Chapter 3

Riemann Mapping Theorem

Let G be an open set in $\mathbb{C}$. We consider families of analytic functions $\{f_n\}$, $f_n : G \to \mathbb{C}$ and ask for condition on $\{f_n\}$ so that we could extract a convergent subsequence $\{f_{n_k}\}$ which converges uniformly in a certain sense. Such consideration is of fundamental importance in complex function theory. As an application, we shall prove the celebrated Riemann mapping theorem at the end of this chapter. We shall develop the theory step by step, first to continuous functions and then to analytic and meromorphic functions. On the other hand, we shall consider functions with values in a general complete metric space Ω although $\Omega = \mathbb{C}$ or $\Omega = \widehat{\mathbb{C}}$ is our primary considerations.

3.1 Metric Space

Definition 3.1.1. Let (Ω, d) to denote a *complete metric space* with the metric d on Ω. Suppose G is an open subset of $\mathbb{C}$, then $C(G, \Omega)$ denotes the *set of all continuous functions* from G to Ω.

In order to develop $C(G, \Omega)$ to have a meaning of *compactness*, we have to clarify several issues, such as how to turn $C(G, \Omega)$ into a metric space, what are the topology on it etc.

Let us first recall some basic facts about point-set topology.

Definition 3.1.2. (i) A metric space S is *complete* if every Cauchy sequence converges;

(ii) A subset X of a metric space S is *compact* if and only if every open covering of X contains a finite subcovering. (Heine-Borel property) (See Ahlfors p.60)

Proposition 3.1.3. *Let X be a compact subset of a metric space. Then X is complete and bounded.*

Proof. Let $\{x_n\}$ be a Cauchy sequence and suppose that $x_n \not\to y$ for any $y \in X$ as $n \to \infty$. Then there exists an $\epsilon > 0$ such that $d(x_n, y) > 2\epsilon$ for infinitely many n. With the same ϵ, there exists n_0 such that $d(x_n, x_m) < \epsilon$ for $n, m > n_0$. We choose a $n > n_0$ such that $d(x_n, y) > 2\epsilon$. Then $2\epsilon < d(x_n, y) \le d(x_n, x_m) + d(x_m, y) < \epsilon + d(x_m, y)$ for all $m > n_0$. So $d(x_m, y) > \epsilon$ for all $m > n_0$, i.e. all open balls $B(y, \epsilon)$ contains only finitely many x_n.

Let U be the union of open balls which contain only a finite number of x_n. If we suppose $\{x_n\}$ dose not converge, then U is an open covering of X all open balls contains only finitely many x_n by the preceding paragraph, or considering if any one of the open balls contain an infinite number of x_n, then $\{x_n\}$ will converge by the preceding paragraph.

Then, since X is compact, we could find a finite subcovering of the original covering. But this implies $\{x_n\}$ is a finite sequence. A contradiction. Hence x_n must converge.

Fix an $x_0 \in X$. Then $\cup_{r>0} B(x_0, r)$ is am open covering of X. Thus $X \subset B(x_0, r_1) \cup \cdots \cup B(x_0, r_m)$. Let $\tilde{r} = \max_{1 \le i \le m} r_i$. So for any $x, y \in X$, $d(x, y) \le d(x, x_0) + d(x_0, y) < 2\tilde{r}$ and thus X is bounded. □

In fact, a compact set is not just bounded, but totally bounded.

Definition 3.1.4. A subset X of a metric space S is *totally bounded* if for every $\epsilon > 0$, X can be covered by *finitely* many balls of radius ϵ.

Theorem 3.1.5. *A metric space is compact if and only if it is complete and totally bounded.*

Proof. It remains to prove a compact set is totally bounded in " $\Longrightarrow$ ". But this is easy, since $\cup_{x\in X} B(x,\epsilon)$ is an open cover of X. We extract a finite subcover $B(x_1,\epsilon)\cup\cdots\cup B(x_m,\epsilon)$ of X by compactness.

" $\Longleftarrow$ " We now assume X to be complete and totally bounded. Suppose X has an open covering U which does not contain any finite subcovering. Let $\epsilon_n = 1/2^n$. We know that X can be covered by finitely many $B(x,\epsilon_1)$, hence there must exist a $B(x_1,\epsilon_1)$ has no finite subcovering otherwise X must have a finite subcovering. But $B(x_1,\epsilon_1)$ is itself totally bounded (why?), hence there exists a ball $B(x_2,\epsilon_2)$ which does not admit a finite subcovering. Continuing the process, we obtain a sequence $\{x_n\}$ with the property that $B(x_n,\epsilon_n)$ has no finite subcovering and $x_{n+1}\in B(x_n,\epsilon_n)$. But then

$$\begin{aligned} d(x_n,x_{n+p}) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{x+2}) + \cdots + d(x_{n+p-1},x_{n+p}) \\ &< \epsilon_n + \epsilon_{n+1} + \cdots + \epsilon_{n+p-1} < \frac{1}{2^{n-1}}. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and suppose $x_n \to y$. This y must belong to a $B(y,\delta)$ which belongs to an open set in the original cover U. We choose n so large that $d(x_n,y)<\delta/2$ and $\epsilon_n<\delta/2$. But $d(x,y)\le d(x,x_m)+d(x_n,y)<\delta/2+\delta/2$ whenever $d(x,x_n)<\epsilon_n<\delta/2$. That is $B(x_n,\epsilon_n)\subset B(y,\delta)\subset$ an open subset of U. A contradiction since $B(x_n,\epsilon_n)$ has no finite subcovering by construction. □

We state the following results without proofs.

Corollary 3.1.5.1. *A subset of* $\mathbb{R}$ *or* $\mathbb{C}$ *is compact is and only if it is closed and bounded.*

Theorem 3.1.6. *A metric space is compact if and only if every infinite sequence has a limit point.*

Corollary 3.1.6.1. *Any infinite sequence in a closed and bounded subset of* $\mathbb{R}$ *and* $\mathbb{C}$ *has a convergent subsequence.*

Theorem 3.1.6 can be rephrased as a metric space is compact if and only if every infinite sequence has a convergent subsequence. We called such space to have the *Bolzano-Weierstrass property.*

We shall return to the question asked at the beginning of this chapter namely how to make $C(G, \Omega)$ to have the Bolzano-Weierstrss property. But for $C(G, \Omega)$ we have another name.

Definition 3.1.7. A family $\mathcal{F} \subset C(G, \Omega)$ is *normal* if each infinite sequence in F contains a convergent subsequence converges to a function in $C(G, \Omega)$. (Note that the precise definition is not given at this stage.)

Note that this definition differs to a subset to be sequentially compact (i.e. Theorem 3.1.6) in a metric space, because we do not require the limit of the infinite sequence to be in the subset.

Our first question is how to turn $C(G, \Omega)$ into a metric space. The problem being that G is an open set and even continuous functions may not behave well on an open set. So compact sets are much more suitable for our consideration especially for an infinite sequence. We shall first investigate some fundamental point-set topology result to see how one can *approximate* an open set by compact subsets.

Proposition 3.1.8. *Suppose that G is an open set, then there exists a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$. Moreover, the sequence can be chosen so that*

(i) $K_n \subset \mathit{int}\, K_{n+1}$

(ii) for each compact subset K of G, we can find an n such that $K \subset K_n$;

(iii) every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus G$.

Proof. Let $A \subset X$ and $x \in X$, recall that the *distance* from x to A is defined by

$$d(x, A) = \inf\{d(z, a) : a \in A\}$$

, where (X, d) is any metric space.

