3.5 Boundary Correspondence of Conformal Mappings

Suppose f is a conformal mapping from the unit disc Δ to a simply connected domain D. We are concerned with under what circumstance that we could extend the f to the boundary $|z| = 1$.

Lemma 3.5.1. *Let $f : \Delta \to \mathbb{C}$ be continuous, $f(\Delta) = D$. Suppose $\lim_{z\to\xi} f(z)$ exists for every ξ with $|\xi| = 1$. Then the function $\tilde{f} : \Delta \to \overline{\mathbb{C}}$ defined by*

$$\tilde{f}(z) = \begin{cases} f(z), & |z| < 1, \\ \lim_{z\to\xi} f(z), & |\xi| = 1, \end{cases}$$

is the unique continuous extension of f to $|z| \le 1$. Moreover, $f(\bar{\Delta}) = \bar{D}$.

The lemma provides a way to define a possible meaning of a continuous extension of f to $|z| = 1$. Interested reader can consult Palka's book [7, Chap. XI] or Ahlfors' [1].

Definition 3.5.2. A plane domain/region G is *finitely connected along its boundary* if corresponding to each point z of ∂G and each $r > 0$, there exists an $s \in (0, r)$ such that $G \cap B(z, s)$ intersects at most finitely many components of the open set $G \cap B(z, r)$.

Theorem 3.5.3 (Väisälä & Näkki)**.** *Let $f : \Delta \to \mathbb{C}$ be conformal. The f can be extended to a continuous mapping $\tilde{f}$ of $\overline{\Delta}$ onto $\overline{f(\Delta)}$ if and only if $f(\Delta)$ is finitely connected along its boundary.*

Definition 3.5.4. A plane domain/region G is *locally connected along its boundary* if corresponding to each point z of ∂G and each $r > 0$, there exists an $s \in (0, r)$ such that $G \cap B(z, s)$ intersects exactly one component of $G \cap B(z, r)$.

Theorem 3.5.5. *Let $f : \Delta \to \mathbb{C}$ be conformal. Then f can be extended to a homeomorphism $\tilde{f}$ of $\overline{f(\Delta)}$ if and only if $f(\Delta)$ is locally connected along its boundary.*

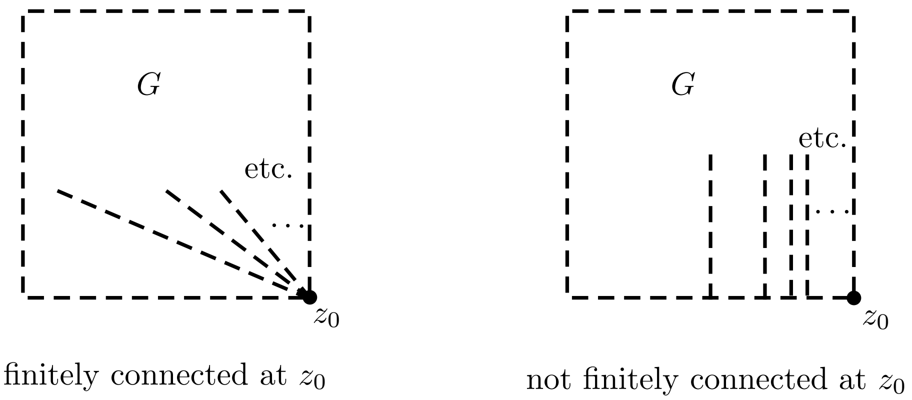

Figure 3.4: Finitely connectedness along different boundaries

Definition 3.5.6. A set J of points in $\mathbb{C}$ is called a *Jordan curve* if J is the boundary of some simple closed path. (J is compact and hence bounded.)

Theorem 3.5.7 (Jordan Curve Theorem, Jordan 1887)**.** *The complement of a Jordan curve J has exactly two components, each having J as its boundary. One of these components is a bounded set (the* inside *of J), while the other is unbounded (the* outside *of J).*

Definition 3.5.8. A domain/region $G \subset \mathbb{C}$ with the property that ∂G is a Jordan curve is called a *Jordan domain*.

Theorem 3.5.9 (Caratheodory-Osgood Theorem)**.** *A conformal mapping f of Δ onto a domain D can be extended to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Delta}$ if and only if D is a Jordan domain.*

3.6 Space of Meromorphic Functions

Definition 3.6.1. Let $M(G) \subset C(G, \widehat{\mathbb{C}})$ denote the space of meromorphic functions on the region G.

