2.8 Orthogonal circles

We follow the ideas of Riemann and Klein to visualise the effects of conformal mappings. We use the toy models of Möbius transformations to allow us to have a glimpse.

Consider the map

$$w = h(z) = k\,\frac{z-a}{z-b},$$

where k is some non-zero constant to be chosen later. The map carries $z = a$ to $w = 0$ and $z = b$ to $w = \infty$. This means that any straight-line passing through the origin in the $w-$plane has its preimage to pass through the points $z = a$ and $z = b$, and this preimage must be a circle (may be a generalised circle, i.e., a straight-line) in the $z-$plane.

On the other hand, the circles centred at the orgin in the $w-$plane are of the form $|w| = \rho$ for some $\rho > 0$. That is,

$$\left|\frac{z-a}{z-b}\right| = \rho/|k|.$$

Hence the loci of the $h^{-1}\{|w| = \rho/|k|\}$, which must also be a circle, also lies on the $z-$plane. The relation

$$|z-a| = (\rho/|k|)\,|z-b|$$

describes the loci of the point z so that the distances of it to a and b are in a constant ratio. Such circles, denoted by C_2, are called **Apollonius' circles** and the points a and b are called the **limit points**. It is clear that the family of concentric circles $|w| = \rho/|k|$ are always at right angles with any straight-line through the origin in the $w-$plane. So their preimages, denoted by C_1 are orthogonal to the Apollonius circles C_2. In general, we denoted by C_1' and by C_2' the images of $C_!$ and Apollonius circles C_2, respectively, under a Möbius transformation in the $w-$plane. Obviously, the C_1' and C_2' are orthogonal to each other at their intersections.

We have the following theorem.

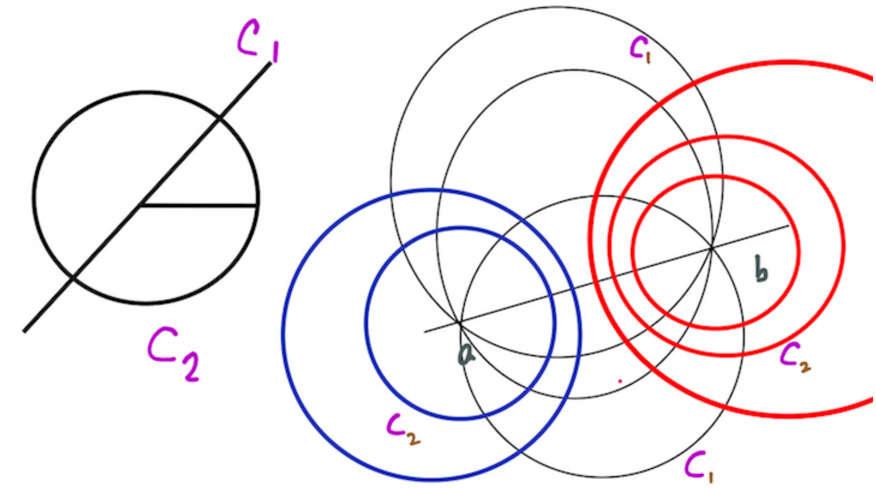

Figure 2.6: Orthogonal circles

Theorem 2.8.1. *Let a and b be two given points, C_1 and C_2 as defined above. Then*

(i) there is exactly one C_1 and one C_2 through each point in $\mathbb{C}$ except at the limit point a and b in the $z-$plane;

(ii) the tangent of each C_1 and that of each C_2 are orthogonal to each other at the points of intersections;

(iii) reflection in C_1 transforms every C_2 into itself and every C_1 into another C_1;

(iv) reflection in a C_2 transforms every C_1 into itself and every C_2 into another C_2;

(v) the limit points are symmetric with respect to each C_2, but not with respect to any other circle.

Proof. We consider the special case that $a = 0$ and $b = 0$ so that the circles passing through 0 and ∞ become straightlines passing through the origin in the $z-$plane. Then

(i) it is clear since there is only one straightline passing through any non-zero finite point and the origin, and only one circle intersecting with the straightline and orthogonal to it at that point;

(ii) follows since the C_2 are concentric circles;

(iii) also follows since it is clearly that any reflection of a concentric circle C_2 with respect to any straight line passing through the origin remains unchange. Reflection of any C_1 (straightline) with respect to a C_1 is obviously another C_1;

(iv) follows from Theorem 2.6.3 when considering symmetric points lying on a straightline is reflected upon each other lying on the same straightline with respect to a C_2. So a C_1 is mapped onto itself with respect to any C_2. Let C_2 relfect with respect to another C_2. Then parts (i) and (ii) imply that each point of the image of C_2 upon reflection must be orthogonal to each C_1 and this implies the image must be a circle. The image circle C_2 must be different from its preimage except itself because of the symmetric principle Theorem 2.6.3;

(v) this is obvious because of the choice.