One way to construct the compact subset K_n is to let K_n consist of all points in G at distance $\leq n$ from the origin, *and* at distance $\geq 1/n$ from the boundary ∂G. That is, we define

$$K_n = \{z \in G : |z| \leq n\} \cap \{z \in G : d(z, \mathbb{C} \setminus G) \geq 1/n\}$$

which is bounded; and being the intersection of two closed sets must itself be closed. The interior $int\, K_n$ is just $\{z \in G : |z| < n\} \cap \{z \in G : d(z, \mathbb{C} \setminus G) > 1/n\}$. Hence $int\, K_{n+1} \supset K_n$ and (i) is satisfied. It is also easy to see from the definition of K_n that $G = \cup_1^\infty K_n$.

But since also $K_{n+1} \supset int\, K_{n+1}$, we get $G = \cup_1^\infty int\, K_n$ as well. Suppose now K is a compact subset of G. $G = \cup_1^\infty int\, K_n$ implies that $\{int\, K_n\}$ forms an open cover of G and also of K. But K is compact so we can find a finite subcovering $\cup_1^N int\, K_n$ of K. Since $\cup_1^N int\, K_n \subset int\, K_N \subset K_N$, there exists an N such that $K \subset K_N$.

To prove part (iii), we need to show every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus G$. Since $K_n \subset G$ for each n, we have $\widehat{\mathbb{C}} \setminus G \subset \widehat{\mathbb{C}} \setminus K_n$. It follows that the *unbounded* component of $\widehat{\mathbb{C}} \setminus G$ must be a subset of the unbounded component of $\widehat{\mathbb{C}} \setminus K_n$ for each n. It also follows from the definition of K_n that the unbounded component of $\widehat{\mathbb{C}} \setminus K_n$ must contain $\{z : |z| > n\}$ as a subset. So for any *bounded* component D (open) of $\widehat{\mathbb{C}} \setminus K_n$, it must contain a point z such that $d(z, \mathbb{C} \setminus G) < 1/n$. By definition we can therefore find a $w \in \mathbb{C} \setminus G$ such that $|w - z| < 1/n$. But then $z \in B(w, 1/n) \subset \widehat{\mathbb{C}} \setminus K_n$. Since disks are connected and z is in the component D of $\widehat{\mathbb{C}} \setminus K_n$, $B(w, 1/n) \subset D$. If D_1 is the component of $\widehat{\mathbb{C}} \setminus G$ that contains w, then it follows that $D_1 \subset D$. $\square$

The sequence of compact sets K_n such that $\cup K_n = G$, $K_n \subset K_{n+1}$ is called an *exhaustion* of G by compact sets.

Metric Space $C(G, \Omega)$

Suppose (S, d) is a metric space then it is easy to show that

$$d'(s, t) = \frac{d(s, t)}{1 + d(s, t)} \quad (s, t \in S)$$

is also a metric on S, and hence (S, d') is another metric space. (Verify that $d'(s,t) \le d'(s,q) + d'(q,t)$ and $d'(s,t) = 0 \iff s = t$.)

It is also not difficult to check that d and d' induce the same topology on S i.e. a subset T is open in (S, d) if and only if it is open in (S, d'); a sequence is a Cauchy sequence in (S, d) if and only if it is a Cauchy sequence in (S, d'), etc.

Let G be an open set in $\mathbb{C}$ and according to Proposition 3.1.8, there is an exhaustion of G by the compact set $\{K_n\}$, $K_n \subset int\, K_{n+1}$, $G = \cup_1^\infty K_n$. Suppose $f, g \in C(G, \Omega)$, and we recall that $C(G, \Omega)$ denotes the set of all continuous functions $f : G \to \Omega$. We define

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in K_n\}.$$

It is easy to see that ρ_n is a metric on $C(K_n, \Omega)$ for each n since (Ω, d) is a metric space. We further define

$$\begin{aligned} \rho(f, g) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \\ &\le \sum_{n+1}^{\infty} \frac{1}{2^n} = 1 \end{aligned}$$

since $\rho_n(f, g)/(1+\rho_n(f, g)) \le 1$. By the above discussion ρ satisfies the triangle inequality, $\rho(f, g) = \rho(g, f)$. Finally suppose $\rho(f, g) = 0$. Then $\rho_n(f, g) = 0$ and $f = g$ on K_n. But $G = \cup K_n$. So $f = g$ identically on G. So ρ is a metric on $C(G, \Omega)$ and $(C(G, \Omega), \rho)$ is a metric space. (We shall see later that $(C(G, \Omega), \rho)$ is in fact a complete metric space.)

If $f_m \to f$ in $C(G, \Omega)$ with sequence to ρ, then $f_m \to f$ uniformly on each compact subset K_n of G. (See later if this is unclear to you at this point.)

Since the construction of the metric space $(C(G, \Omega), \rho)$ depends on a particular exhaustion $\{K_n\}$, we naturally ask will $\{K_n\}$ affects the topology on $(C(G, \Omega), \rho)$ i.e. if O is open with respect to $\{K_n\}$, would O be still open with respect to another exhaustion? To do so, we require the following characterization of open sets in $(C(G, \Omega), \rho)$ in terms of the metric d on Ω.

Proposition 3.1.9. *Let ρ be the above metric defined on $C = C(G, \Omega)$.*

(i) For every $\epsilon > 0$, there exist a $\delta > 0$ and a compact set $K \subset G$ such that for $f, g \in C$, $\sup\{d(f(z), g(z) : z \in K\} < \delta$ implies $\rho(f, g) < \epsilon$.

(ii) Conversely, if we are given a $\delta > 0$ and *a compact set $K \subset G$, there exists an $\epsilon > 0$ such that for $f, g \in C$, $\rho(f, g) < \epsilon$ implies* $\sup\{d(f(z), g(z) : z \in K\} < \delta$.

Proof. (i) Let $\epsilon > 0$ be given, we choose an integer p so large such that $\sum_{p+1}^{\infty} 1/2^n < \epsilon/2$. Let $\delta > 0$ be chosen so small such that for $0 < t < \delta$, we have $t/(t+1) < \epsilon/2$. Recall that $G = \cup K_n$, now let $K = K_p$, and consider those f and g such that $\sup\{d(f(z), g(z)); z \in K\} < \delta$. But $\rho_k(f, g) \leq \rho_p(f, g)$ for $1 \leq k \leq p$. Hence

$$\begin{aligned}
\rho(f, g) = \sum_{1}^{\infty} \frac{\rho_k(f, g)}{2^k(1 + \rho_k(f, g))} &= \left(\sum_{1}^{p} + \sum_{p+1}^{\infty}\right) \frac{\rho_k(f, g)}{2^k(1 + \rho_k(f, g))} \\
&\leq \sum_{1}^{p} \frac{1}{2^k} \cdot \frac{\epsilon}{2} + \sum_{p+1}^{\infty} \frac{1}{2^k} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

as required.

(ii) Suppose now a $\delta > 0$ and a compact set $K \subset G$ is given. Suppose $\cup K_n = G$ is an exhaustion of G by compact set. Then there exists an integer p such that $K \subset K_p$. Choose $\epsilon > 0$ so small such that $\dfrac{2^p \epsilon}{1 - 2^p \epsilon} < \delta$.