Theorem 3.6.2. *Let $\{f_n\} \subset M(G)$, $f_n \to f$ in $C(G, \widehat{\mathbb{C}})$. Then either f is meromorphic or $f \equiv \infty$. If each $\{f_n\}$ is analytic or $f \equiv \infty$.*

Corollary 3.6.2.1. *$M(G) \cup \{\infty\}$ is a complete metric space. (w.r.t. spherical metric)*

Corollary 3.6.2.2. *$H(G) \cup \{\infty\}$ is closed in $C(G, \widehat{\mathbb{C}})$.*

Example 3.6.3. $f_n(z) = n(z^2 - n)$ is analytic on $\mathbb{C}$ for each n. The $f_n \to \infty$ uniformly on each compact subset of $\mathbb{C}$. While $\{f_n'(z)\} = \{2nz\}$ is not a normal family, since $f_n'(0) = 0$ and $f_n'(z) \to \infty$ for $z \neq 0$. So $\mathcal{F}$ is normal $\not\Longrightarrow$ $\mathbb{F}'$ is normal.

Definition 3.6.4. $\rho(f)(z) = \dfrac{2|f'(z)|}{1 + |f(z)|^2}$ is called the *spherical derivative of f*. It is defined even at the poles of f.

Recall that the chordal distance under the stereographic projection is given by

$$\begin{aligned} d(f(z_1), f(z_2)) &= \frac{2|f(z_1) - f(z_2)|}{\sqrt{(1 + |f(z_1)|^2)(1 + |f(z_2)|^2)}} \\ &\sim \frac{2|f'(z_1)|dz}{1 + |f(z_1)|^2} \quad \text{as } z_2 \to z_1. \end{aligned}$$

Let γ be the curve in $\mathbb{C}$. The length of $f(\gamma)$ under the stereographic projection on the Riemann sphere is given by

$$\int_\gamma \rho(f)(z)\,|dz|.$$

Theorem 3.6.5. *$\mathcal{F} \subset M(G)$ is normal in $C(G, \widehat{\mathbb{C}})$ if and only if $\rho(f)(z)$ is locally bounded on $\mathcal{F}$.*

3.7 Schwarz's reflection principle

Let $G \subset \mathbb{C}$ be a region, and $\bar{G} = \{\bar{z} : z \in G\}$. Clearly if a region G is symmetrical with respect to $\mathbb{R}$, then $\bar{G} = G$.

Theorem 3.7.1. *Suppose $\bar{G} = G$. We denote $G_+ = \{z \in G : \Im z > 0\}$, $G_- = \{z \in G : \Im z < 0\}$ and $G_0 = G \cap \mathbb{R}$. Suppose $f : G_+ \cup G_0 \to \mathbb{C}$ is continuous, analytic on G_+ such that f is* real *on G_0. Then*

$$g(z) := \begin{cases} f(z) & z \in G_+ \\ \overline{f(\bar{z})} & z \in G_0 \cup G_- \end{cases} \tag{3.4}$$

is analytic on G.

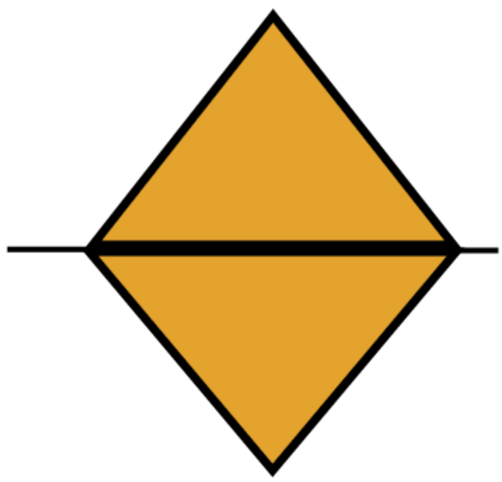

Figure 3.5: Schwarz's relfection along the $\mathbb{R}$

Remark. We note that if f is only defined on G_+ and continuous and real on G_0, then we can use the above g to extend f across to G_- by reflection. By the identity theorem applied to $\mathbb{R}$, so that such an extension is unique.