Having established the special case $a = 0$ and $b = 0$, the general case (i-v) for arbitrary a and b follow since one can map a C_1 by a Möbius transformation to a straightline C_1' passing through the origin and then each corresponding C_2 becomes a circle C_2' centred at the origin so that C_2 must be orthogonal to C_1 because any Möbius transformation is conformal on $\mathbb{C}$. $\square$

Fixed points

The general Möbius transformation T that carries a to a' and b to b' can be written as

$$\frac{w-a'}{w-b'} = k\,\frac{z-a}{z-b}$$

which is an application of cross-ratios. Suppose we impose the requirement that $a = a'$ and $b = b'$. That is, we assume that

$$z = T(z) = \frac{az + b}{cz + d},$$

which will have two **fixed points** $Ta = a$ and $Tb = b$ since we have a quadratic equation in z. In the exceptional circumstance, we have a double root from the quadratic equation so that we are left with one double root. The transformation T maps C_1 to C_1', C_2 to C_2' and a, b to a', b'.

Theorem 2.8.2. *Let $w = T(z)$ be a Möbius transformation that satisfies,*

$$\frac{w - a}{w - b} = k\,\frac{z - a}{z - b}.$$

Then

(i) the whole circular net consists of C_1 and C_2 are mapped onto itself. That is, the union of C_1' and C_2' are the same as the union of C_1 and C_2;

(ii) when the images C_1' and C_2' are plotted on the same graph as C_1 and C_2, then

(a) the $\arg k$ represents the difference of the angle made by the tangents at the point of intersections between the circles C_1 and C_1';

(b) the

$$|k| = \frac{|w - a|/|w - b|}{|z - a|/|z - b|}$$

measures the ratio of the above right-hand side concerning the Apollonius circles C_2 and C_2',

*(iii) $C_1 = C_1'$ if $k > 0$ (with orientation reversed if $k < 0$), where the points on Tz on C_1 flow toward b upon increasing the value of k, and we call T **hyperbolic**;*

(iv) *$C_2 = C_2'$ if $|k| = 1$, then as* $\arg k$ *increase, the Tz circulates along C_2, and we call T* ***elliptic****.*

Proof. Exercise. □

Definition 2.8.3. If two fixed points of a Möbius transformation T coincide, then we call the transformation **parabolic**.

Rotations of the Riemann sphere

Let us consider a subgroup R of the set of all Möbius transformation that represent the rotation of the Riemann sphere S about its centre. Let us assume that the axis of rotation passes through the antipodal points Z_0 and Z_1 whose images on $\mathbb{C}$ are z_0 and z_1. Then we know that they are z_0 and $z_1 = -1/\bar{z}_0$ since $z_0\bar{z}_1 + 1 = 0$.

Theorem 2.8.4. *The Möbius transformation*

$$\frac{w - z_0}{1 + \bar{z}_0 w} = k\,\frac{z - z_0}{1 + \bar{z}_0 z}, \quad k = \cos\alpha + i\sin\alpha \tag{2.13}$$

(i) *leaves the points z_0 and $-1/\bar{z}_0$ invariant;*

(ii) *leaves the points Z_0 and Z_1 corresponding to z_0 and $-1/\bar{z}_0$ respectively, on the Riemann sphere S invariant;*

(iii) *rotates the plane that intersects the S in a great circle passing through Z_0 and Z_1 by an angle of α.*

Proof. The statements (i) and (ii) are clear. It remains to verify the (iii). It is left as an exercise for the reader to check that if $Tz = w$,

$$\Big|\frac{w - z_0}{1 + \bar{z}_0 w}\Big| = \Big|\frac{z - z_0}{1 + \bar{z}_0 z}\Big| = \rho > 0$$

then their chordal distance is

$$\chi(z,\, z_0) = \chi(w,\, z_0) = \frac{\rho}{\sqrt{1+\rho^2}}.$$

Let Z and W be the images of z and w respectively. Then it follows from (2.13) that the T is a rotation of the Riemann sphere S through the plane containing the great circle passing through the points Z_0, Z and Z_1 to the plane containing the great circle Z_0, W and Z_1. □

2.9 Extended Maximum Modulus Theorem

Let us recall some knowledge about metric spaces. Let (X, d) be a metric space. Then $F \subset X$ is *closed* if $X \setminus F$ is open. Let $A \subset X$ be a subset, the *closure* $\overline{A}$ of A is defined by

$$\cap\{F : F \text{ is closed and } A \supset F\}.$$

The *boundary* ∂A of A is defined by $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Let G be a subset of $\widehat{\mathbb{C}}$. We write

$$\partial_\infty G = \begin{cases} \partial G & \text{if } G \text{ is bounded;} \\ \partial G \cup \{\infty\} & \text{if } G \text{ is unbounded.} \end{cases}$$

to be the extended boundary of G in $\widehat{\mathbb{C}}$. If $a = \infty$, then the $B(a, r)$ is understood in terms of chordal metric.