Suppose $\rho(f, g) < \epsilon$, then

$$\frac{\rho_p(f, g)}{2^p(1 + \rho_p(f, g)} < \epsilon,$$

i.e.

$$\rho_p(f, g) < \frac{2^p \epsilon}{1 - 2^p \epsilon} < \delta.$$

Thus $\sup\{d(f(z), g(z)) : z \in K\} \leq \rho_p(f, g) < \delta$ as required. $\square$

What is an *open ball* in (C, ρ)?
Ans: $B(f, \epsilon) = \{g : \rho(g, f) < \epsilon\}$.

What about an *open set* in (C, ρ)?
Ans: Since open set is an union of open balls, or for each f in the open set, there exists an $\epsilon > 0$ such that $B(f, \epsilon)$ is a proper subset of the open set.

We immediately obtain:

Proposition 3.1.10. *A set $U \subset (C, \rho)$ is open if and only if for each $f \in U$, there exist a compact set $K \subset G$ and a $\delta > 0$ such that*

$$U \supset \{g : d(f(z), g(z)) < \delta : z \in K\}.$$

Proposition 3.1.10 clearly indicates that any open set U of (C, ρ) is independent of the particular exhaustion $\{K_n\}$ used to define ρ_n and hence ρ. This answers the question raised before Proposition 3.1.9.

Here we again answer a claim made before Proposition 3.1.9.

Proposition 3.1.11. *Let $\{f_n\}$ be an infinite sequence in $(C(G, \Omega), \rho)$. Then $f_n \to f \in (C(G, \Omega), \rho)$ if and only if $\{f_n(z)\}$ converges to $f(z)$ uniformly on every compact subset of G.*

Proof. " $\Longrightarrow$ " Let $K \subset G$ be an arbitrary compact set. By (ii) of Proposition 3.1.8, there exists a compact set K_N in the exhaustion $\cup K_n = G$ so that $K \subset K_N \subset K_n$ for all $n \geq N$. Thus $\rho_N(f_m, f) \to 0$ as $m \to \infty$ since

$$\frac{\rho_N(f_m, f)}{2^N(1 + \rho_N(f_m, f))} \leq \sum_1^\infty \frac{\rho_N(f_m, f)}{2^N(1 + \rho_N(f_m, f))} = \rho(f_m, f) \to 0$$

as $m \to \infty$. But

$$\sup\{d(f_m(z), f(z)) : z \in K\} \leq \sup\{d(f_m(z), f(z)) : z \in K_N\} \to 0$$

as $m \to \infty$ by Proposition 3.1.9(ii). Hence $f_m \to f$ on any compact set $K \subset G$.

The converse is left as an exercise. □

So far we have not used the assumption at the beginning that Ω is a complete metric space.

Theorem 3.1.12. $(C(G,\Omega),\rho)$ *is a complete metric space.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $(C(G,\Omega),\rho)$. That is , given $\epsilon > 0$, there exists a $N > 0$ such that $\rho(f_n, f_m) < \epsilon$ whenever $n, m > N$.

By Proposition 3.1.9(ii), given any compact set $K \subset G$ and $\delta > 0$, we have

$$\sup\{d(f_n(z), f_m(z) : z \in K\} < \delta \tag{3.1}$$

whenever $n, m > N$. That is, $\{f_n(z)\}$ is a Cauchy sequence in $\mathbb{C}$. Thus $f_n(z)$ must converge to a complex number $f(z)$, say. This is true for every $z \in K$. So we obtain a function by $f : K \to \mathbb{C}, z \mapsto f(z)$.

We need to verify that $f_n \to f$ with respect to ρ and that $f \in C(G,\Omega)$. Let z be an arbitrary element of K, then there exists an $m_0 = m_0(z)$ such that $d(f_m(z), f(z)) < \delta$ for $m > m_0$.

Let $n > N$ and $z \in K$, we have

$$d(f_n(z), f(z)) \leq d(f_n(z), f_m(z)) + d(f_m(z), f(z)) \leq \delta + \delta = 2\delta \tag{3.2}$$

by choosing $m > m_0$ sufficiently large. It follows from (3.1) that (3.2) holds uniformly for all $z \in K$ and $n > N$. That is, $f_n \to f$ uniformly on every compact subset K of G. Proposition 3.1.10 implies that $\rho(f_n, f) \to 0$ as $n \to \infty$. Moreover since $f_n \to f$ uniformly on K, f must be continuous. Since K is arbitrary, f must be continuous on G by Proposition 3.1.8, i.e. $f \in C(G,\Omega)$. □

Recall that a family $\mathcal{F} \subset C(G,\Omega)$ is normal if every infinite sequence has a subsequence which converges to a function in $C(G,\Omega)$. Note that the limit is not required to be a member of $\mathcal{F}$. This and Theorem 3.1.6 imply that

Proposition 3.1.13. *A family $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if $\overline{\mathcal{F}}$ is compact (or $\mathcal{F}$ is relatively compact in $C(G, \Omega)$).*

We now relate the concepts of normality and total boundedness. We recall, from Theorem 3.1.5 that, a subset is compact if and only if it is complete and totally bounded. Hence Proposition 3.1.13 can be rephrased as: $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if $\overline{\mathcal{F}}$ is complete and totally bounded. $\mathcal{F}$ being a subset of $\overline{\mathcal{F}}$ is also totally bounded, i.e. given $\epsilon > 0$, $\mathcal{F} \subset \cup_1^N B(f_i, \epsilon)$ for some $\{f_1, \ldots, f_N\}$ of $\mathcal{F}$. So for every $\epsilon > 0$, there exist $f_1, \ldots, f_N \in \mathcal{F}$ such that for every $f \in \mathcal{F}$, there exist an i such that $\rho(f, f_i) < \epsilon$.

We now state this in terms of the original metric d.

Exercise.

Let $S = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R},$ only finitely many $x_i \neq 0\}$. Then (S, d) is a metric space, where $d(x, y) = \max\{|x_i - y_i|\}$. Is (S, d) complete? Show that the $\delta-$neighbourhoods are not totally bounded.

Theorem 3.1.14. *A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if for every compact set $K \subset G$ and $\delta > 0$, there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that for each $f \in \mathcal{F}$, there exists an i among $\{1, \ldots, n\}$ with*

$$\sup\{d(f(z), f_i(z)) : z \in K\} < \delta. \tag{3.3}$$

Proof. Suppose $\mathcal{F}$ is normal; hence $\overline{\mathcal{F}}$ is compact and thus totally bounded. So for each $\epsilon > 0$, there exist $f_1, \ldots, f_n$ among $\mathcal{F}$ such that $\mathcal{F} \subset \cup_1^n B(f_i, \epsilon)$.

Let $K \subset G$ be compact and $\delta > 0$ be given. According to Proposition 3.1.9(ii), we may choose $\epsilon > 0$ such that for each $f \in B(f_i, \epsilon)$, we have

$$\sup\{d(f(z), f_i(z)) : z \in K\} < \delta.$$

Conversely, suppose $\mathcal{F}$ has the property (3.3), then it is clear that $\overline{\mathcal{F}}$ also has this property (3.3). By Proposition 3.1.13, it is equivalent to show that $\overline{\mathcal{F}}$ is a compact subset of (C, ρ) in order to show that $\mathcal{F}$ is normal. But $\overline{\mathcal{F}}$ is compact if and only if it is complete and totally bounded. Since $\overline{\mathcal{F}}$ satisfies (3.3), $\overline{\mathcal{F}}$ is totally bounded by Proposition 3.1.9(i). But $\overline{\mathcal{F}}$ is a closed subset of the complete metric space (C, ρ), so it must be complete also. This proves that $\mathcal{F}$ is normal. □

We have essentially established the *theory part* of Normal family. However, it is still too general to be applicable. For example, one main result is by Montel: A family of analytic functions is normal if and only if the family is locally bounded. We shall define the term *locally bounded* precisely later. It essentially means each f in the family is bounded on every ball. To make the connection, we still need to establish several *links*, some of them are very important on their own.