Proof. It is clear that g is analytic on G_+ and G_-. It remains to consider if g is analytic on G_0. That is, if g is analytic in a neighbourhood $B(x_0,,r)$, where x_0 real and for every $x_0 \in G_0$ and a corresponding $r > 0$. We could achieve this by proving for each triangle T within $B(x_0, r)$ the integral $\int_T g\, dz = 0$. Then g is analytic in $B(x_0, r)$ by Morera's theorem. Thus, if the triangle T lies entirely in G_+ with no intersection with G_0, then $\int_T f = 0$ since f is analytic there. Similarly if T lies entirely in G_-. So we assume that $T \cap G_0 \neq \emptyset$.

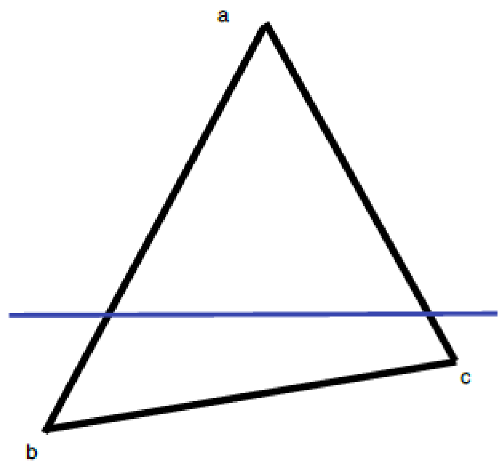

Figure 3.6: One triangle and one quadrilaterial

In general, either $T \cap G_0$ is a single point or it is a line segment. The former consideration obviously gives $\int_T f = \int_T g = 0$. The latter means that the G_0 deivdes the T into two pieces. Without loss of generality, we may assume that $G_+ \cup G_0$ contains the triangle $T' = [a, b, c, a]$ part of T and $[a, b]$ lies on G_0, leaving the quadrilateral part in $G_- \cup G_0$.

Notice that $g = f$ is uniformly continues on T' since T' is a compact set. That is, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $z, z' \in T'$, and $|z - z'| < \delta$, then

$$|f(z) - f(z')| < \varepsilon.$$

We construct a sub-triangle $T'' = [\alpha, \beta, c, \alpha]$ of T' such that one of

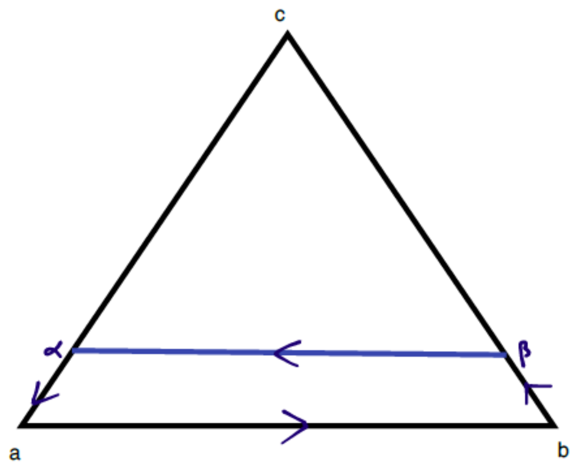

Figure 3.7: Integration along the quadrilaterial

the sides $[\alpha, \beta]$ is parallel and close to $[a, b]$ and hence to the $\mathbb{R}$. We may parametrise the horizontal line segments $[a, b]$ and $[\beta, \alpha]$ by

$$(1-t)ta + tb, \quad (1-t)\alpha + t\beta, \qquad (0 \le t \le 1).$$

So now with the given $\varepsilon > 0$, we choose $\delta > 0$ so that

$$|\alpha - a| < \delta, \quad |\beta - b| < \delta \qquad (0 \le t \le 1),$$

hence

$$\begin{aligned} |(1-t)\alpha + t\beta - ((1-t)ta + tb)| &\le (1-t)|\alpha - a| + t|\beta - b| \\ &\le \delta(1 - t + t) \\ &= \delta. \end{aligned}$$

This implies

$$|f[(1-t)\alpha + t\beta] - f[(1-t)ta + tb)]| < \varepsilon, \qquad (0 \le t \le 1).$$

