\triangle = \{z : |z| < 1\}$. *Show that a Möbius transformation* f *that* $f : \triangle \to \triangle$ *if and only if there exists* θ_0, $|\alpha| < 1$ *such that*

$$w = f(z) = e^{i\theta_0}\,\frac{z - \alpha}{1 - \overline{\alpha}z}.$$

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