3.2 Arzela-Ascoli Theorem

Definition 3.2.1. A set $\mathcal{F} \subset C(G, \Omega)$ is *equicontinuous at a point* $z_0 \in G$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every $f \in \mathcal{F}$.

Similarly, $\mathcal{F}$ is *equicontinuous over a set* $E \subset G$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $|z - z'| < \delta$, $d(f(z), f(z')) < \epsilon$ whenever $z, z' \in E$ and for every $f \in \mathcal{F}$.

Remark. If $\mathcal{F} = \{f\}$, then $\mathcal{F}$ is equicontinuous at z_0 means just f is continous at z_0. And $\mathcal{F} = \{f\}$ is equicontinous over a set $E \subset G$ if f is uniformly continuous over E.

Lemma 3.2.2 (Lebesgue's Covering Lemma)**.** *Let (X, d) be a compact metric space. If $\mathcal{G}$ is an open covering of X, then there is an $\epsilon > 0$ such that for each $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x, \epsilon) \subset G$.*

Proof. Since X is compact, Theorem 3.1.6 implies that every infinite sequence has a convergent subsequence. Let $\mathcal{G}$ be an open cover of X, suppose on the contrary that there is no such $\epsilon > 0$ can be found. In particular, for every integer n there is a point $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any member G of $\mathcal{G}$. But $\{x_n\}$ must have a subsequence $\{x_{n_k}\}$ converging to $x_0 \in X$, say. There must be a $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Choose $\epsilon > 0$ such that $B(x_0, \epsilon) \subset G_0$. Let $N > 0$ such that $d(x_0, x_{n_k}) < \epsilon/2$ for all $n_k > N$. We further choose n_k such that $n_k \geq \max\{N, 2/\epsilon\}$, $y \in B(x_{n_k}, 1/n_k)$. Then $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \epsilon/2 + \epsilon/2 = \epsilon$. That is $B(x_{n_k}, 1/n_k) \subset B(x_0, \epsilon) \subset G_0 \in \mathcal{G}$. A contradiction. $\square$

Remark. The $\epsilon > 0$ in the above lemma is known as *Lebesgue's number.*

Proposition 3.2.3. *Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Then $\mathcal{F}$ is equicontinuous over each compact subset of G.*

Proof. Let $K \subset G$ be a compact set and fix $\epsilon > 0$. $\mathcal{F}$ is equicontinuous at each point w of K means that there exists a $\delta_w > 0$ such that $d(f(w), f(w')) < \epsilon/2$, for all $f \in \mathcal{F}$ and $|w - w'| < \delta_w$.

The set $\{B(w, \delta_w) : w \in K\}$ forms an open cover of K. By Lebesgue's Covering Lemma, there exists a $\delta > 0$ such that for each $z \in K$, $B(z, \delta)$ is contained in one of these $B(w, \delta_w)$. So if $z' \in B(z, \delta)$, then $d(f(z), f(z')) \leq d(f(z), f(w)) + d(f(w), f(z')) < \epsilon/2 + \epsilon/2 = \epsilon$ for all $f \in F$ whenever $z' \in B(z, \delta)$. Hence $\mathcal{F}$ is equicontinuous over K. $\square$

Theorem 3.2.4 (Arzela-Ascoli Theorem)**.** *A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if*

(i) $\mathcal{F}$ is equicontinuous at each point of G;

(ii) for each $z \in G$, $\overline{\{f(z) : f \in \mathcal{F}\}}$ is compact in Ω.

We shall postpone the proof of Arzela-Ascoli Theorem and give an application first. (Full detail will be given later.)

Theorem 3.2.5 (Montel's Theorem)**.** *Let $H(G)$ be a subset of $C(G, \Omega)$ of all analytic functions $f : G \to \Omega = \mathbb{C}$. (Note that $H(G)$ is complete.) Then $F \subset H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.*

In order to prove the Arzela-Ascoli Theorem, we need the following lemma.

Lemma 3.2.6 (Cantor Diagonalization Process)**.** *Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$, and let $X = \prod_1^\infty X_n$ be their Cartesian product. Let $\xi = (x_n)$, $\eta = (y_n) \in X$. Then*

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))}$$

defines a metric on X ((X, d) is a metric space). Let

$$\xi^k = (x_n^k)_{k=1}^{\infty} = (x_1^k, x_2^k, x_3^k, \ldots) \in X,$$

then $\xi^k \to \xi = (x_n)$ say, in (X, d) if and only if $x_n^k \to x_n \in X_n$ for each n as $k \to \infty$.

Moreover (X, d) is compact if (X_n, d) is compact for each n.

Proof. It is left to the reader to verify that (X, d) is a metric space.

" $\Longrightarrow$ " Suppose first that $\xi^k \to \xi$ in (X, d), i.e. $d(\xi^k, \xi) \to 0$ as $k \to \infty$. Then, for each $n \in \mathbb{N}$, $d_n(x_n^k, x_n) \to 0$ as $k \to \infty$ since

$$\lim_{k\to\infty} \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq \lim_{k\to\infty} d(\xi^k, \xi) 2^n = 0.$$

" $\Longleftarrow$ " Suppose now that $d_n(x_n^k, x_n) \to 0$ for each $n \in \mathbb{N}$ as $k \to \infty$.

Given $\epsilon > 0$, we choose l so large that $\sum_{n=l+1}^{\infty} 1/2^n < \epsilon/2$, and choose a $\delta > 0$ so small that $\dfrac{t}{1+t} < \dfrac{\epsilon}{2}$ if $t < \delta$. Since $d_n(x_n^k, x_n) \to 0$ as $k \to \infty$, there exists a $K > 0$ such that $d_n(x_n^k, x_n) < \delta$ if $k > K$ for $1 \leq n \leq l$. Hence

$$\begin{aligned} d(\xi^k, \xi) &= \left(\sum_{1}^{l} + \sum_{l+1}^{\infty} \right) \frac{d_n(x_n^k, x_n)}{2^n(1 + d_n(x_n^k, x_n))} \\ &< \sum_{1}^{l} \frac{1}{2^n} \cdot \frac{\epsilon}{2} + \sum_{l+1}^{\infty} \frac{1}{2^n} < \epsilon \end{aligned}$$

by the choice of l and k above. Hence $d(\xi^k, \xi) \to 0$ as $k \to \infty$. This proves the first part of the lemma.

Suppose now that (X_n, d_n) is compact for each $n \in \mathbb{N}$. By Theorem 3.1.6 it suffices to prove that every infinite sequence contains a convergent subsequence. We now come to describe the famous *Cantor diagonalization process.* Let $\xi^k = (x_n^k) = (x_1^k, x_2^k, x_3^k, \ldots)$, $k = 1, 2, 3, \ldots$, be a sequence in (X, d) where each $x_n^k \in (X_n, d_n)$

Since X_1 is assumed to be compact, so $(x_1^k)_1^\infty$ has a convergent subsequence converges to a point x_1 say, in X_1 (by Theorem 3.1.6). So there is a subset of $\mathbb{N}$ denoted by $\mathbb{N}_1$ such that $k \in \mathbb{N}_1$. Similarly since X_2 is compact, we can find a subset of $\mathbb{N}_1$ denoted by $\mathbb{N}_2$ such that $x_2^k \to x_2 \in X_2$ as $k \to \infty, k \in \mathbb{N}_2$. It is to be noted that $x_1^k \to x_1$ and $x_2^k \to x_2$ as $k \to \infty, k \in \mathbb{N}_2$. By the same method we may repeat the above procedure for $X_3, X_4, \ldots$ and obtain $\mathbb{N}_2 \supset \mathbb{N}_3 \supset \mathbb{N}_4 \supset \mathbb{N}_5 \supset \cdots$.