Thus

$$\begin{aligned}
&\Big|\int_{[a,b]} f - \int_{[\alpha,\beta]}\Big| \\
&\qquad = \Big|(b-a)\int_0^1 f[(1-t)ta+tb) - (\beta-\alpha)\int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\qquad \le |b-a|\Big|\int_0^1 f[(1-t)ta+tb) - (\beta-\alpha)\int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\qquad\quad + |(b-a)-(\beta-\alpha)|\Big|\int_0^1 f[(1-t)\alpha+t\beta]\,dt\Big| \\
&\qquad \le |b-a|\,\varepsilon + |(b-a)-(\beta-\alpha)|\,M \\
&\qquad \le \varepsilon\ell(T') + |(b-a)-(\beta-\alpha)|\,M \\
&\qquad \le \varepsilon\ell(T') + 2\delta\,M
\end{aligned}$$

where $\ell(T')$ stands for the length of the parameter of T', and $M = \max\{|f(z)| : z \in T'\}$. The estimates of the remaining integrals are easy:

$$\Big|\int_{[a,\alpha]} f\Big| \le |\alpha - a|M \le M\delta, \quad \Big|\int_{[b,\beta]} f\Big| \le |\beta - b|M \le M\delta.$$

We finally deduce

$$\begin{aligned}
\Big|\int_T f\Big| &= \Big|\int_{T'} f + \int_{[a,b,\beta,\alpha,a]} f\Big| \\
&= \Big|\int_{[a,b,\beta,\alpha,a]} f\Big| \\
&= \Big|\int_{[a,b]} f - \int_{[\alpha,\beta]}\Big| + \Big|\int_{[a,\alpha]} f\Big| + \Big|\int_{[b,\beta]} f\Big| \\
&\le \varepsilon\,\ell(T') + 4\delta\,M \\
&\le \varepsilon\,(\ell(T') + 4\,M)
\end{aligned}$$

since we may choose $\delta < \varepsilon$. This shows that $\int_{T'} f = 0$. We conclude that f is analytic in $B(x_0, r)$. Hence g is analytic on G. □

The above is called Schwarz's[1] reflection principle. We can map the above upper half-plane onto a circle and the real-axis $\mathbb{R}$ to $|z-a| = r$.

---

[1] H. A. Schwarz (1843-1921): advisor Karl Weierstrass

Theorem 3.7.2 (Schwarz reflection principle: second version)**.** *Let G_1 denote a simply-connected domain interior to $C_a := \{z : |z-a| = r\}$ with an arc γ on C_a such that every point of* $\text{int}(\gamma)$ *has a semi-circular neighbourhood in $B(a, r) \cap \gamma$. Let $f : G_1 \to \mathbb{C}$ be analytic and continuous on $G_1 \cup \gamma$. Suppose $f(\gamma) = \Gamma$ consists of an arc of the circle $C_b := \{w : |w-b| = R\}$. Then we can extend f to the region G_2, obtained by reflecting G_1 with respect to C_a, mapping every $z \in G_1$ to*

$$z^* = a + \frac{r^2}{\bar{z} - \bar{a}}$$

being the symmetric (inverse) point of z in G_1, and

$$f(z^*) = b + \frac{R^2}{\overline{f(z)} - \bar{b}},$$

in G_2 so that the new function is analytic in $G = G_1 \cup \gamma \cup G_2$.

Proof. Let $z \in G_1$. Then we recall that the symmetric point z^* with respect to the circle C_a is given by

$$z^* = a + \frac{r}{\bar{z} - \bar{a}}.$$

Let M_{C_a} be the Möbius transformation that maps the circle onto $\mathbb{R}$ with the notaton $z \mapsto Z$. We also denote the inverse point of $w = f(z)$ with respect to the circle$|w-b| = R$ to be

$$w^* = b + \frac{R}{\overline{f(z)} - \bar{b}}.$$

We also denote the Möbius transformation that maps the circle

$|w-b| = R$ onto $\mathbb{R}$ by M_{C_b} with the notaton $w \mapsto W$. Then we have

$$\begin{aligned} f(z^*) &= f \circ M_{C_a}(Z^*) \\ &= F(Z^*) = F(\bar{Z}) \\ &= \overline{F(\bar{Z})} \\ &= \overline{W} \ (= W^*) \\ &= M_{C_b}(w^*) \\ &= M_{C_b}(f(z)^*) \\ &= b + \frac{R^2}{\overline{f(z)} - \bar{b}}, \end{aligned}$$

