Example 3.8.3. Construct a one-one conformal map from the upper half-plane $\mathbb{H}^+$ to a rectangle with coordinates $[-K,\ K,\ K+iK',\ -K+iK']$ for some $K>0$.

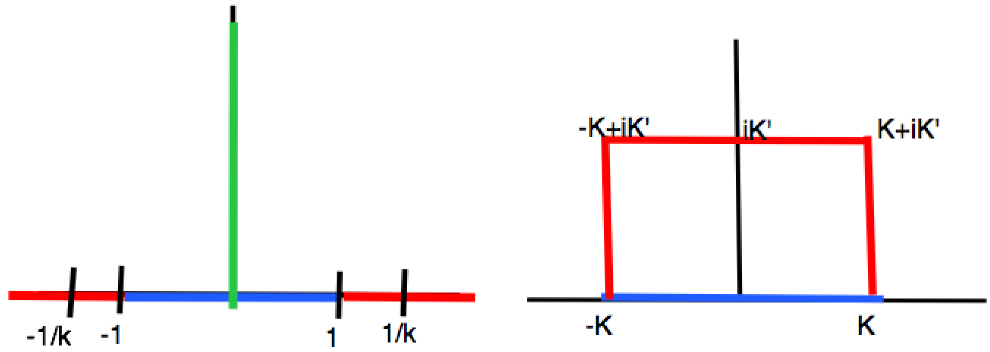

Figure 3.12: Elliptic function of the 1st kind

We recall that a slight variation of Riemann mapping theorem allows us to assert tht there is a one-one conformal mapping from the first quadrant of the $z-$plane to the rectangle with vertices $[0,\ K,\ K+iK',\ iK']$ such that the points 0, 1 and ∞ in the $z-$plane are mapped onto the points $0,\ K,\ iK$ respectively. So we have the following correspondences:

$$[0,\ 1]\mapsto[0,\ K],\quad [1,\ \infty)\mapsto[K,\ K+iK']\cup[K+iK',\ iK'].$$

So there is a $0<k<1$ so that the point $z=1/k>1$ is mapped onto the point $K+iK'$. This also implies that the positive imaginary axis $\{z=iy:y>0\}$ is being mapped onto the line segment $[0,\ iK']$.

So we obtain the desired mapping $\mathbb{H}^+\to[-K,\ K,\ K+iK',\ -K+iK']$ after reflecting the Riemann mapping obtained above with respect to the imaginary axis, so that the real-axis $\mathbb{R}$ is mapped onto $[-K,\ K,\ K+iK',\ -K+iK']$, and the points $-1/k, -1,\ 1,\ 1/k$ are mapped onto the points $-K+iK',\ -K,\ K,\ K+iK'$ respectively. The explicit formula is therefore given by

$$\begin{aligned} f(z) &= \alpha \int_0^z \Big(z+\frac{1}{k}\Big)^{\frac{1}{2}-1}(z-1)^{\frac{1}{2}-1}(z+1)^{\frac{1}{2}-1}\Big(z-\frac{1}{k}\Big)^{\frac{1}{2}-1} + \beta \\ &= \alpha' \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + \beta \end{aligned}$$

Let $z = 0$ in the variable above. Then clearly $\beta = 0$. We choose the branch of square root above in accord to positive value when z lies in $(0, 1)$. But $f(1) = K$. So

$$K = \alpha' \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

This allows us to determine the constant $\alpha' > 0$ provided we know the value of k. Moreover, since $f(\frac{1}{k}) = K + iK'$, so

$$\begin{aligned} K + iK' &= \alpha' \int_0^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &= \alpha' \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &\quad + \alpha' i \int_1^{1/k} \frac{dz}{\sqrt{(z^2-1)(1-k^2z^2)}} \end{aligned}$$

since there is a change of $\arg(1-z)$, amongst all the factors of $(1-z^2)(1-k^2z^2)$, by $-\pi$. It follows that

$$K' = \alpha' \int_1^{1/k} \frac{dz}{\sqrt{(z^2-1)(1-k^2z^2)}}.$$

Let

$$z = \frac{1}{\sqrt{1-k'^2t^2}}$$

in the above integration, where ${k'}^2 = 1 - k^2$ and $0 < k' < 1$. It is routine to check that the above substitution yields

$$K' = \alpha' \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

We therefore deduce the relationship:

$$\frac{K'}{2K} = \frac{\displaystyle\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}}{\displaystyle 2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}}. \tag{3.9}$$

We see that both the numerator and denominator have similar integrands. As k increases from 0 to 1, the integral

$$\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}$$

increases from

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2} \quad \text{to} \quad \int_0^1 \frac{dz}{1-z^2} = +\infty.$$

That is the interval $(0, 1)$ is being mapped onto $[\frac{\pi}{2}, +\infty)$. While k increases from 0 to 1, its complementary value k' decreases from 1 to 0. So the numerator

$$2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

behaves in a similar behaviour but in the opposite direction, namely, it decreases monotonically from $+\infty$ to π. We deduce that the ratio $K'/2K$, increases monotonically, as a function of k, from 0 to $+\infty$. So there is a unique $0 < k < 1$ such that (3.9) holds for a given K and K'. This allows us to compute an approximate (and hopefully to know exactly) value of k, and hence α'.