We now let k_j be the $j-$th element in $\mathbb{N}_j$, then

$$\xi^{k_j} = (x_1^{k_j}, x_2^{k_j}, x_3^{k_j}, \ldots)$$

converges to $\xi = (x_n) = (x_1, x_2, x_3, \ldots)$ as $k_j \to \infty$ with j. To see this, we note that $\lim_{k_j \to \infty} x_n^{k_j} = x_n$ for each n, since $k_j \in \mathbb{N}_j \subset \mathbb{N}_n$ when $j \geq n$. This completes the proof. $\square$

Now we are ready to prove the Arzela-Ascoli Theorem (Theorem 3.2.4).

Proof of Arzela-Ascoli Theorem. " $\Longrightarrow$ " Let us first assume that $\mathcal{F}$ is normal. We deal with (ii) first. So fix a $z \in G$ and define a map $F : C(G, \Omega) \to \Omega$ by $f \mapsto f(z)$. We aim to prove that F is a continuous mapping. Proposition 3.1.9(ii) implies that given $f, g \in C(G, \Omega)$ and $\epsilon > 0$, we can find a $\delta > 0$ such that

$$d(f(z), g(z)) < \epsilon \quad \text{whenever} \quad \rho(f, g) < \delta. \quad (K = \{z\})$$

The statement is equivalent to

$$d(F(f), F(g)) < \epsilon \quad \text{whenever} \quad \rho(f, g) < \delta.$$

That is, F is a continuous mapping from $C(G, \Omega)$ to Ω. Since $\mathcal{F}$ is normal, and so $\overline{\mathcal{F}}$ is compact, it follows $F(\overline{\mathcal{F}})$ is also compact in Ω.

Since this argument works for each $z \in G$, it completes the argument.

We now show that $\mathcal{F}$ is equicontinuous at each point z_0 of G. Fix $z_0 \in G$, and let $\epsilon > 0$ be given. We choose $R > 0$ such that $\overline{B(z_0, R)} \subset G$. Let $K = \overline{B(z_0, R)}$ which is a compact set. According to Theorem 3.1.14, there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that for each $f \in \mathcal{F}$, there exists a $k \in \{1, \ldots, n\}$ with

$$\sup\{d(f(z), f_k(z)) : z \in \overline{B(z_0, R)} = K\} < \frac{\epsilon}{3}.$$

We now make use of the fact that f_k is continuous at z_0. That is, there exists a $0 < \delta < R$ such that $|z - z_0| < \delta$ implies

$$d(f_k(z), f_k(z_0)) < \frac{\epsilon}{3}$$

for $1 \le k \le n$. Therefore given $\epsilon > 0$, $f \in \mathcal{F}$, there exists a $\delta > 0$ (with a suitable k) such that $|z - z_0| < \delta$ implies

$$\begin{aligned} d(f(z), f(z_0)) &\le d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

" $\Longleftarrow$ " We now prove the converse. So suppose (i) and (ii) of the theorem hold. Let $\{z_n\}$ be an rational enumeration of G (i.e. z_n has rational real and imaginary parts, $z_n \in G$). We define

$$X_n = \overline{\{f(z_n) : f \in \mathcal{F}\}} \subset \Omega$$

for every n. By (ii) of the hypothesis (X_n, d) is a compact metric space. Hence Lemma 3.2.6 implies $X = \prod_1^\infty X_n$, with the metric as defined in Lemma 3.2.6, is again a compact metric space .

For each $f \in \mathcal{F}$ we define a sequence

$$\tilde{f} = (f(z_1), f(z_2), f(z_3), \ldots) \in X.$$

Suppose $\{f_k\}$ is an infinite sequence in $\mathcal{F}$, we shall prove $f_k \to f \in C(G, \Omega)$ by proving that $\{f_k\}$ is a Cauchy sequence in the $C(G, \Omega)$. But $C(G, \Omega)$ is complete and hence $\mathcal{F}$ must be normal.

As for $\widetilde{f}$, we define

$$\widetilde{f_k} = (f_k(z_1), f_k(z_2), \ldots)$$

which is an infinite sequence in the compact metric space X. By Theorem 3.1.6 $\{\widetilde{f_k}\}$ has a convergent subsequence which we still denote by $\{\widetilde{f_k}\}$. Suppose $\lim_{k\to\infty} f_k(z_n) = w_n$, Lemme 3.2.6 implies $\lim_{k\to\infty} \widetilde{f_k} = \xi = (w_n)$.

So our strategy is to show given $\epsilon > 0$, K is an arbitrary compact subset, there exists a $J > 0$ such that

$$d(f(_k(z), f_j(z)) < \epsilon \quad \text{whenver } k, j > J$$

and for $z \in K$. Then by Proposition 3.1.9(i), $\{f_k\}$ will be a Cauchy sequence in $C(G, \Omega)$.

Since K is compact, let $R = \text{dist}(K, \partial G) > 0$, and

$$K_1 = \left\{ z \in G : d(z, K) \le \frac{R}{2} \right\}.$$

So K_1 is again compact and $K \subset int\, K_1 \subset K_1 \subset G$.

We clearly have the values of f_k at z_n when k is large, $f_k(z_n) \sim w_n$ (k sufficiently large). We use the hypothesis that $\mathcal{F}$ is equicontinuous over K to gain control of $f_k(z)$ when z is close to one of z_n. Since $\mathcal{F}$ is equicontinuous at each point of G, it is equicontinuous over K_1. That is, with the $\epsilon > 0$ given above, we can find a $\delta > 0$ such that $\delta < \dfrac{R}{2}$ and

$$d(f(z), f(z')) < \frac{\epsilon}{3}$$

for all $f \in \mathcal{F}$ whenever $|z - z'| < \delta$ and z, $z' \in K_1$. Let $D = \{z_n\} \cap K_1 = \{\xi_i\}$. Then the open sets $\{B(\xi_i, \delta) : \xi_i \in D\}$ is an open cover of K. (See Figure 3.1)

But K is compact, so we can find a subcovering of disks with centres $\xi_1, \xi_2, \ldots, \xi_n \in D$.

Note that $\lim_{k\to\infty} f_k(\xi_i)$ exists for each i, hence there exists a $J > 0$ such that for $j, k > J$, $d(f_k(\xi_i), f_j(\xi_i)) < \dfrac{\epsilon}{3}$ for each of $i = 1, \ldots, n$.