where $F = f \circ M_{C_a}$. □

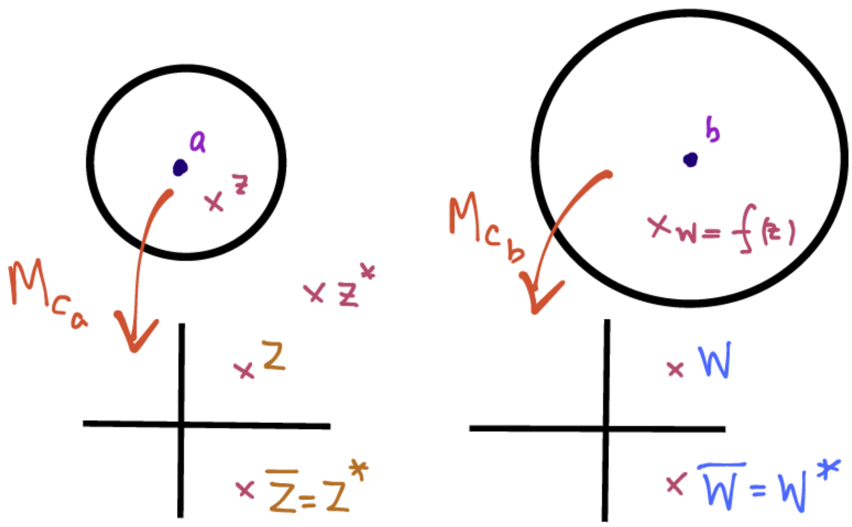

Figure 3.8: Schwarz reflection with respect to circles

One can achieve a more general reflection below.

Theorem 3.7.3. *Let G_1 and G_2 be two simply-connected domains such that*

1. $G_1 \cap G_2 = \emptyset$;

2. *$\bar{G}_1 \cap \bar{G}_2 = \gamma$ where γ is a smooth curve such that every interior point int(γ) of γ has a neighbourhood lying entirely inside $G := G_1 \cup int(\gamma) \cup G_2$.*

Let $f_j(z)$ be analytic in G_j, continuous in $G_j \cup \gamma$, $j = 1, 2$ such that for every point $\xi \in \gamma$

$$\lim_{D_1 \ni z \to \xi} f_1(z) = h(\xi) = \lim_{D_2 \ni z \to \xi} f_2(z)$$

for some complex-valued function $h : \gamma \to \mathbb{C}$. Then there exists an analytic function f in G such that $f(z) = f_j(z)$ for each $z \in G_j$, $j = 1, 2$.

3.8 Schwarz-Christoffel formulae

The Riemann mapping theorem that we discussed is an existence result. It is rather difficult to construct explicit formulae that actually realise the theorem for reasonable shape simple-connected regiona given simply connected can be approximated by polygons, so it becomes of interest to find explicit formulae for conformal of polygons.

Theorem 3.8.1 (Schwarz (1869), Christoffel (1867))**.** *Let f be a one-one conformal mapping that maps the upper half-plane $\mathbb{H}^+$ onto the interior of the a polygon $D = [w_1, w_2, \cdots w_n]$ with the interior angles*

$$0 < \alpha_k \pi := (1 - \nu_k)\pi < 2\pi,$$

at each of the vertex w_k $k = 1, \cdots n$. Suppose $-\infty < a_1 < a_2 < \cdots < a_n < \infty$ are real numbers on $\mathbb{R}$ such that $f(a_k) = w_k$, $k = 1, \cdots n$. Then f is given by

$$\begin{aligned} f(z) &= \alpha \int_0^z \frac{dz}{(z-a_1)^{1-\alpha_1}(z-a_2)^{1-\alpha_2}\cdots(z-a_n)^{1-\alpha_n}} + \beta \\ &= \alpha \int_0^z \frac{dz}{(z-a_1)^{\mu_1}(z-a_2)^{\mu_2}\cdots(z-a_n)^{\mu_n}} + \beta \end{aligned} \tag{3.5}$$

where α, β are two integration constants, where the ν_k, $\quad k = 1, \cdots, n$ are the corresponding exterior angles.