Example 2.9.1. Let $G = \{z : |\arg z| < \frac{\pi}{2}\}$. Then

$$\partial G = \{z = x + iy : x = 0\}, \qquad \partial_\infty G = \partial G \cup \{\infty\}.$$

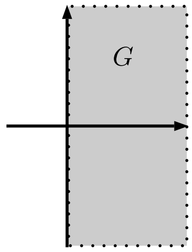

Figure 2.7: $G = \{z : |\arg z| < \frac{\pi}{2}\}$

Definition 2.9.2. Let $G \subset \mathbb{C}$ and $f : G \to \mathbb{R}$ be continuous. Suppose $a \in \partial_\infty G$, then we define

$$\limsup_{z\to a} f(z) = \lim_{r\to 0}\Big(\sup_z\{f(z) : z \in G \cap B(a,r)\}\Big) = L$$

and

$$\liminf_{z\to a} f(z) = \lim_{r\to 0}\Big(\inf_z\{f(z) : z \in G \cap B(a,\, r)\}\Big) = l.$$

If $a \neq \infty$, the above definition can be written as:
Given $\epsilon > 0$, there exists $r > 0$ such that

$$L - \epsilon < \sup_z\{f(z) : z \in G \cap B(a,r)\} < L + \epsilon.$$

In particular, $f(z) < L + \epsilon$ for all $z \in G \cap B(a,r)$.

Similarly, given $\epsilon > 0$, there exists $r > 0$ such that

$$l - \epsilon < \inf_z\{f(z) : z \in G \cap B(a,r)\} < l + \epsilon.$$

In particular, $f(z) > l - \epsilon$ for all $z \in G \cap B(a,r)$.

If $a = \infty$, we understand $B(a,r)$ is with the chordal metric and the $\limsup$, $\liminf$ have similar interpretations.

Note also that, it follows easily $\lim_{z\to a} f(z)$ exists if and only if $L = l$ ($a \in \partial_\infty G$).

Theorem 2.9.3 (Maximum Modulus Theorem - Extended version)**.** *Let $G \subset \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ is analytic. Suppose $\limsup_{z\to a} |f(z)| \leq M$ for some $M > 0$ and all $a \in \partial_\infty G$. Then $|f(z)| \leq M$ for all $z \in G$.*

Proof. Let

$$H = \{z \in G : |f(z)| > M + \delta\}$$

for a fixed $\delta > 0$. We aim to show that $H = \emptyset$. Since then $|f| \le M$ because $\delta > 0$ is arbitrary. It follows from the elementary fact in real analysis that H is open because $|f|$ is continuous. We next show that H has no intersection with a region near the ∞ and in particular $H \cap \partial_\infty G = \emptyset$, and hence H is a bounded set.

By the hypothesis $\limsup_{z\to a} |f(z)| \le M$ for all $a \in \partial_\infty G$, for the above $\delta > 0$, there exists $r > 0$ such that

$$|f(z)| < M + \delta$$

for all $z \in G \cap B(a, r)$. Hence $\overline{H} \subset G$. This argument works whether G is bounded or unbounded, and $a = \infty$. Thus $H \cap \partial_\infty G = \emptyset$ and hence H is bounded. Therefore $\overline{H}$ is a compact set.

Note that $|f(z)| = M + \delta$ when $z \in \partial H$ since $\overline{H} \subset \{z \in G : |f(z)| \ge M + \delta\}$. Thus either f is constant on H by Theorem 1.7.2 (hence f is constant on G by Identity theorem since H is open and non-empty) or $H = \emptyset$. But if f is constant on G, where $|f| = M + \delta$, then it contradicts the hypothesis that $|f| < M + \delta$ near $\partial_\infty G$. Thus $H = \emptyset$. This completes the proof. □

We shall apply the maximum modulus theorem to characterize certain analytic map of unit disk. We first recall

Theorem 2.9.4 (Schwarz's Lemma)**.** *Let $\Delta = \{z : |z| < 1\}$ be the unit disk. Suppose $f : \Delta \to \mathbb{C}$ is analytic such that $|f(z)| \le 1$ for each $z \in \Delta$, and $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in \Delta$ and $|f'(0)| \le 1$.*

Moreover, $f(z) = e^{i\theta} z$ for a fixed θ whenever $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$.