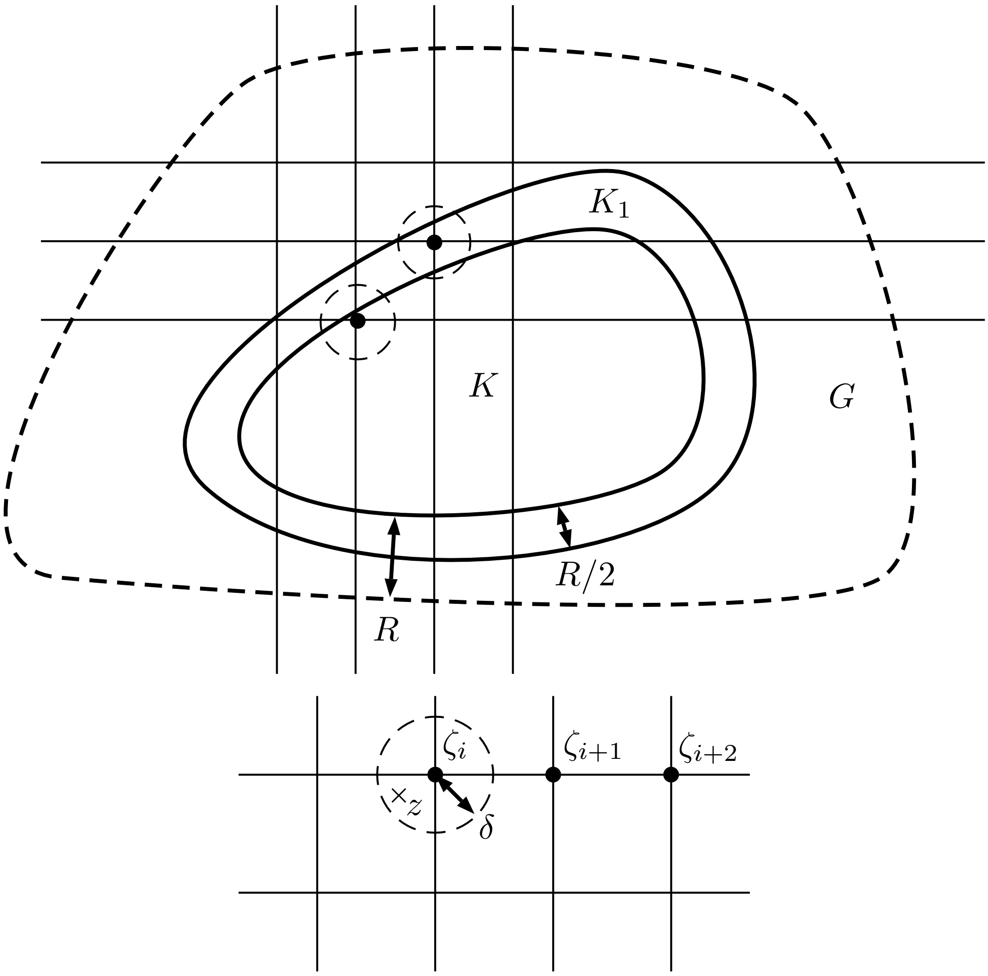

Figure 3.1: $\{B(\xi_i,\delta) : \xi_i \in D\}$

Now let z be an arbitrary point in K, $z \in B(\xi_i, \delta)$ for some i, so

$$\begin{aligned} d(f_k(z), f_j(z)) &\leq d(f_k(z), f_k(\xi_i)) + d(f_k(\xi_i), f_j(\xi_i)) + d(f_j(\xi_i), f_j(z)) \\ &< \underbrace{\frac{\epsilon}{3}}_{\text{equicontinuous}} + \underbrace{\frac{\epsilon}{3}}_{\text{convergence}} + \underbrace{\frac{\epsilon}{3}}_{\text{equicontinuous}} = \epsilon \end{aligned}$$

provided $j, k > J$. This completes the proof. $\square$

3.3 Normal Family of Analytic Functions

Let G be an open subset of $\mathbb{C}$ and let $H(G)$ be a subset of $C(G, \mathbb{C})$ consisting of analytic functions $f : G \to \mathbb{C}$. Thus almost all basic properties of $C(G, \Omega)$ are carried over to $H(G)$. However, it is not clear that if $H(G)$ is closed (and hence complete).

Theorem 3.3.1. *Suppose $\{f_n\}$ is a sequence in $H(G)$ and $f \in C(G, \Omega)$ such that $f_n \to f$. Then $f \in H(G)$, and $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$.*

Proof. Let T be a triangle contained inside a disk $D \subset G$. Since T is a compact set, $\{f_n\}$ converges to f uniformly over T. Hence $\oint_T f = \lim \oint_T f_n = 0$ by Cauchy's Theorem. But this is true for every T, Morera's Theorem implies that f must be analytic on every disk $D \subset G$. That is, f is analytic on G.

To show $f_n^{(k)} \to f^{(k)}$, this follows from Cauchy's integral formula. Let $a \in G$. Then there exists $R > r$ such that $B(a, r) \subset B(a, R) \subset G$. Let $\gamma = \partial B(a, R)$ then Cauchy's integral formula gives, for $z \in B(a, r)$,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w - z)^{k+1}}\, dw.$$

Let $M_n = \max\{|f_n(w) - f(w)| : w \in \gamma\}$. Then $M_n \to 0$ as $n \to \infty$ since $f_n \to f$ in $C(G, \Omega)$. Thus

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &\leq \frac{k!}{2\pi} M_n \int_0^{2\pi} \frac{1}{(R - r)^{k+1}} R\, d\theta \\ &= \frac{k! M_n R}{(R - r)^{k+1}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $f_n^{(k)} \to f^{(k)}$ uniformly on $B(a, r)$. Suppose K is an arbitrary compact set of G. Then we can find $a_1, \ldots, a_m$ such that $K \subset \cup_1^m B(a_i, r)$. So $f_n^{(k)} \to f^{(k)}$ uniformly on K and thus $\rho(f_n^{(k)}, f^{(k)}) \to 0$ in $H(G)$ by Proposition 3.1.11. $\square$

Corollary 3.3.1.1. *(i) $H(G)$ is a complete metric space;*

(ii) *If each $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to f, then*

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note that both Theorem 3.3.1 and Corollary 3.3.1.1 have no analogues in real variable theory. Can you think of some examples?

Here is again an unusual theorem.

Theorem 3.3.2 (Hurwitz's Theorem)**.** *Let G be a region and $f_n : G \to \mathbb{C}$ are in $H(G)$. Suppose $f_n \to f \not\equiv 0$, $\overline{B(a, R)} \subset G$ and $f(z) \neq 0$ on $|z - a| = R$, then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a, R)$.*

Proof. Let us recall *Rouché's Theorem*: (see Conway p.125) Suppose f and g are analytic in a neighborhood of $\overline{B(a, R)}$ and have no zeros on $|z - a| = R$. Suppose further that

$$|f(z) + g(z)| < |f(z) + |g(z)|$$

for all $|z - a| = R$, then f and g have the same number of zeros with due count of multiplicities of multiple zeros.