We recall that from elementary geometry that if the above polygon D is convex, that is, $0 < \nu_k < 1$, then

$$\sum_{k=1}^{n} \nu\pi_k = 2\pi.$$

Proof. Since the boundary of the proposed polygon D is certainly a Jordan curve, we immediately deduce from Theorem 3.5.9 that there is a conformal mapping f from the upper half-plane $\mathbb{H}^+$ onto the D such that f can be extended continuously to the real-axis $\mathbb{R}$ and $f(\mathbb{R}) = \partial D$. Let us label

$$f(a_k) = w_k, \quad k = 1, \cdots, n$$

$w_{n+1} = w_1$ that are the vertices of the polygon D. Let us denote $f(a_k, a_{k+1}) = L_k$, $k = 1, \cdots n$. Then we can apply Schwarz's reflection principle (Theorem 3.7.2) to a chosen $\mathbb{H}^+ \cup (a_k, a_{k+1})$ for some $k \in \{1, \cdots, n\}$ and reflect along (a_k, a_{k+1}) to continue f to the lower half-plane $\mathbb{H}^-$. But this corresponds to a reflection image D' obtained from D after a reflection of D along its side L_k. In fact, the $D' = f(\mathbb{H}^-)$. where we have reused the notation for the extension of f onto the domain $\mathbb{H}^+ \cup (a_k, a_{k+1}) \cup \mathbb{H}^-$. But the Riemann mapping theorem again asserts that there is a one-one conformal mapping $\hat{f}$ that maps $\mathbb{H}^-$ onto D'. So we may apply the Schwarz reflection principle (Theorem 3.7.2) again to reflect $\mathbb{H}^-$ along one of the other intervals (a_{k+1}, a_{k+2})[2] say, to the upper half-plane $\mathbb{H}^+$. This again corresponds to the reflection of D' along its side L_{k+1} to a symmetrical region. The resulting image, which we denote by D'' is of identical shape as D where we started off, but located in a different position. The Riemann mapping theorem again implies that there is a $\tilde{f}$ that maps the upper half-plane $\mathbb{H}^+$ onto the D''. Since we can superimpose the$D$ to D'' by a translation and a rotation, so we have

$$\tilde{f}(z) = Af(z) + B \tag{3.6}$$

in $\mathbb{H}^+$ for some constants A, B[3].

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[2]Any other side will do.

[3]In fact, $A = e^{i\theta_k}$.

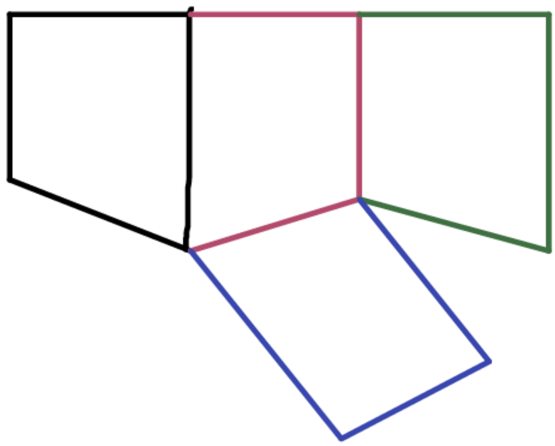

Figure 3.9: Even number of reflections

We deduce

$$\tilde{f}'(z) = Af'(z) \neq 0$$

throughout the $\mathbb{H}^+$ since f is conformal there. Moreover,

$$g(z) := \frac{\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{f''(z)}{f'(z)} \tag{3.7}$$

in $\mathbb{H}^+$. This shows that the function g is analytic in $\mathbb{H}^+$. A similar consideration leads to a similar conclusion that g is analytic in $\mathbb{H}^-$, and hence on

$$\mathbb{H}^+ \cup_{k=1}^n (a_k,\, a_{k+1}) \cup \mathbb{H}^-$$

by the Schwarz reflection principle. Hence g is analytic on $\mathbb{C}$ except perhaps at a_k, $k = 1, \cdots n$. Let us investigate what happens at these a_k. Let us consider the behaviour of f when z changing from the line segment $(a_{k-1},\, a_k)$ to $(a_k,\, a_{k+1})$. We have

$$f(z) = f(a_k) + (z - a_k)^{\alpha_k} h(z)$$

where h is analytic in a neighbourhood at $z = a_k$ and $h(a_k) \neq 0$ (imagine that z lies on a line segment slight above the $\mathbb{R}$. Thus $f(z) -$

$f(a_k)$ changes an angle $\alpha_k\pi$ from L_{k-1} to L_k when z "passes through" a_k.