Definition 3.8.4. The above integral where $\alpha' = 1$,

$$K(k) = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

is called the (Legendre form) of **complete elliptic integral of the first kind**.

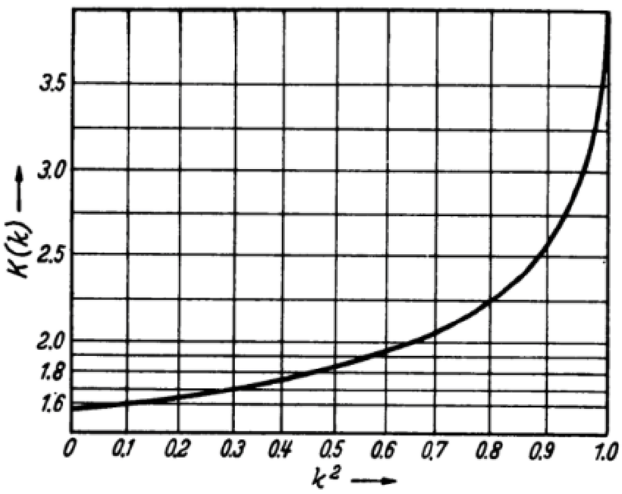

Figure 3.13: Modulus of an elliptic integral: Byrd and Friedman

Theorem 3.8.5 (Schwarz-Christoffel: second version)**.** *Let f be a one-one conformal mapping that maps the upper half-plane $\mathbb{H}^+$ onto the interior of the a polygon $D = [w_1, w_2, \cdots w_n]$ with the interior angles*

$$0 < \alpha_k \pi := (1 - \nu_k)\pi < 2\pi,$$

at each of the given vertex w_k, $k = 1, \cdots n$. Suppose the corresponding points $-\infty < a_1 < a_2 < \cdots < a_{n-1} < \infty$ are real numbers on $\mathbb{R}$ such that $f(a_k) = w_k$, $k = 1, \cdots n-1$, and $a_n = \infty$, $f(\infty) = w_n$. Then f is given by

$$f(z) = \alpha \int_0^z \frac{dz}{(z-a_1)^{1-\alpha_1}(z-a_2)^{1-\alpha_2}\cdots(z-a_{n-1})^{1-\alpha_{n-1}}} + \beta \quad (3.10)$$

where α, β are two integration constants.

Proof. The transformation

$$z = a - \frac{1}{\zeta} \qquad (\text{i.e., } \zeta = -1/(z-a)), \quad a < a_1$$

transforms the upper half-plane $\mathbb{H}^+$ onto itself such that the $a_1 < \cdots < a_{n-1}$ are mapped onto $b_1 < \cdots < b_{n-1}$ and $a_n = \infty$ to $b_n = 0$. Hence we may apply (3.5) to

$$F(\zeta) = f\Big(a - \frac{1}{\zeta}\Big)$$

and this yields

$$\begin{aligned} F(\zeta) &= \alpha' \int_0^\zeta \frac{d\zeta}{(\zeta - b_1)^{1-\alpha_1} \cdots (\zeta - b_{n-1})^{1-\alpha_{n-1}} \zeta^{\alpha_n - 1}} + \beta' \\ &= \alpha' \int_0^\zeta \prod_{k=1}^{n-1} (\zeta - b_k)^{\alpha_k - 1} \zeta^{\alpha_n - 1} \, d\zeta + \beta'. \end{aligned}$$

Hence

$$\begin{aligned} f(z) = F(\zeta) &= \alpha' \int_{z_0}^z \prod_{k=1}^{n-1} \Big(\frac{-1}{z-a} + \frac{1}{a_k - a}\Big)^{\alpha_k - 1} \Big(\frac{-1}{z-a}\Big)^{\alpha_n - 1} \zeta^2 \, dz + \beta' \\ &= \alpha' \int_{z_0}^z \prod_{k=1}^{n-1} \Big(\frac{a_k - z}{(z-a)(a_k - a)}\Big)^{\alpha_k - 1} \Big(\frac{-1}{z-a}\Big)^{\alpha_n - 1} \frac{dz}{(z-a)^2} + \beta' \\ &= \alpha'' \int_{z_0}^z \prod_{k=1}^{n-1} (z - a_k)^{\alpha_k - 1} \frac{1}{(z-a)^{\sum \alpha_k - n + 2}} \, dz + \beta' \\ &= \alpha'' \int_0^z \prod_{k=1}^{n-1} (z - a_k)^{\alpha_k - 1} \, dz + \beta'' \end{aligned}$$