Proof. Define

$$F(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0; \\ f'(0), & z = 0. \end{cases}$$

F is thus analytic on Δ.

Moreover, $|F(z) = \left|\frac{f(z)}{z}\right| \le \frac{1}{|z|} \to 1$ as $|z| \to 1$. It follows from Theorem 2.9.3 that $|F(z)| \le 1$.

If $|F(z)| = 1$ for some $z \in \Delta$ (i.e. either $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$), then F is a constant $e^{i\theta}$ for some $\theta \in [0, 2\pi]$ by the maximum modulus theorem 1.7.2 since $|F| \le 1$ for all $z \in \Delta$. And so $f(z) = e^{i\theta}z$. □

Exercise. Suppose $\phi(z)$ is analytic on $|z| \le R$, where $|\phi(z)| \le 1$ and $\phi(0) = 0$. Show that $|\phi(z)| \le \frac{r}{R}$ on $|z| = r$, where $r < R$.

Proposition 2.9.5. *Suppose $|a| < 1$, then*

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}$$

is a conformal map mapping Δ onto Δ, $\partial\Delta$ to $\partial\Delta$. Moreover, $\varphi_a^{-1} = \varphi_{-a}$, $\varphi_a'(0) = 1 - |a|^2$ and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Proof. Since $|a| < 1$, φ_a is clearly analytic. In fact, φ_a is conformal (Exercise). We only show

$$\begin{aligned} |\varphi_a(e^{i\theta})| &= \left|\frac{e^{i\theta} - a}{1 - \overline{a}e^{i\theta}}\right| \\ &= \left|e^{i\theta} \cdot \frac{e^{i\theta} - a}{e^{-i\theta} - \overline{a}}\right| \\ &= \frac{|e^{i\theta} - a|}{|e^{-i\theta} - \overline{a}|} = 1. \end{aligned}$$

Hence $\varphi_a(\partial\Delta) = \partial\Delta$. The remaining conclusion is left as an exercise. □

Proposition 2.9.6. *Suppose $f : \Delta \to \Delta$ is analytic and $f(a) = \alpha$. Then*

$$|f'(a)| \le \frac{1 - |\alpha|^2}{1 - |a|^2}. \quad (\textit{max. value of } |f'(a)|)$$

Moreover, equality occurs if and only if $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$, $|c| = 1$.

Remark. We may assume $|\alpha| < 1$. Otherwise f is a constant.

Proof. Define $g = \varphi_\alpha \circ f \circ \varphi_{-a}$, Then $g(\Delta) \subset \Delta$, and $g(0) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = \dfrac{\alpha - \alpha}{1 - \overline{\alpha}\alpha} = 0$. Clearly g is analytic and thus $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$ by Schwarz's Lemma. But

$$g'(0) = \frac{1 - |a|^2}{1 - |\alpha|^2} f'(a).$$

Thus

$$|f'(a)| \leq \frac{1 - |\alpha|^2}{1 - |a|^2}. \tag{2.14}$$

Equality will occur if and only if there exists a c such that $|g'(0)| = |c| = 1$ and $g = cz$. $\square$

We can now prove the converse of Proposition 2.9.5.

Theorem 2.9.7. *Let $f : \Delta \to \Delta$ be an one-to-one analytic function onto Δ. Suppose $f(a) = 0$. Then there is a c such that $|c| = 1$ and*

$$f = c\varphi_a = c\frac{z - a}{1 - \overline{a}z}.$$

Proof. Since f is bijective, we let $g : \Delta \to \Delta$ to be f^{-1}. So $g(f(z)) = z$ for all $z \in \Delta$. We apply (2.14) to both f and g to derive the inequalities:

$$|f'(a)| \leq \frac{1}{1 - |a|^2} \quad \text{and} \quad |g'(0)| \leq 1 - |a|^2.$$

On the other hand, $1 = g'(0)f'(a)$. Thus, $|f'(a)| = (1 - |a|^2)^{-1}$ since

$$\frac{1}{1 - |a|^2} \leq |f'(a)| \leq \frac{1}{1 - |a|^2}.$$

Then, since $\varphi_0(z) = z$, Proposition 2.9.6 gives $f = c\varphi_a$ for some c with $|c| = 1$. $\square$

Remark. A simple consequence of the maximum modulus of entire functions is that the function $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ is an increasing function of r, i.e. $M(r_1) \leq M(r_2)$ if $r_1 \leq r_2$.

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