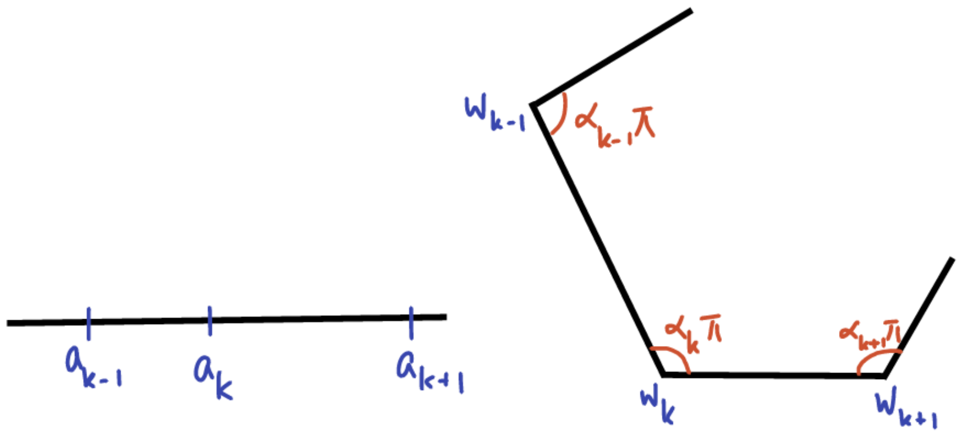

Figure 3.10: "Opening" an angle

Hence

$$\begin{aligned} f'(z) &= \alpha_k(z-a_k)^{\alpha_k-1}h(z) + (z-a_k)^{\alpha_k}h'(z) \\ &= (z-a_k)^{\alpha_k-1}\Big[\alpha_k h(z) + (z-a_k)h'(z)\Big] \\ &:= (z-a_k)^{\alpha_k-1}\phi(z), \end{aligned} \tag{3.8}$$

where $\phi(z)$ is analytic at a_k and $\phi(a_k)\neq 0$. Thus,

$$\frac{f''(z)}{f'(z)} = \frac{\alpha_k-1}{z-a_k} + \frac{\phi'(z)}{\phi(z)}.$$

This shows that the function g defined above is analytic in $\mathbb{C}$ except at the a_k, $k=1,\cdots,n$ where it has a residue α_k-1 at each simple pole a_k. Thus the function

$$\frac{f''(z)}{f'(z)} - \sum_{k=1}^{n}\frac{\alpha_k-1}{z-a_k}$$

is an entire function, and in fact including $z = \infty$. To see this we note that f and its analytic continuation are bounded at ∞, that is, we have the Laurent expansion at ∞:

$$f(z) = f(\infty) + O\Big(\frac{1}{z^m}\Big), \quad z \to \infty$$

for some integer $m \geq 1$. We deduce that g has a simple pole at ∞. This shows that

$$\frac{f''(z)}{f'(z)} - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k} \equiv 0$$

by Liouville's theorem. The above formula implies that

$$f'(z) = \alpha \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1}.$$

integrating the above formula from 0 to z yields the desired formula.

□

Remark. We can continue the above reflection along one $(a_k,\, a_{k+1})$ from $\mathbb{H}^+$ to $\mathbb{H}^-$ and then from $\mathbb{H}^-$ to $\mathbb{H}^+$ via another interval $(a_j,\, a_{j+1})$ any number of times for different k and j in the above construction. The upshoot is that evey time we complete a cycle we end up with a different function valued at the same point in the upper half-plane and similarly in the lower half-plane. This suggests that we should consider that these different values from different "reflected values" to be different branches of an analytic function $w = F(z)$ defined on $\mathbb{C}\setminus \cup_{k=1}^{n} (a_k,\, a_{k+1})$. The above proof shows that the $g = f''/f'(z)$ so constructed is independent of the branches chosen. In fact, we have shown that it is globally defined in $\hat{\mathbb{C}}$.