Since $f(z) \neq 0$ on $|z - a| = R$, therefore

$$\delta = \inf\{|f(z)| : |z - a| = R\} > 0.$$

The hypothesis $f_n \to f$ uniformly on $|z - a| = R$ implies there is an N such that $f_n \neq 0$ for all $n \geq N$. But

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)|$$

for all n sufficiently large. We conclude the theorem by applying Rouché's theorem. □

Corollary 3.3.2.1. *Suppose G is a region and $\{f_n\} \subset H(G)$, $f_n \to f$ in $H(G)$. Suppose $f_n(z) \neq 0$ for each $z \in G$ and n, then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in G$.*

Definition 3.3.3. A family $\mathcal{F} \subset H(G)$ is *locally bounded* if each $a \in G$, there is a $M > 0$ and an $r > 0$ such that for all $f \in \mathcal{F}$,

$$|f(z)| \leq M, \quad \text{for all } z \in B(a, r).$$

We immediately deduce

Proposition 3.3.4. *A family $\mathcal{F} \subset H(G)$ is* locally bounded *if and only if for each compact set $K \subset G$ there is a constant M such that*

$$|f(z)| \leq M, \quad \textit{for all } f \in \mathcal{F} \textit{ and } z \in K.$$

Theorem 3.3.5 (Montel's Theorem)**.** *A family $\mathcal{F} \subset H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.*

Proof. " $\Longrightarrow$ " Suppose $\mathcal{F}$ is normal and not locally bounded. By Proposition 3.3.4, there exists a compact set $K \subset G$ and $f \in \mathcal{F}$ such that $\sup\{|f(z)| : z \in K\} = \infty$. So we can find a sequence $\{f_n\} \subset \mathcal{F}$ such that $\sup\{|f_n(z)| : z \in K\} \geq n$. But $\mathcal{F}$ is normal, so there exist a subsequence $f_{n_k} \to f$ uniformly on any compact subsets. That is $\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$.

Since $f \in H(G)$ and $|f| \leq M$, $z \in K$ for some $M > 0$. But

$$\begin{aligned} n_k &\leq \sup\{|f_{n_k}(z)| : z \in K\} \\ &\leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + \sup\{|f(z)| : z \in K\} \\ &\to 0 + M \quad \text{as} \quad k \to \infty \end{aligned}$$

A contradiction.

" $\Longleftarrow$ " Suppose now that $\mathcal{F}$ is locally bounded. Then the set $\overline{\{f(z) : f \in \mathcal{F}\}}$ is clearly compact, and it remains to show $\mathcal{F}$ is equicontinuous at each point of G. Let $a \in G$ and $\epsilon > 0$ be given. Tt follows from the hypothesis that there exists an $M > 0$ and $r > 0$ such that for all $f \in \mathcal{F}$, $|f(z)| \leq M$ for $z \in \overline{B(a, r)}$. Now choose a z in $|z - a| < \dfrac{r}{2}$ $(z \in B(a, r/2))$. Put $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then we have, for $w \in \gamma$, $|w - z| \geq |w - a| - |a - z| > \dfrac{r}{2}$. An application of Cauchy's integral formula on γ gives

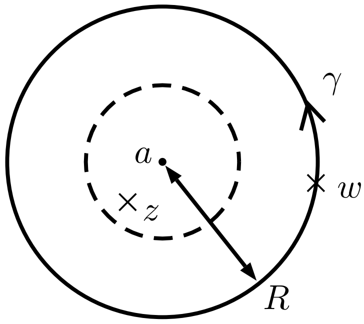

Figure 3.2: $z \in B(a, r/2)$

$$
\begin{aligned}
|f(z) - f(a)| &\le \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(z-a)}{(w-a)(w-z)}\,dw\right| \\
&\le \frac{1}{2\pi}2\pi\frac{M|z-a|}{|re^{it}|\frac{r}{2}}|ire^{it}| = \frac{2M}{r}|z-a| < \epsilon \quad \text{(independent of } f\text{)}
\end{aligned}
$$

provided we choose $\delta < \min\left\{\frac{r}{2}, \frac{r}{2M}\epsilon\right\}$. Hence given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for all $f \in \mathcal{F}$ and $z \in B(a, \delta)$. □

Corollary 3.3.5.1. *$\mathcal{F} \subset H(G)$ is compact if and only if $\mathcal{F}$ is closed and locally bounded.*

Example 3.3.6. Let S be the normalized class of one-to-one conformal mapping on the unit disk with Taylor's expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

It is well-known that

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad \text{for all } |z| < 1 \text{ and } f \in S.$$

Montel's theorem implies that S is a normal family.

Theorem 3.3.7 (Another theorem of Montel). *Let G be a region and $F \subset H(G)$. Suppose each $f \in \mathcal{F}$ omits same two fixed values $a, b \in \mathbb{C}$ in their range. Then $\mathcal{F}$ is normal.*

The above theorem is called as *Fundamental normality test.*

Remark (Bieberbach conjecture). $|a_n| \leq n$, for all $n \geq 2$ and $f \in S$. Proved by de Branges in 1984.

3.4 Riemann Mapping Theorem

Definition 3.4.1. Two regions G_1 and G_2 in $\mathbb{C}$ are said to be *conformally equivalent* if there exists an one-to-one analytic map f with $f(G_1) = G_2$.

We note that Louville's theorem implies that $\mathbb{C}$ is not equivalent to the unit disk Δ.

Theorem 3.4.2 (Riemann Mapping Theorem). *Let $G \subset \mathbb{C}$ be a simply connected region where its complement contains at least one point. Let $a \in G$. Then there is a unique one-to-one analytic mapping $f : G \to \mathbb{C}$ that satisfies $f(G) = \Delta = \{z : |z| < 1\}$ and $f(a) = 0$, $f'(a) > 0$.*

Suppose f and g are Riemann mappings for G_1 and G_2 respectively with $f(G_1) = \Delta$, $g(G_2) = \Delta$. Then $g^{-1} \circ f : G_1 \to G_2$ is an one-to-one analytic map such that $(g^{-1} \circ f)(G_1) = G_2$.

It is clear to see that conformally equivalent is an equivalence relation mapping all simply connected regions where their complements are non-empty.

Proof of Riemann Mapping Theorem. Let G be a region as assumed in the theorem. We shall divide the proof into five stages. Let $a \in G$, we define the family

$$\mathcal{F} = \{f \in H(G) : f \text{ one-to-one}, \ f(G) \subset \Delta, \ f(a) = 0, \ f'(a) > 0\}.$$

The theorem will be proved if we can find a $f \in \mathcal{F}$ such that $f(G) = \Delta$.

(A) ($\mathcal{F}$ *is non-empty*). Let $b \in \mathbb{C} \setminus G$ is non-empty by the hypothesis. Since G is simply connected, Theorem 1.10.13 asserts that we can find an analytic function g with

$$g(z) = \sqrt{z-b} = \exp\left(\frac{1}{2}\log(z-b)\right), \quad g(z)^2 = z-b.$$

It is easily observed that g is one-to-one analytic function.

Then the open mapping theorem (Theorem 1.11.4) asserts that there is a real number $r > 0$ with $B(g(a), r) \subset g(G)$. We next show $B(-g(a).r) \cap g(G) = \emptyset$. For suppose there exists a $z \in G$ with $g(z) \in B(-g(a), r)$, then

$$|g(z) - (-g(a))| < r.$$

This inequality can be written as

$$|-g(z) - g(a)| < r.$$

In other words, $-g(z) \in B(g(a), r)$. Hence there exists a $w \in G$ such that $g(w) = -g(z)$, squaring both sides yields $w - b = g(w)^2 = g(z)^2 = z - b$. So $w = z$, and $2g(z) = 0$. A contradiction. Hence $B(-g(a), r) \cap g(G) = \emptyset$.