since $\sum_{k=1}^n \alpha_k = n - 2$. □

Example 3.8.6. Let us apply the above formula to obtain an equilateral triangle of side length ℓ. That is, we may assume the three points on the real axis are

$$a_1 = 0, \; a_2 = 1, \; a_3 = \infty.$$

Then $\alpha_k = \pi/3$, $k = 1, 2, 3$. The formula (3.10) yields

$$f(z) = \alpha \int_0^z \frac{dz}{(z-1)^{\frac{2}{3}} \, z^{\frac{2}{3}}} + \beta.$$

The side length ℓ can be expressed as integration of arc-lenght:

$$\begin{aligned}\ell &= |\alpha|\int_0^1 |f'(z)|\,|dz| \\ &= |\alpha|\int_0^1 |z^{\frac{1}{3}-1}(z-1)^{\frac{1}{3}-1}||dz| \\ &= |\alpha|\int_0^1 t^{\frac{1}{3}-1}(t-1)^{\frac{1}{3}-1}\,dt \\ &= |\alpha|\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3}+\frac{1}{3})} = |\alpha|\frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})},\end{aligned}$$

where $\Gamma(z)$ denotes the Euler-Gamma function (see later) and it is known that

$$B(\alpha,\,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

provided that $\Re\alpha > -1$ and $\Re\beta > -1$.

Example 3.8.7. In general, if we consider the image of 0, 1, ∞ to be the general triangle ABC with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ with side lengths a, b, c respectively, then we have the Schwarz-Christoffel map to be

$$f(z) = \int_0^a z^{\alpha-1}(1-z)^{\beta-1}\,dz$$

where we have chosen $C_1 = 1$ and C_2 so that $f(0) = 0$. Then we can compute the side length of, say,

$$c = \int_0^1 |f'(z)|\,|dz| = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

But since $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$, so

$$c = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1-\gamma)} = \frac{1}{\pi}\sin(\gamma\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)$$

since $\alpha+\beta+\gamma = 1$. Similarly, the side lengths of the other two sides are given by

$$a = \frac{1}{\pi}\sin(\alpha\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)$$

and

$$b = \frac{1}{\pi}\sin(\beta\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma).$$

Example 3.8.8. Apply a SC-formula to show that the conformal mapping f that maps $\mathbb{H}^+$ onto the half vertical strip:

$$-\frac{\pi}{2} < \Re(w = f(z)) < \frac{\pi}{2}; \quad \Im(w) > 0.$$

such that $-1 \mapsto -\frac{\pi}{2}$, $1 \mapsto \frac{\pi}{2}$, $\infty \mapsto \infty$ is given by

$$f(z) = \int_0^z \frac{dz}{\sqrt{1-z^2}} = \sin^{-1} z.$$

Exercise 3.8.2. Show that the formula

$$f(z) = \int_0^z \frac{dz}{\sqrt{z(1-z^2)}}$$

maps the upper half-plane $\mathbb{H}^+$ onto the interior of a square of side length

$$\frac{1}{2\sqrt{2\pi}}\Gamma\Big(\frac{1}{4}\Big)^2.$$

Exercise 3.8.3. Given a polygon D with vertices $w_1, \cdots, w_n$ and interior angles α_k $k = 1, \cdots, n$, has one of its angles, $\alpha_2 = 0$, say. See the figure below. Derive a Schwarz-Christoffel formula mapping the upper half-plane to this polygon. (Hint: Consider the polygon with $n+1$ sides constructed from that of the original polygon with a line segment drawn from new vertices w_{21} and w_{22} each on the parallel sides of D with $\alpha_2 = 0$ and perpendicular to the parallel sides. Use the Schwarz-Christoffel formula of this polygon to approximate the desired mapping).

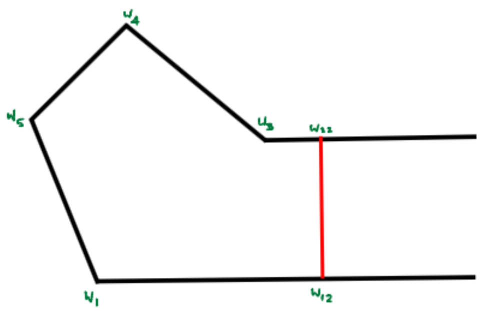

Figure 3.14: The second angle is 0

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