Remark. The reader may notice that we did not discuss the actual locations of the real numbers $-\infty < a_1 < a_2 < \cdots < a_n < \infty$ and the constants α, β in the Schwarz-Christoffel formula above. This turns out to be a difficult unsolved problems. However, we can still prescribe a_1, a_2, a_n to w_1, w_2, w_n say after a suitably chosen Möbius transformation. However, given a polygon with more than three vertices, it becomes a non-trivial problem to determine the other points $a_4, \cdots, a_n$

on the real axis. This is partly due to the fact that the Schwarz-Christoffel formula only precribed the angles α_k, but not the length of $(a_k,\, a_{k+1})$ (recall that conformal map does not preserve lenghts in general). The remaining unknowns are $a_4, \cdots, a_n$ real numbers and two complex numbers α and β. We deduce from the formula (3.5) that when $z = x > a_n$, then

$$\arg f'(x) = \arg \alpha,$$

and the line segment $(a_n,\, a_1)$ (via $\hat{\mathbb{C}}$) corresponds to the side $L_n = [w_n,\, a_1]$ of the polygon D. But $\arg f'(x) = \arg \alpha$ corresponds to the angle that L_n makes with the real-axis $\mathbb{R}$. This shows that $\arg \alpha$ is known. On the other hand, putting $z = a_1$ in (3.5) yields $f(a_1) = \beta$. This implies $\beta = w_1$ is therefore also known. We are left with $n - 2$ real unknown constants

$$a_4, \cdots, a_n,\ |\alpha|$$

to be determined. On the other hand, we have a further $n-2$ equations

$$\ell([w_k,\, w_{k+1}]) = |\alpha| \int_{a_k}^{a_{k+1}} \Big| \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} \Big| \, |dz|$$

$k = 4, \cdots, n$ (with $a_{n+1} = a_n$) that can be used to compute the $a_4, \cdots, a_n,\ |\alpha|$. But it is generally difficult in not impossible.

Example 3.8.2. Find a conformal mapping from the upper half-plane onto an equilateral triangle of side lenght ℓ.

That is the three angels of the triangle are all equal to $\alpha_k \pi = \pi/3,\ k = 1,\, 2,\, 3$. According to the last remark, the Schwarz-Christoffel formula completely determine the $a_j,\, w_j = f(a_j),\ k = 1,\, 2,\, 3$. So let us choose

$$a_1 = -1,\, a_2 = 0,\, a_3 = 1.$$

Then the SC-formula (3.5) yields

$$w = f(z) = \alpha \int_0^z \frac{dt}{(t - (-1))^{1-1/3} t^{1-1/3} (t-1)^{1-1/3}} + \beta,$$

Without loss of generality, we may choose $f(a_2) = f(0) = 0$. Hence $\beta = 0$. Moreover, we have

$$\ell = \left|\alpha \int_0^1 \frac{dt}{\sqrt[3]{t^2(t^2-1)}}\right|,$$

implying that

$$\alpha = \frac{\ell}{\displaystyle\int_0^1 \frac{dt}{\sqrt[3]{t^2(1-t^2)}}}.$$

Hence

$$f(z) = \ell \frac{\displaystyle\int_0^z \frac{dt}{\sqrt[3]{t^2(t^2-1)}}}{\displaystyle\int_0^1 \frac{dt}{\sqrt[3]{t^2(1-t^2)}}}$$

is the desired mapping.

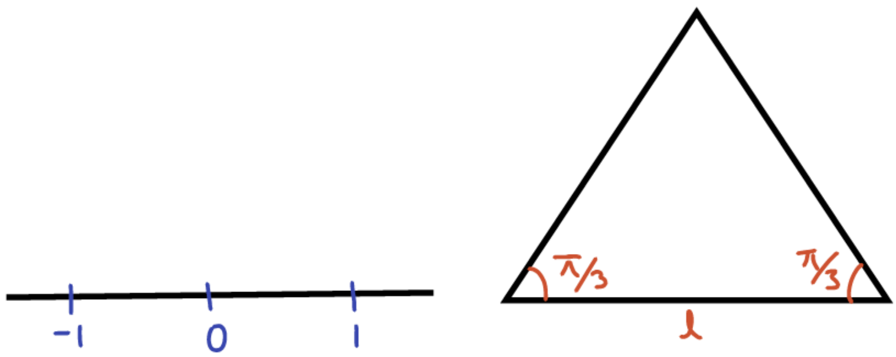

Figure 3.11: Schwarz equaliterial triangle

Exercise 3.8.1. Replace the above equilateral triangle with an isosceles right trangle with $\alpha_2 = \frac{1}{2}$, $\alpha_1 = \alpha_3 = \frac{1}{4}$, with the length of the hypotenuse ℓ.

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