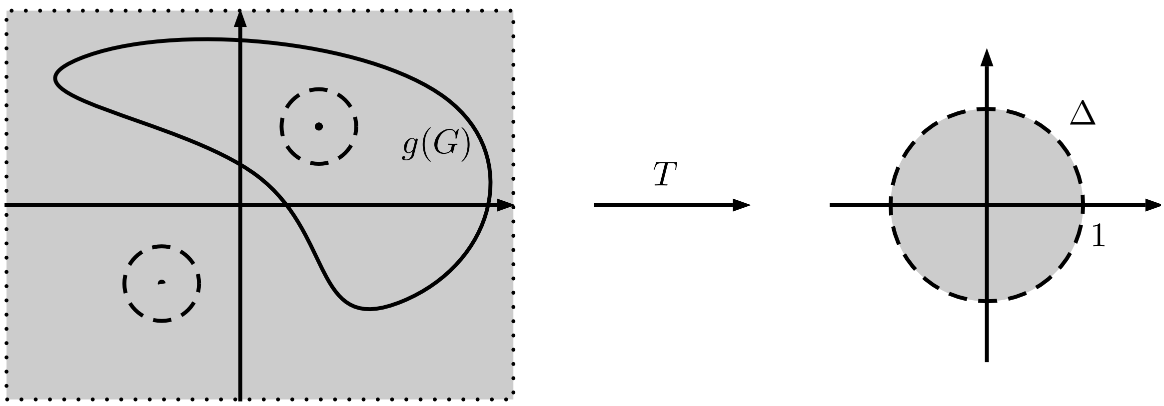

Figure 3.3: $T \circ g : G \to \Delta$

For any three points fixed on $\partial B(-g(a), r)$, we can always find a unique Möbius mapping $T(z) = \dfrac{az+b}{cz+d}(: \mathbb{C} \to \mathbb{C})$ such that $T(\partial B(-g(a), r)) = \partial\Delta$ and $T(\mathbb{C} \setminus \overline{B(-g(a), r)}) = \Delta$. Hence $T \circ g : G \to \Delta$. It remains to make $T \circ g$ a member of $\mathcal{F}$. But this is easy. Suppose $T \circ g(a) = \alpha$, then we define $\varphi_\alpha = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ which is an automorphism with $\varphi_\alpha(\alpha) = 0$. Hence $(\varphi_\alpha \circ T \circ g)(G) \subset \Delta$ with $(\varphi_\alpha \circ T \circ g)(a) = 0$.

Since each of φ_α, T and g is conformal, so is $\varphi_\alpha \circ T \circ g$. That is, $(\varphi_\alpha \circ T \circ g)'(z) \neq 0$ for all $z \in G$. We finally choose a suitable θ, so that $e^{i\theta}(\varphi_\alpha \circ T \circ g) \in \mathcal{F}$. Hence $\mathcal{F}$ is non-empty.

(B) ($\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$). Note that the zero function 0 is not conformal. Let $\{f_n\}$ be a sequence in $\mathcal{F}$. Suppose $f_n \to f$. We show either $f \in \mathcal{F}$ (not identically zero) or $f \equiv 0$. We first deduce that $f(a) = 0$ and $f'(a) \geq 0$ since the convergence is uniform on every compact subsets of G.

Let $z_1, z_2 \in G$. We choose an $r > 0$ so small that $z_1 \notin \overline{B(z_2, r)}$. Then $f_n(z) - f_n(z_1) \neq 0$ on $\overline{B(z_2, r)}$ since $f_n \in \mathcal{F}$ and so one-to-one. According to Corollary 3.3.2.1, we have

$$f_n(z) - f_n(z_1) \to f(z) - f(z_1) = \begin{cases} \neq 0, & \text{for all } z \in \overline{B(z_2, r)}; \\ \equiv 0, & \text{for all } z \in \overline{B(z_2, r)}. \end{cases}$$

If $f(z) \equiv f(z_1)$ for all $z \in \overline{B(z_2, r)}$, then $f(z) \equiv 0$ on G since $f(a) = 0$. If, however, $f(z) \neq f(z_1)$ for all $z \in \overline{B(z_2, r)}$, this means $f(z_2) \neq f(z_1)$ whenever $z_1 \neq z_2$. So f is one-to-one on G. But this implies $f'(z) \neq 0$ for each $z \in G$, and in particular $f'(a) > 0$. Hence $f \in \mathcal{F}$ as required.

(C) (*Existence of the largest* $f'(a) > 0$). Note that (C) and (D) below are related. Consider the mapping $H(G) \to \mathbb{C}$ given by $f \mapsto f'(a)$ (a is already fixed in G). By Theorem 3.3.1 the mapping $f \to$

$f'(a)$ is continuous. But $\mathcal{F}$ is locally bounded (since $|f| < 1$ for each $f \in \mathcal{F}$) and so normal. That is, $\overline{\mathcal{F}}$ is compact by Proposition 3.1.13. The image of $\overline{\mathcal{F}}$ under the above continuous mapping must also be compact in $\mathbb{C}$. Hence there exists a $f \in \overline{\mathcal{F}}$ such that $f'(a) \geq g'(a) > 0$ for all $g \in \mathcal{F}$. But $\mathcal{F} \neq \emptyset$ by (A) so there exists a non-constant $f \in F$ such that $f'(a) \geq g'(a) > 0$ for *all* $g \in \mathcal{F}$.

(D) (*The f found in (C) has $f(G) = \Delta$*). We suppose that there exists a $w \in \Delta$ such that $f(z) \neq w$ for all $z \in G$. Then the function

$$\frac{f - w}{1 - \overline{w} f} \neq 0$$

for all $z \in G$. We may define an analytic branch $h : G \to \mathbb{C}$ by

$$(h(z))^2 = \frac{f(z) - w}{1 - \overline{w} f(z)}.$$

Let

$$k(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}.$$

It is not difficult to observe that $h(G) \subset \Delta$ and $k(G) \subset \Delta$. We also have $k(a) = 0$ and $k'(z) \neq 0$. In fact, $k \in \mathcal{F}$ since

$$\begin{aligned} k'(a) &= \frac{|h'(a)|}{h'(a)} h'(a) \frac{1 - |h(a)|^2}{(1 - |h(a)|^2)^2} \\ &= \frac{|h'(a)|}{1 - |h(a)|^2} > 0. \end{aligned}$$

On the other hand, $|h(a)|^2 = \left| \dfrac{f(a) - w}{1 - \overline{w} f(a)} \right| = \left| \dfrac{0 - w}{1 - 0} \right| = |w|$.

Notice that

$$2h(z)h'(z) = \frac{d}{dz}(h(z))^2 = \frac{f'(z)(1 - |w|^2)}{(1 - \overline{w} f(z))^2}.$$

Thus

$$2h(a)h'(a) = f'(a)(1 - |w|^2).$$

Finally,

$$k'(a) = \frac{|h'(a)|}{1-|h(a)|^2} = \frac{\dfrac{f'(a|(1-|w|^2)}{2|h(a)|}}{1-|h(a)|^2}$$
$$= f'(a)\left(\frac{1+|w|}{2\sqrt{|w|}}\right)$$
$$> f'(a).$$

A contradiction. This completes the proof of (D).

(E) (*Uniqueness of f*). Suppose g also satisfies (A)-(D), then $f \circ g^{-1} : \Delta \to \Delta$ is an one-to-one, onto analytic map. Notice that $f \circ g^{-1}(0) = f(a) = 0$. So Theorem 2.9.7 shows that there is a constant $c = e^{i\theta}$ and $f \circ g^{-1}(z) = cz$ for all $z \in \Delta$. That is $f(z) = cg(z)$ for all $z \in G$ which gives $0 < f'(a) = cg'(a)$. But $g'(a) > 0$, so $c = 1$ and $f(z) = g(z)$.

Remark. The simply connectedness implies the existence of analytic square root function which is all we need to prove the conclusion.

□

Corollary 3.4.2.1. *Among the simply connected regions, there are only two equivalence classes; one consisting of $\mathbb{C}$ alone and the other containing proper simply connected regions.*